1. Let $X = S^2 \cup A$, where $A$ is an axis connecting the north and south poles of $S^2$. Describe the universal cover $\tilde{X}$ of $X$ and the action of $\pi_1(X)$ on the universal cover.

2. A covering map $p : Y \to X$ is called regular if the corresponding subgroup $p_\ast \pi_1(Y, y_0)$ is normal in $\pi_1(X, x_0)$. Otherwise, the cover is irregular. Construct irregular covers of the Klein bottle by the Klein bottle and by the torus.

3. Let $M$ and $N$ be (topological) manifolds, with a covering map $p : N \to M$. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas on $M$, such that $\varphi_\alpha(U_\alpha)$ is an open ball $B_\alpha \subset \mathbb{R}^n$.

   a) For each $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$, prove that $p^{-1}U_\alpha$ is a disjoint union of slices, each mapped homeomorphically to $U_\alpha$.

   b) Prove that
   \[ \mathcal{B} = \{(V_\alpha, \varphi_\alpha \circ p) : V_\alpha \text{ is a component of } p^{-1}U_\alpha\} \]
   is a smooth atlas on $N$.

   c) Prove that the smooth atlas $\mathcal{B}$ makes $p$ into a smooth map, and furthermore that $p : V_\alpha \to U_\alpha$ is a diffeomorphism.

   d) Let $Y$ be a smooth vector field on $M$. Construct a smooth vector field $Z$ on $N$, such that $p_\ast Z_x = Y_{p(x)}$, for all $x \in N$. (Here, $p_\ast$ denoted the pushforward of tangent vectors, not the induced action on $\pi_1$.) Note that this gives a natural way to pull back vector fields to a cover, whereas normally vectors push forward and co-vectors (and differential forms) pull back.