RESEARCH OVERVIEW

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The central aim of my research program is to weave together several different points of view in low-dimensional topology. These are: the combinatorial topology of knots and 3–manifolds, their hyperbolic geometry, quantum invariants such as the Jones polynomial, and geometric group theory. At present, each of these viewpoints is fairly well developed, and has led to the resolution of many open problems. However, connections among them are still new and being freshly constructed. My work to build such connections can be organized into several themes:

Effective hyperbolic geometry. The work of Thurston, Perelman, and Mostow (see also Maher [69]) implies that almost every 3–manifold admits a hyperbolic metric, and that this metric is unique up to isometry. Thus there should be a dictionary between combinatorial descriptions of a manifold and geometric measurements such as volume and the lengths of geodesics. In recent years, such a dictionary has started to come into view [15, 75]. Harder still is the search for an *effective* dictionary – that is, estimates on geometry with explicit constants – which would greatly aid both computation and classification.

My results with several collaborators provide some of the first and strongest estimates of this sort. These include diagrammatic estimates on the volumes of knot and link complements [36, 37, 38, 39, 41, 59]; combinatorial estimates on the geometry of fibered 3–manifolds [35, 38, 50]; constructions of manifolds that are hard to distinguish by their geometric invariants [46]; and bounds on cosmetic Dehn surgeries [48]. See Section 1 for a detailed description.

The topology and geometry of quantum invariants. Ever since Jones introduced his polynomial in the mid-1980s, it has been an open and tantalizing question to decode what exactly this invariant and its relatives say about the geometry and topology of knot complements. For instance, Kashaev, Murakami, and Murakami conjectured that the asymptotic growth rate of colored Jones polynomials predicts the hyperbolic volume of a knot complement [66, 76]. While these conjectures are still wide open, my work with Kalfagianni and Purcell [34, 41, 42, 43] has described systematic connections between knot polynomials, certain associated graphs, and the geometric topology of incompressible surfaces. These results are described in Section 2.

3–Manifolds and group theory. In the last several years, spectacular advances in geometric group theory have been employed to resolve several long-standing conjectures about finite covers of 3–manifolds. Following the work of Agol [3], Kahn–Markovic [65], and Wise [91], we know that all aspherical 3–manifolds are virtually Haken, and all finite-volume hyperbolic 3–manifolds are virtually fibered. All of these results run via Wise's theory of non-positively cube complexes and special covers of these complexes. My recent work with Cooper [21] contributes to a streamlined second-generation proof of these important results, by providing the requisite cube complexes for noncompact hyperbolic manifolds. In current work with Bering, we also provide an effective estimate on the degree of a special cover [13]. See Section 3 for details.

Random groups and random manifolds. In addition to exploring what statements hold true for *all* manifolds or groups in a particular class, it is often interesting to see what properties are satisfied by a *generic* object produced using some well-defined random process. In a recent joint project with Taylor and Worden, we study the mapping tori of random mapping classes [51]. In a current project with Wise, we construct cubical actions of random quotients of hyperbolic groups [52]. These results are described in Section 4.

1. Effective hyperbolic geometry

1.1. **Link diagrams and geometry.** The idea that combinatorics should have an explicit and effective translation into hyperbolic geometry — often summarized as "what you see is what you get" topology, or "effective geometrization" — takes a particularly clear and visual form in the study of knots in S^3. This is because knots and links are easy to describe via planar diagrams, and it is natural to look for combinatorial features of the diagrams that will predict geometric invariants, such as volume or the length of geodesics.

One diagrammatic invariant that appears closely connected to hyperbolic geometry is called the *twist number*. A *twist region* is a section of a diagram $D(K)$ in which two strands of K wrap around each other maximally. (See Figure 1, left.) The number of these regions is called the *twist number* of $D(K)$, and is denoted $\mathrm{tw}(D)$. With this definition, Lackenby showed that the the twist number of an alternating diagram is approximately equal to the hyperbolic volume of the link complement [68]. More precisely, volume is bounded above and below by explicit linear functions of $\mathrm{tw}(D)$.

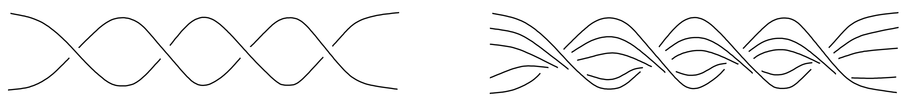

Figure 1. Left: a twist region. Right: a generalized twist region.

In a series of joint papers with Kalfagianni and Purcell, I have extended Lackenby's result to many families of non-alternating links. These include highly twisted links (where every twist region contains at least 7 crossings) [36], positive braids [41] Montesinos links [41], Conway sums of alternating tangles [37], and symmetric links [37, 45]. Some of these estimates are quite sharp. For instance, here is a sample theorem.

Theorem 1.1 ([38]). *Let K be a hyperbolic link obtained as the closure of a 3–string braid, and let $D(K)$ be the diagram corresponding to the Schreier normal form of the braid word. Then*

$$2v_3\,\mathrm{tw}_{\mathrm{gen}}(D) - 279 \;<\; \mathrm{vol}(S^3 \smallsetminus K) \;<\; 2v_8\,\mathrm{tw}_{\mathrm{gen}}(D),$$

where $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

Furthermore, the multiplicative constants in both the upper and lower bounds are optimal.

The additive constant of 279 is presumably far from sharp. Nonetheless, it is striking to see geometry and combinatorics so closely intertwined, and satisfying to know the sharp constants that constrain their relationship. In addition, Theorem 1.1 can also be applied to express hyperbolic volume in terms of certain coefficients of the Jones and colored Jones polynomials (see Section 2).

