Midterm Exam 3
Math 153H, Spring 2008

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions

ID #: ________________________________

1. _________ (/20 points)

2. _________ (/20 points)

3. _________ (/20 points)

4. _________ (/20 points)

5. _________ (/20 points)

Total _________ (/100 points)

Homework Average _____________

Course Average _____________
1. [20 points] Evaluate the following integrals.

(a) \[ \int_{0}^{1} \frac{x}{\sqrt{1-x^2}} \, dx = \int_{0}^{\pi/2} \frac{\sin \theta}{\cos \theta} \, d\theta \]

\[ x = \sin \theta \]
\[ dx = \cos \theta \, d\theta \]
\[ \sqrt{1-x^2} = \cos \theta \]

\[ = \left[ \int_{0}^{\pi/2} \sin \theta \, d\theta \right]_{0}^{\pi/2} \]
\[ = 0 - (-1) \]
\[ = 1. \]

(b) \[ \int \frac{8x-10}{(x+1)(x-1)(x-2)} \, dx \]

\[ \frac{8x-10}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2} \]

\[ 8x-10 = A(x-1)(x-2) + B(x+1)(x-2) + C(x+1)(x-1) \]

\[ x = -1 \Rightarrow 8(-1) - 10 = A(-2)(-3) \]
\[ -18 = 6A \]
\[ A = -3 \]

\[ x = 1 \Rightarrow 8(1) - 10 = B(2)(-1) \]
\[ 2 = -2B \]
\[ B = 1 \]

\[ x = 2 \Rightarrow 8(2) - 10 = C(3)(1) \]
\[ 6 = 3C \]
\[ C = 2 \]

\[ \int \frac{8x-10}{(x+1)(x-1)(x-2)} \, dx = \int \frac{-3}{x+1} + \frac{1}{x-1} + \frac{2}{x-2} \, dx \]

\[ = -3 \ln \left| x+1 \right| + \ln \left| x-1 \right| + 2 \ln \left| x-2 \right| + C. \]
2. [20 points] Do the following improper integrals converge or diverge? Justify your answer.

(a) \[ \int_0^\infty x e^{-x} \, dx = -x e^{-x} \bigg|_0^\infty + \int_0^\infty e^{-x} \, dx \]

Integrate by parts:

\[ u = x, \quad dv = e^{-x} \, dx \]
\[ du = dx, \quad v = -e^{-x} \]

\[ \int_0^\infty (x e^{-x}) \, dx \]

\[ = \lim_{b \to \infty} \left( -x e^{-x} - e^{-x} \right) \bigg|_0^b \]
\[ = \lim_{b \to \infty} \left( -b - 1 \right) e^{-b} - 0 - (-1) \]
\[ = \lim_{b \to \infty} \frac{-b-1}{e^b} + 1 \]
\[ = \lim_{b \to \infty} \frac{-1}{e^b} + 1 \]
\[ = 1. \]

So the integral converges.

(b) \[ \int_0^\infty \frac{3 + \cos x}{x} \, dx \]

\[ > \int_0^\infty \frac{2}{x} \, dx \]

which diverges (both as \( x \to 0 \) and \( x \to \infty \)).

2 \leq 3 + \cos x \leq 4

Thus, by the comparison test, \[ \int_0^\infty \frac{3 + \cos x}{x} \, dx \]

diverges.
3. [20 points] Compute the following limits.

(a) \( \lim_{{x \to 0^+}} (\sin x)^x = L \).

\[
\lim_{{x \to 0^+}} x \ln (\sin x) = \ln L.
\]

\[
= \lim_{{x \to 0^+}} \frac{\ln (\sin x)}{1/x}
\]

\[
L' = \lim_{{x \to 0^+}} \frac{(\cos x)/(\sin x)}{(-1/x^2)}
\]

\[
= \lim_{{x \to 0^+}} \frac{-x^2 \cos x}{\sin x}
\]

\[
L' = \lim_{{x \to 0^+}} \frac{x^2 \sin x - 2x \cos x}{\cos x} = 0.
\]

(b) \( \lim_{{n \to \infty}} \frac{1}{\sqrt{n} + \sqrt{n - \sqrt{n}} + \sqrt{n}} \cdot \frac{\sqrt{n + \sqrt{n}} + \sqrt{n}}{\sqrt{n} + \sqrt{n} + \sqrt{n}} \cdot \frac{\sqrt{n + \sqrt{n}} + \sqrt{n}}{\sqrt{n} + \sqrt{n} + \sqrt{n}} \)

\[
= \lim_{{n \to \infty}} \frac{\sqrt{n + \sqrt{n}} + \sqrt{n}}{(n + \sqrt{n} - \sqrt{n}) - n}
\]

\[
= \lim_{{n \to \infty}} \frac{\sqrt{n + \sqrt{n}} + \sqrt{n}}{\sqrt{n}}
\]

\[
= \lim_{{n \to \infty}} \sqrt{1 + \frac{1}{\sqrt{n}}} + 1 = 2.
\]
4. [20 points] Consider the function \( f(x) = \sin(x^2) \) on the interval \([0, \pi]\).

(a) Set up an integral that expresses the length of the curve \( y = f(x) \). You do not have to evaluate the integral!

\[
f'(x) = 2x \cos(x^2)
\]

So \( ds = \sqrt{1 + (f'(x))^2} \, dx = \sqrt{1 + 4x^2 \cos^2(x^2)} \, dx \)

\[
\text{Arc Length} = \int_0^\pi \sqrt{1 + 4x^2 \cos^2(x^2)} \, dx
\]

(b) Suppose the graph of \( f(x) \) is revolved around the line \( y = -1 \). Set up an integral that expresses the area of the resulting surface. Again, you do not have to evaluate the integral!

\[
\text{Area} = \int_0^\pi 2\pi f(x) \cdot ds
\]

\[
= \int_0^\pi 2\pi (\sin(x^2) + 1) \sqrt{1 + 4x^2 \cos^2(x^2)} \, dx
\]
5. [20 points] Suppose you are trying to find the root of \( y = \sqrt{x} \). Describe what will happen when you search for the root with Newton’s method, starting from an initial guess of \( x_0 = 1 \).

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
= x_n - \frac{(x_n)^{1/3}}{\frac{1}{3} (x_n)^{-2/3}}
\]

\[
= x_n - 3 x_n
\]

\[
= -2 x_n.
\]

So, if \( x_0 = 1 \), we have

\[
x_1 = -2
\]

\[
x_2 = 4
\]

\[
x_3 = -8
\]

\[
x_4 = 16
\]

etc.

Thus Newton’s method takes us further and further away from the root at \( x=0 \).