Review Questions for the Final Exam
Math 320, Fall 2006

1. Are the following true or false? Give a brief explanation or a counterexample.
   - If $\sup A \leq \inf B$, and $A$ does not have a maximum, then $a < b$ for all $a \in A$ and $b \in B$.
   - If the sequences $(a_n)$ and $(b_n)$ converge, then $(a_nb_n)$ converges.
   - Every bounded, monotonic sequence is Cauchy.
   - If $\sum a_n$ converges, and $(b_n)$ is a bounded sequence, then $\sum a_nb_n$ converges.
   - An open set cannot contain any isolated points.
   - If $A$ is a bounded set, then $\sup A$ is a limit point of $A$.
   - Every non-empty compact set contains a non-empty open set.
   - If $f : A \to \mathbb{R}$ is differentiable, and $f'(x) > 0$ for all $x$, then $f$ is 1-to-1.
   - If $f : A \to \mathbb{R}$ is differentiable, $A$ is connected, and $f'(x) > 0$ for all $x$, then $f$ is 1-to-1.
   - If $f_n$ converges to $f$ on an interval $A$, and each $f_n$ is an increasing function, then $f$ is increasing.
   - If $f_n \to f$ uniformly on an interval $A$, and each $f_n$ is differentiable, then $f$ is differentiable.

2. A Buddhist monk leaves his monastery at 7 A.M. and climbs the neighboring mountain, arriving at the top at 7 P.M. After a night of meditation on the mountaintop, he starts descending at 7 A.M. the next day, and arrives at his monastery at 7 P.M. Prove that there is a time $t$, such that at time $t$ the monk was at the same elevation on both days.

3. Prove that the function $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$. (Hint: show that $|f'(x)| \leq 1$ on this interval, and use the Mean Value Theorem.) Is $f(x)$ uniformly continuous on $(0, \infty)$?

4. Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{3^n}$.
   - Prove that the sum converges on $\mathbb{R}$, and that $g(x)$ is continuous on $\mathbb{R}$. Is $g$ differentiable? Twice differentiable?

5. Let $h(x) = \sum_{n=1}^{\infty} nx^{n-1}$.
   - Prove that this series converges and defines a continuous function on $(-1, 1)$. (Hint: what function has $h(x)$ as its derivative?) Make sure that you reference all necessary theorems in your argument.