Matrix Basic Concepts

Topics:

• What is a matrix?
• Matrix terminology
  • Elements or entries
    • Diagonal entries
  • Address/location of entries
  • Rows and columns
  • Size of a matrix
  • A column matrix; vectors
• Special types of matrices
  • Square
  • Diagonal
  • Triangular
  • Identity matrix
• Operations on matrices
  • Equal matrices
  • Addition, subtraction
  • A number (scalar) times a matrix
  • A matrix times a column matrix
Common Matrix Terminology

A **matrix** is a rectangular array of rows and columns. For example

\[
\begin{bmatrix}
4 & -2 & \pi \\
7 & 3 & 1/10
\end{bmatrix}.
\]

We usually name matrices using letters; for instance, \( B = \begin{bmatrix} 4 & -2 & \pi \\ 7 & 3 & 1/10 \end{bmatrix} \). (Capital letters are often used.)

This convention lets us easily refer to a matrix using the letter assigned, here \( B \). The numerical values within a matrix are called **elements** or **entries** of the matrix. **Each** matrix entry has an address/location given by the number of the row and the number of the column in which it appears.

Entries of a matrix are often denoted by a lower case letter of it’s assigned name with a **pair of subscripts denoting the row, column position of the entry**.

**Example:** For \( B = \begin{bmatrix} 4 & -2 & \pi \\ 7 & 3 & 1/10 \end{bmatrix} \) we have \( b_{11} = 4 \), \( b_{13} = \pi \), and \( b_{21} = 7 \).

The entries occupying like numbered row, column position are referred to as **diagonal entries**. For the matrix \( B \) in the preceding example the diagonal entries are \( b_{11} = 4 \) and \( b_{22} = 3 \).
The size of a matrix is denoted by listing the number of rows followed by the number of columns. Matrix \( B \) is \( 2 \times 3 \); it has two rows and 3 columns. The number of rows is always stated first.

A column matrix consists of a number of rows and a single column. Each of the following is a column matrix. It is common practice to use lower case letters for column matrices. We have indicated the size of the column below the matrix.

\[
\begin{align*}
\mathbf{w} &= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, & \mathbf{z} &= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 9 \end{bmatrix}, & \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*}
\]

3×1  4×1  2×1

Column matrices are often called vectors. This is from the geometric notions of directed line segments in the plane or 3-space. Such geometric vectors are designated by the coordinates of the end point of the line segment when we draw it starting from the origin.
Special types of matrices.

• A **square matrix** has the same number of rows as column. 

\[
\begin{bmatrix}
3 & -2 \\
1 & 8 \\
e^3 & 12 & -0.007
\end{bmatrix}
\]

\[2 \times 2 \quad 3 \times 3\]

• A **diagonal matrix** is a square matrix in which the non-diagonal entries are all zero.

\[
\begin{bmatrix}
7 & 0 & 0 \\
0 & -9 & 0 \\
0 & 23 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-3 & 0 & 0 \\
0 & 89 & 0 \\
0 & 0 & 17 \\
\end{bmatrix}
\]

Called zero matrices.

• A diagonal matrix with all diagonal entries equal to 1 is called an **identity matrix**.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Denoted \(I_2\) and \(I_3\) respectively.

The \(n \times n\) identity is denoted \(I_n\).

• An **upper triangular** matrix is a square matrix with all zero entries below the diagonal.

\[
\begin{bmatrix}
1 & 2 & 3 & -2 & 0 \\
0 & 4 & 0 & 7 & -9 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\]

• An **lower triangular** matrix is a square matrix with all zero entries above the diagonal.

\[
\begin{bmatrix}
6 & 0 \\
2 & -1 \\
1 & 0 & 0 \\
2 & -3 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Matrix Operations

Definition  Two m \times n matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are said to be equal matrices if \( a_{ij} = b_{ij} \) for \( 1 \leq i \leq m, 1 \leq j \leq n \), that is, if corresponding entries are equal.

Definition  If \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are both \( m \times n \) matrices, then their sum, \( A + B \), is the matrix whose \((i,j)\)-entry is \( a_{ij} + b_{ij} \); that is, we add corresponding entries. The difference, \( A - B \), is the matrix whose \((i,j)\)-entry is \( a_{ij} - b_{ij} \); that is, we subtract corresponding entries.

\[
A = \begin{bmatrix} 1 & 4 \\ 6 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 9 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad A + B = \begin{bmatrix} 11 & 13 \\ 6 & -7 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} -9 & -5 \\ 6 & -9 \end{bmatrix}
\]

The next operation involves numbers (possibly real or complex), which we call scalars and a matrix. The entries of the matrix could be real numbers, complex numbers, or even functions; we have not restricted the entries of a matrix.

Definition  If \( A = [a_{ij}] \) is an \( m \times n \) matrix and \( k \) is a scalar, then the scalar multiple of \( A \) by \( k \), \( kA \), is the \( m \times n \) matrix whose entries are \( ka_{ij} \); that is, each entry of matrix \( A \) is multiplied by \( k \).

\[
7 \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 35 & 0 \end{bmatrix}, \quad 2i \begin{bmatrix} 4 & 1+i & 6 \\ 0 & -2 & i \\ 3 & -1 & 2-3i \end{bmatrix} = \begin{bmatrix} 8i & 2i - 2 & 12i \\ 0 & -4i & -2 \\ 6i & -2i & 4i + 6 \end{bmatrix}
\]
The matrix operations developed so far involved element-by-element manipulations. Here we introduce an operation that involves row-by-column dot products.

**Definition** Let \( A \) be an \( m \times n \) matrix and \( c \) be a column matrix with \( n \) entries, that is, an \( n \times 1 \) matrix. Then the matrix \( A \) times column \( c \) is the \( n \times 1 \) matrix whose entries are obtained by taking dot product of each row of \( A \) with the column matrix \( c \).

\[
Ac = \begin{bmatrix}
\text{row}_1(A) \cdot c \\
\text{row}_2(A) \cdot c \\
\vdots \\
\text{row}_n(A) \cdot c
\end{bmatrix}
\]

**Warning:** If the number of columns of \( A \) is not equal to the number of rows of \( c \), then the product of \( A \) times \( c \) is **not defined**.

\[
A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} c = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \implies Ac = \begin{bmatrix} 2 \cdot 6 - 4 \cdot (-2) \\ 3 \cdot 6 - 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 20 \\ 16 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 3 & -2 & 4 \\ -5 & 0 & 1 \end{bmatrix} c = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \implies Ac = \begin{bmatrix} 3 \cdot 3 - 2 \cdot -5 + 4 \cdot 1 \\ -2 \cdot 3 \\ 4 \cdot 3 \end{bmatrix} = \begin{bmatrix} 22 \\ -7 \end{bmatrix}
\]