Math 8061 Homework 9
Due Wednesday, 12/12/12

1. Suppose $M$ and $N$ are connected, oriented smooth $n$–manifolds, and $f : M \to N$ is an immersion. Prove that $f$ is orientation–preserving everywhere or orientation–reversing everywhere.

2. Let $T^2 \subset \mathbb{R}^4$ be a torus, parametrized by the map

$$f(\theta, \varphi) = (\cos \theta, \sin \theta, \cos \varphi, \sin \varphi).$$

That is, charts on $T^2$ are local inverses of $f$. Then $T^2$ is oriented by the 2–form $d\theta \wedge d\varphi$.

Compute $\int_{T^2} \omega$, where $(w, x, y, z)$ are the coordinates on $\mathbb{R}^4$, and $\omega = xyz \, dw \wedge dy$.

3. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let $\omega$ be a closed 1–form on $M$. Let $C$ be the unit circle, oriented counterclockwise. Prove that $\omega$ is an exact form if and only if $\int_C \omega = 0$.

*Hint for the “only if” argument:* define a function $f : M \to \mathbb{R}$ by

$$f(p) = \int_\gamma \omega,$$

where $\gamma$ is an arbitrary piecewise–smooth curve from $(1,0)$ to $p$. You will need to argue that this is well–defined, i.e., that the definition does not depend on the choice of $\gamma$. This is where Stokes’ theorem (and invariance under homotopy) should be of help.