Midterm Exam 1
Math 320-02, Fall 2006

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions

ID #: _____________________________

1. __________ ( /20 points)

2. __________ ( /30 points)

3. __________ ( /25 points)

4. __________ ( /25 points)

Total __________ ( /100 points)

Homework Average __________

Course Average __________
1. [20 points] State the definitions of the following terms or expressions.

(a) \( \inf A \)

\[ p = \inf A \text{ if} \]

- \( p \) is a lower bound on \( A \) (for all \( a \in A \), \( p \leq a \))
- every lower bound \( b \) for \( A \) satisfies \( b \leq p \).

(b) A set \( S \) is uncountable. (Please give a direct definition of this, without relying on the notion of countable.)

\( S \) is uncountable if \( S \) is infinite and there does not exist a 1-1, onto function \( f : \mathbb{N} \to S \).

(c) Cauchy sequence

A sequence \( (a_n) \) is Cauchy if \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. for all } m, n \geq N, |a_m - a_n| < \varepsilon \).

(d) The series \( \sum_{k=1}^{\infty} a_k \) converges.

Let \( S_n = \sum_{k=1}^{n} a_k \), for each \( n \in \mathbb{N} \). Then \( \sum_{k=1}^{\infty} a_k \) converges if \( (S_n) \to L \) for some \( L \in \mathbb{R} \).

That is, \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. if } n \geq N, |a_n - L| < \varepsilon \).
2. [30 points] **True/False/Explain.** State whether each of the following statements is true or false. Then explain your answer, in one or two sentences. Provide a counterexample where it’s relevant. *This problem does not need complete proofs – don’t spend time writing them!*

(a) If \( \sup A \leq \sup B \), then some number \( b \in B \) is an upper bound for \( A \).

**False.** For example, \( A = B = (0, 1) \).

Then \( \sup A = \sup B = 1 \), but no number \( b \in B \) is an upper bound for \( A \), since \( A = B \) does not have a maximum.

(b) The union of countably many countable sets is countable.

**True.** This was Theorem 1.4.13.

(c) If the sequence \( (a_n) \) diverges, then \( (1/a_n) \) converges.

**False.** For example, \( a_n = (-1)^n \). Then \( \frac{1}{a_n} = a_n = (-1)^n \), so \( (1/a_n) \) also diverges.
True/False/Explain, continued.

(d) There exists a sequence \((b_n)\), with the property that every real number \(x \in [0, 1]\) is the limit of a subsequence of \((b_n)\).

**True.** For example, we can use the sequence \((b_n) = (0.1, 0.2, \ldots, 0.9, 0.01, 0.02, \ldots, 0.99, 0.001, \ldots, 0.999, \ldots)\). Then every decimal expansion (hence every number \(x \in [0, 1]\)) is the limit of a subsequence of \((b_n)\).

Alternatively, we could use the fact that \(\mathbb{Q}\) is countable to arrange the rational numbers in a sequence.

(e) Every bounded sequence contains a monotonic subsequence.

**True.** By Bolzano-Weierstrass, a bounded sequence \((a_n)\) has a convergent subsequence \((a_{n_k})\). If \(L = \lim (a_{n_k})\), then the terms of \((a_{n_k})\) either to the left or right of \(L\) form a monotonic subsequence of \((a_n)\).

(f) If the series \(\sum a_n\) converges, then \(\sum a_n^2\) also converges.

**False.** For example, if \(a_n = \frac{(-1)^n}{\sqrt{n}}\), then \(\sum a_n\) converges by the Alternating Series Test. On the other hand, \(\sum a_n^2 = \sum \frac{1}{n}\) diverges.
3. [25 points] Let \( S = \left\{ \frac{n - 1}{2n} : n \in \mathbb{N} \right\} \).

(a) Find \( \sup S \).

\[
\sup S = \frac{1}{2}.
\]

(b) Prove that your answer is, in fact, the supremum of \( S \).

There are two properties to check:

1) Because \( \frac{n - 1}{2n} < \frac{n}{2n} = \frac{1}{2} \) for all \( n \), \( \frac{1}{2} \) is an upper bound for \( S \).

2) Let \( b \in \mathbb{R} \) be an upper bound for \( S \).

Suppose, for a contradiction, that \( b < \frac{1}{2} \). Then \( \frac{1}{2} - b > 0 \). By the Archimedean property, there is an \( n \in \mathbb{N} \) s.t.

\[
0 < \frac{1}{n} < \frac{1}{2} - b.
\]

In particular, \( 0 < \frac{1}{2n} < \frac{1}{2} - b \).

Then \( b < \frac{1}{2} - \frac{1}{2n} \)

\[
b < \frac{n}{2n} - \frac{1}{2n} = \left( \frac{n-1}{2n} \right) \in S. \quad (\text{Contradiction})
\]

Thus any number \( b < \frac{1}{2} \) can't be an upper bound.

By (1) and (2), \( \sup S = \frac{1}{2} \).
4. [25 points] Consider the sequence \((a_n)\), where

\[ a_n = (-1)^n \left( \frac{n-1}{2n} \right). \]

So \((a_n) = \left( \frac{0}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{10}, \frac{7}{12}, \ldots \right)\).

(a) Prove that \((a_n)\) does not converge to any real number.

We will prove that \((a_n)\) is not Cauchy.

Note that, for \(n > 1, |a_n| = \frac{n-1}{2n} \geq \frac{1}{4}\).

Thus \(a_n \geq \frac{1}{4}\) for \(n\) even,
\(a_n \leq -\frac{1}{4}\) for \(n\) odd (and \(n > 1\)).

So, choose \(\varepsilon = \frac{1}{4}\). Then, for any \(N \in \mathbb{N}\), no matter how large \(N\) is, we can always find \(m > N\) and \(n > N\), \(m\) odd, \(n\) even, so that \(|a_m - a_n| \geq \frac{1}{2} = \varepsilon\).

Since \((a_n)\) is not Cauchy, it cannot converge.

(b) In a sentence or two, describe an alternate way to prove that \((a_n)\) diverges. (There are at least three different ways to do this problem.)

Two alternate approaches:

1. From the definition. For any putative limit \(L \in \mathbb{R}\), we have to choose \(\varepsilon > 0\) such that there are infinitely many terms with \(|a_n - L| \geq \varepsilon\). \(\varepsilon = \frac{1}{4}\) will always work.

2. Subsequences. We could prove that the subsequence of odd terms converges to \(\frac{1}{2}\) and the subsequence of even terms to \(\frac{1}{2}\).