1. Use the parametrization
\[
\Phi(s, \theta) = (s, \sqrt{1 + s^2} \cos \theta, \sqrt{1 + s^2} \sin \theta)
\]
of the surface
\[
M = \{(x, y, z) : 1 + x^2 - y^2 - z^2 = 0\}
\]
to compute \(dA\). Use the orientation in which the normal field at \(p_0 = (0, 0, 1)\) is \(\langle p_0; 0, 0, 1 \rangle\). Next, compute the area of the part of \(M\) between the planes \(x = -1\) and \(x = 1\).

In the following three problems, \(V\) is a vector space over \(\mathbb{R}\) and \(\lambda_1, \lambda_2, \ldots, \eta_1, \eta_2, \ldots\) are linear maps \(V \to \mathbb{R}\).

Recall that \(\lambda_1 \otimes \lambda_2 : V \to V\) is defined as
\[
(\lambda_1 \otimes \lambda_2)(v_1, v_2) = \lambda_1(v_1) \lambda_2(v_2),
\]
similarly one defines
\[
\lambda_1 \otimes \lambda_2 \otimes \lambda_3 : V \times V \times V \to \mathbb{R},
\]
as \((\lambda_1 \otimes \lambda_2) \otimes \lambda_3\) and so on. We don’t need parenthesis because of the associativity property of multiplication in \(\mathbb{R}\). You will need the following formulas:
\[
\lambda_1 \wedge \lambda_2 = \frac{1}{2} (\lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1),
\]
and
\[
\lambda_1 \wedge \lambda_2 \wedge \lambda_3 = \frac{1}{3!} (\lambda_1 \otimes \lambda_2 \otimes \lambda_3 + \lambda_3 \otimes \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_3 \otimes \lambda_1
\]
\[- \lambda_2 \otimes \lambda_1 \otimes \lambda_3 - \lambda_1 \otimes \lambda_3 \otimes \lambda_2 - \lambda_3 \otimes \lambda_2 \otimes \lambda_1)
\]

Suggestion for all problems below: you mostly only need to use the definition of the wedge product \(\wedge\), not of the tensor product \(\otimes\).

2. Verify that \(\lambda_1 \wedge (\eta_1 + \eta_2) = \lambda_1 \wedge \eta_1 + \lambda_1 \wedge \eta_2\). (This is just a warm-up exercise.)

3. Suppose \(\lambda_1 = a_{11}\eta_1 + a_{12}\eta_2\), \(\lambda_2 = a_{21}\eta_1 + a_{22}\eta_2\) for some numbers \(a_{ij}\). Compute \(\lambda_1 \wedge \lambda_2\) in terms of \(\eta_1 \wedge \eta_2\). If you recognize the coefficient, make a statement.

4. Verify that
\[
\lambda_1 \wedge \lambda_2 \wedge \lambda_3 = \frac{1}{3!} (\lambda_2 \wedge \lambda_3 \otimes \lambda_1 \wedge \lambda_3 \otimes \lambda_2 + \lambda_2 \wedge \lambda_1 \otimes \lambda_3 \wedge \lambda_1)
\]

5. Suppose \(n \in V\) is some fixed element. Write a formula for the function
\[
V \times V \ni (v_1, v_2) \mapsto (\lambda_1 \wedge \lambda_2 \wedge \lambda_3)(v_1, v_2, n) \in \mathbb{R}
\]
involving only the numbers \(\lambda_k(n)\) and wedge products of pairs of elements of \(\{\lambda_1, \lambda_2, \lambda_3\}\). Hint: Take advantage of the formula in the last problem (Problem 4). This function is denoted \(n \upharpoonright \lambda_1 \wedge \lambda_2 \wedge \lambda_3\) (there is a \((−1)^2\) in front of this expression which we don’t see for obvious reasons).

6. Let \(M\) be the graph of some smooth function \(f : U \to \mathbb{R}\). The domain \(U\) is an open set in \(\mathbb{R}^2\). Parametrize \(M\) by
\[
\Phi(s, t) = (s, t, f(s, t))
\]
Let
\[ \vec{S}(s,t) = (\Phi(s,t); \frac{\partial \phi}{\partial s}(s,t)) \quad \text{and} \quad \vec{T}(s,t) = (\Phi(s,t); \frac{\partial \phi}{\partial t}(s,t)) \]

let
\[ \vec{V}(s,t) = \vec{S}(s,t) \times \vec{T}(s,t), \]

finally let
\[ \vec{N}(s,t) = \frac{1}{\|\vec{V}(s,t)\|} \vec{V}(s,t). \]

We will use \( \vec{N} \) for the orientation of \( M \). Compute \( \vec{S}(s,t), \vec{T}(s,t) \) and \( \vec{N}(s,t) \).

Write \( x, y, z \) for the coordinates in \( R^3 \). The following computations are independent from each other, but the results are related.

1. Compute \( \vec{N}(s,t) \] \( dx \wedge dy \wedge dz \). (Use your answer in Problem 5.)
2. Compute \( (dx \wedge dy \wedge dz)(\vec{S}(s,t), \vec{T}(s,t), \vec{N}(s,t)) \). Part (1) may seem redundant but it helps organize the calculations.
3. Compute
\[
\begin{vmatrix}
\vec{S}(s,t) \\
\vec{T}(s,t) \\
\vec{N}(s,t)
\end{vmatrix}
\]
4. The formula in Part 1 has the form
\[ \alpha(s,t) dx \wedge dy + \beta(s,t) dx \wedge dz + \gamma(s,t) dy \wedge dz \]
for some functions \( \alpha, \beta, \gamma \). Replace \( dx \) by the differential of the first component of \( \Phi \), \( dy \) by that of the second, \( dz \) by that of the third in this expression. Combine the result so that it is expressed as \( h(s,t) ds \wedge dt \) for some function \( h \). Draw conclusions as to the relations with your answers in Parts 2 and 3.