Math 8062 Homework 3
Due Wednesday, 2/27/11

1. There is a standard way to glue together two (connected) manifolds $M$ and $N$ of the same dimension. Remove an open ball $B^n$ from each of $M$ and $N$, and glue $M \setminus B^n$ to $N \setminus B^n$ along the two $(n-1)$ dimensional boundary spheres. The resulting manifold is called the connected sum of $M$ and $N$, and is denoted $M \# N$.

   a) Prove that when $n \geq 3$, $\pi_1(M \setminus B^n) \cong \pi_1(M)$.

   b) Prove that when $n \geq 3$, $\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$.

2. Let $M$ be a manifold of dimension $n \geq 4$, and let $C$ be a knot (i.e., an embedded circle) in $M$.

   a) Prove that $\pi_1(M \setminus C) \cong \pi_1(M)$.

   b) Is the hypothesis that $n \geq 4$ necessary?

3. Let $S$ be a compact, connected surface with boundary. Let $X = X^2$ be a cell complex structure on $S$, in which every 2–cell is an embedded polygon. Prove that $X$ deformation retracts into the 1–skeleton $X^1$, and conclude that $\pi_1(S)$ is a free group.

   For extra credit, extend this argument to dimension $n$: that is, show an $n$–manifold with boundary deformation retracts into its $(n-1)$ skeleton.

4. Do problem 8 on page 53 of Hatcher.

5. Do problem 14 on page 54 of Hatcher.

   General hint: for some of these problems, van Kampen’s theorem will be more useful than Proposition 1.26 (the application to cell complexes).