1. Consider the following vector fields on \( \mathbb{R}^2 \):

\[
V = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}, \quad W = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.
\]

Prove that these vector fields do not commute in two ways:

a) by checking that \([V,W] \neq 0\).

b) by computing their flows, and checking that they do not commute.

2. Let \( f : M \to N \) be a diffeomorphism, and let \( X \) and \( Y \) be vector fields on \( M \). Prove that \( f_*[X,Y] = [f_*X,f_*Y] \).

3. Let \( V \) and \( W \) be the vector fields that span the contact structure on \( \mathbb{R}^3 \):

\[
V = Y = \frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}.
\]

Let \( \gamma : [0,1] \to \mathbb{R}^2 \) be an embedded smooth curve in the \( xy \) plane, such that \( \gamma(1) = \gamma(0) \).

a) Prove that there is a smooth curve \( \delta : [0,1] \to \mathbb{R}^3 \), such that \( \delta'(t) \) is a linear combination of \( V \) and \( W \), and such the projection \( \pi_{xy} \) to the \( xy \) plane sends \( \delta(t) \) to \( \gamma(t) \).

Hint: \( \delta(t) \) can be defined by integrating \( \delta'(t) \). So, define \( \delta'(t) \) so that all the requirements are satisfied.

b) Note that since \( \delta \) projects to \( \gamma \), the point \( \delta(1) \) must be directly above or below \( \delta(0) \). Prove that the vertical distance \( |\delta(1) - \delta(0)| \) is equal to the area enclosed by \( \gamma \).

Hint: First, prove this in the special case where \( \gamma \) is the boundary of a rectangle. Then, use rectangles to approximate an arbitrary smooth curve \( \gamma \).

This is a special case of Stokes’ theorem, and the argument in part (b) mimics the proof.

4. Let \( \mathcal{D} \) be the distribution on \( \mathbb{R}^3 \) spanned by

\[
X = \frac{\partial}{\partial x} + yz \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.
\]

a) Find an integrable submanifold of \( \mathcal{D} \) passing through the origin.

b) Is \( \mathcal{D} \) involutive? Explain your answer in light of part (a).