1. Let $T$ be the torus in $\mathbb{R}^3$ defined by the equation $(r-2)^2 + z^2 = 1$, in cylindrical coordinates. One way to parametrize $T$ is via charts of the form

$$\sigma(\varphi, \theta) = ((2 + \cos \varphi) \cos \theta, (2 + \cos \varphi) \sin \theta, \sin \varphi),$$

for coordinates $(\varphi, \theta)$ that each vary in an interval of length less than $2\pi$. One way to picture the dependence on coordinates is the following. If $\theta$ is fixed and $\varphi$ varies, we are walking around a circle of radius 1 in a vertical plane through the $z$–axis. If $\varphi$ is fixed and $\theta$ varies, we are walking around a circle in a horizontal plane, centered on the $z$–axis.

(a) How many charts of this type are needed to get an atlas for $T$?

(c) For given $(\varphi, \theta)$, find a unit normal vector at $\sigma(\varphi, \theta)$. Do these normal vectors depend continuously (and smoothly) on the coordinates?

(b) Prove that $T$ is orientable.

2. Let $S$ be a surface of revolution about a line $L$. Prove that rotation about $L$ by any angle is a diffeomorphism of $S$.

3. Let $S$ be a smooth surface in $\mathbb{R}^3$, and let $P$ be a plane that intersects $S$ at the origin and only the origin: $P \cap S = \{0\}$. Prove that $P = T_0 S$, the tangent plane to $S$ at 0.

4. Let $S$ be a smooth surface in $\mathbb{R}^3$, and suppose that every normal line to $S$ (that is, every line through $p \in S$ spanned by the normal vector to $T_p S$) passes through the $z$–axis.

(a) Let $\gamma$ be a curve contained in $S$, whose $z$–coordinate stays at a constant height $h$. (In other words, $\gamma$ is contained in a horizontal plane.) Prove that, for every unit–speed parametrization of $\gamma$, the vector $\gamma'(t)$ points toward $(0, 0, h)$.

(b) Prove that the curve $\gamma$ is part (a) is a circle centered at $(0, 0, h)$.

(c) Prove that $S$ is a surface of revolution about the $z$–axis.