Midterm Exam 2
Math 320-02, Fall 2006

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name:  

ID #:  

1. _________ (/20 points)

2. _________ (/30 points)

3. _________ (/25 points)

4. _________ (/25 points)

Total _________ (/100 points)

Homework Average _________

Course Average _________
1. [20 points] State the definitions of the following terms or expressions.

(a) limit point of a set

Let $A \subset \mathbb{R}$, $\alpha \in \mathbb{R}$. Then $\alpha$ is a limit point of $A$ if for all $\varepsilon > 0$, the neighborhood $U_{\varepsilon}(\alpha)$ intersects $A$ in a point other than $\alpha$.

(b) compact set

A set $K$ is compact if every sequence contained in $K$ has a convergent subsequence that converges to a limit in $K$.

(c) $\lim_{x \to c^+} f(x) = L$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(d) $f(x)$ is uniformly continuous

$f(x)$ is uniformly continuous on $A$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. 
2. [30 points] True/False/Explain. State whether each of the following statements is true or false. Then explain your answer, in one or two sentences. Provide a counterexample where it's relevant. This problem does not need complete proofs - don't spend time writing them!

(a) Every finite set is closed.

True. A finite set has no limit points, and thus contains all its (nonexistent) limit points.

(b) If \( K \) is a non-empty compact set, then \( \text{sup} \ K \) exists, and is contained in \( K \).

True. If \( K \) is compact, it is closed and bounded. Since \( K \) is bounded, \( \text{sup} \ K \) exists. If \( \text{sup} \ K \) is a limit point, it must be in \( K \) because \( K \) is closed. Otherwise, it's an isolated point of \( K \), and is thus also in \( K \).

(c) Every non-empty open set is uncountable.

True. A non-empty open set contains an interval \( V_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \). This interval has the same cardinality as \( \mathbb{R} \), so the open set is uncountable.
True/False/Explain, continued.

(d) If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} f(x)g(x) \) both exist, then \( \lim_{x \to c} g(x) \) exists also.

**False**. Let \( c = 0 \), \( f(x) = x \), \( g(x) = \frac{1}{x} \).

Then \( \lim_{x \to 0} f(x) g(x) = \lim_{x \to 0} (1) \) exists, and

\( \lim_{x \to 0} f(x) = \lim_{x \to 0} x \) also exists, but

\( \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x} \) does not exist.

(e) There is a non-empty compact set \( K \) and a continuous function \( h : K \to \mathbb{R} \), such that \( h(K) \) is an open set.

**False**. If \( K \) is compact and \( h \) is continuous, \( h(K) \) is also compact. Thus \( h(K) \) is closed, a non-empty, and bounded (hence \( h(K) \neq \mathbb{R} \)). The only sets that are both open and closed are \( \emptyset \) and \( \mathbb{R} \).

(f) There is an increasing function \( f : \mathbb{R} \to \mathbb{R} \) whose set of discontinuity \( D_f \) is the Cantor set.

**False**. \( D_f \) for an increasing function is finite or countable. On the other hand, \( C \) is uncountable.
3. [25 points] Let \( A \) and \( B \) be closed sets. Prove that \( A \cup B \) is closed.

We need to prove that \( A \cup B \) contains its limit points. So let \( x \) be a limit point of \( A \cup B \). Then there is a sequence \( (x_n) \rightarrow x \), where \( x_n \in A \cup B \) for all \( n \). \((x_n)\) must have a subsequence \((x_{n_k})\) that is contained in just \( A \) or just \( B \). Then

\[
\lim (x_{n_k}) = \lim (x_n) = x,
\]
so \( x \) is a limit point of \( A \) or \( B \).
Since \( A \) and \( B \) are closed, \( x \in A \) or \( x \in B \).
Therefore, \( x \in A \cup B \), and \( A \cup B \) is closed.

Alternate approach: use the definition of "limit point" instead of sequences.

Alternate approach: take complements of \( A \) and \( B \), which are open sets. Then prove that the intersection of two open sets is open.
4. [25 points]
(a) Prove that the function \( f(x) = 2|x| \) is uniformly continuous on \([-1, 1]\).

Choose an arbitrary \( \varepsilon > 0 \), and let \( \delta = \varepsilon / 2 \). Now, suppose that \( x, y \in [-1, 1] \) and \( |x - y| < \delta \). Then:

\[
|f(x) - f(y)| = |2|x| - 2|y|| \\
\leq 2|x - y| \\
< 2 \delta \\
= \varepsilon.
\]

Thus, \( f(x) \) is uniformly continuous (and in particular, continuous).

(b) Use part (a) to prove that \( g(x) = \frac{2|x|}{4|x| + 1} \) is uniformly continuous on \([-1, 1]\).

Since \( f(x) \) is continuous on \([-1, 1]\),

\[4|f(x)| = 4f(x) = \frac{2|x|}{4|x| + 1}\]

is continuous.

\( \Rightarrow \)

\( g(x) \) is continuous (and \( g(x) \neq 0 \) on \([-1, 1]\))

\( \Rightarrow \) \( g(x) = \frac{2|f(x)|}{2f(x) + 1} \) is continuous on \([-1, 1]\).

Since \([-1, 1]\) is a compact set and \( g(x) \) is continuous on \([-1, 1]\), it must be uniformly continuous.

(Note: combinations of uniformly continuous functions are not automatically uniformly continuous.)