1. Let \( \{ A_i \} \) be a family of connected subsets of a topological space \( X \), such that \( A_i \cap A_j \neq \emptyset \) for all \( i, j \). Show that \( \bigcup_i A_i \) is connected.

2. Let \( A \subset X \) be a connected set, and suppose that \( A \subset Y \subset \overline{A} \). Prove that \( Y \) is connected.

3. Is a product of path-connected spaces necessarily path-connected?

4. Let \( X = \mathbb{R}^N \) be the set of all sequences of real numbers. Thus a point of \( X \) has the form \( x = (x_1, x_2, x_3, \ldots) \). Define a metric \( D \) on \( X \) by

\[
D(x, y) = \begin{cases} 
1, & \text{if } |x_n - y_n| \geq 1 \text{ for some } n \in \mathbb{N}, \\
\sup \{|x_n - y_n| : n \in \mathbb{N}\}, & \text{otherwise.}
\end{cases}
\]

Prove that \( x \) and \( y \) lie in the same path component of \( \mathbb{R}^N \) if and only if the sequence

\[
x - y = (x_1 - y_1, x_2 - y_2, \ldots)
\]

is bounded.

5. For each of the following spaces, find an atlas of finitely many charts to \( \mathbb{R}^n \) (with the appropriate \( n \)).
   a) The circle \( S^1 \).
   b) The torus \( T^2 = S^1 \times S^1 \).
   c) The projective plane \( \mathbb{RP}^2 \), obtained as the quotient of \( S^2 \) with antipodal points identified.
   d) The 3–torus \( T^3 = S^1 \times S^1 \times S^1 \).

6. Extra credit. Let \( X \) be the set of all lines in \( \mathbb{R}^2 \), with the topology (or the metric) coming from the charts that we discussed in class. Let \( Y \subset X \) be the set of all lines whose Euclidean distance to the origin is at most 1. Prove that \( Y \) is homeomorphic to a Möbius strip.

   Hint: the correspondence between lines and polar coordinates for points is key here.