1. Find the differential equations for a geodesic in the cone

\[ S = \{(x, y, z) : x^2 + y^2 = z^2, \quad z > 0\} \]

(Write a system of equations like in the displayed formula at the top of page 42 of the textbook)

2. Let \( S \) be the set of zeros of the function \( f(x, y, z) = z - x^2 + y^2 \), that is,

\[ S = \{(x, y, z) : z = x^2 - y^2\} \]

It seems clear that the curve \( s \mapsto (s, 0, s^2) \) is the trajectory of a geodesic (i.e., a geodesic after reparametrization). Verify that this is the case as follows. Let

\[ \gamma(t) = (s(t), 0, s(t)^2) \]

with some function \( s(t) \). Assume that \( \gamma \) has constant speed:

\[ s'(t)^2 + (2s(t)s'(t))^2 = c^2 \]

with some constant \( c > 0 \). Differentiating this with respect to \( t \) you get an equation from which you can find a formula for \( s''(t) \) in terms of \( s(t) \) and \( s'(t) \). Use that formula for \( s''(t) \) to write \( \dot{\gamma}(t) \) using only \( s(t) \) and \( s'(t) \), and verify that the resulting expression is indeed proportional to \( \nabla f(\gamma(t)) \). (You may repeat this problem with the curve \( (0, s, -s^2) \).)

3. Let \( S^2 \) be the the unit sphere in \( \mathbb{R}^3 \): \( S = \{x^2 + y^2 + z^2 = 1\} \). Let \( v_1 \) and \( v_2 \) be elements of \( S^2 \) (so have unit length) that are orthogonal to each other, i.e.,

\[ v_1 \cdot v_2 = 0 \]

Verify first that

\[ \gamma(t) = \cos(t)v_1 + \sin(t)v_2 \]

is a curve in \( S^2 \), then that it is a geodesic of \( S^2 \).