The proof of Theorem 1.1, as well as many of its relatives [36, 37, 39], uses Dehn surgery techniques. The main idea is that every link K in braid position has a *braid axis*, namely an unknot $A \subset S^3 \smallsetminus K$, whose complement $S^3 \smallsetminus A \cong D^2 \times S^1$ is foliated by disks that intersect K a constant number of times. Drilling out the axis produces a manifold $M = S^3 \smallsetminus (K \cup A)$ that fibers over S^1 with fiber an n–punctured disk, whose geometry is controlled via my work with Guéritaud [59, Appendix]. Finally, the volume of $S^3 \smallsetminus K$ can be understood in terms of this fibered manifold M using Theorem 1.2 below.

See also Theorem 2.3 for a different approach to estimating knot volumes using guts of surfaces.

1.2. **Geometric control via Dehn surgery.** Every closed 3–manifold can be constructed by a process called *Dehn surgery* or *Dehn filling*. Start with a manifold with boundary consisting of tori (such as the complement of a knot or link in S^3), and attach a solid torus to each boundary torus. The result depends only on the *slope* (isotopy class of simple closed curve) chosen to bound a disk in the solid torus. When the initial manifold M is hyperbolic, there is a natural way to measure the *length* of this slope. This is because each toroidal end of M is a *cusp*, homeomorphic to $T^2 \times [0, \infty)$, such that each cross-sectional torus $T^2 \times \{x\}$ is a quotient of a horosphere and carries a Euclidean metric. The maximal embedded neighborhood of this sort is called a *maximal cusp*, and the metric on its boundary is uniquely determined by the topology of M. We may measure the length of a slope in this Euclidean metric.

The length of the slope typically controls the outcome of the surgery. A foundational result of Thurston [86] is that if one starts with a hyperbolic manifold M and performs Dehn filling along a sufficiently long slope s, the resulting manifold $M(s)$ will have a hyperbolic structure very close to the original. While this result is very powerful, it is not effective in that it does not quantify the meaning of "sufficiently long" or bound the change in metric.

Some quantitative control on the change in geometry is provided in the work of Hodgson–Kerckhoff [61, 62] and Brock–Bromberg [17]. My collaborators and I have strengthened and extended the reach of their methods. In addition, we have added new tools to the suite of theorems that bound how much the geometry changes under filling. For instance, Kalfagianni, Purcell, and I proved the following explicit estimate on the change in volume [36].

Theorem 1.2 ([36])**.** *Let M be a cusped hyperbolic 3–manifold, and s a slope on a boundary torus of M. If the length of s on a maximal horospherical torus satisfies $\ell(s) > 2\pi$, then the filled manifold $M(s)$ has a complete hyperbolic metric satisfying*

$$\left(1-\left(\frac{2\pi}{\ell(s)}\right)^2\right)^{3/2} \mathrm{vol}(M) \;\leq\; \mathrm{vol}(M(s)) \;<\; \mathrm{vol}(M).$$

Thus, when $\ell(s)$ is sufficiently long, the volume of the filled manifold $M(s)$ is bounded below by $\mathrm{vol}(M)$ times an explicit constant close to 1, while being bounded above by $\mathrm{vol}(M)$.

To prove Theorem 1.2, we carefully construct a negatively curved metric on $M(s)$, with explicit control over sectional curvatures, volume, and the areas of surfaces. Then, we apply volume comparison theorems of Besson, Courtois, and Gallot [14] to relate the volume of the complete hyperbolic metric on $M(s)$ to the volume of our metric.

Since its appearance, Theorem 1.2 has found wide applications to both geometric and topological problems. Gabai, Meyerhoff, and Milley needed our estimate in the endgame of their project to identify the smallest–volume closed hyperbolic 3–manifold [53, 73]. Atkinson and I used this theorem in identifying the lowest volume orbifolds in certain families [8, 9]. Francaviglia, Frigerio, and Martelli used our result to bound the number of tetrahedra required to triangulate a particular family of manifolds [33]. Kalfagianni, Purcell, and I used Theorem 1.2 give diagrammatic estimates for the volumes of several large families of knots and links [36, 37, 38, 39]. Finally, the negatively curved metric constructed in the course of the proof has turned out to be useful in my work with Cooper and Purcell on Heegaard splittings and unknotting tunnels [22, 47].

In very recent work with Purcell and Schleimer [49, 48], I have extended the quantitative methods of Hodgson and Kerckhoff [61, 62] to give explicit bilipschitz estimates on the change in metric during hyperbolic Dehn filling. To control the fillings of a manifold M, we need to know the *systole* $\mathrm{sys}(M)$, namely the length of the shortest closed geodesic in M. The control also depends on the *normalized length* of a slope s on a cusp torus T, which is the scaling-invariant quantity $L(s) = \ell(s)/\sqrt{\mathrm{area}(T)}$. We prove the following.

Theorem 1.3 ([48])**.** *Let M be a cusped hyperbolic 3-manifold. Suppose that s is a slope on a cusp C, whose normalized length satisfies*

$$L(s) \geq \max\left\{10.1,\ \sqrt{\frac{2\pi}{\mathrm{sys}(M)} + 58}\right\}.$$

Then the filled manifold $M(s)$ is hyperbolic, and the core curve γ_s of the added solid torus is the unique shortest geodesic in $M(s)$.

Furthermore, if $s' \neq s$ is another slope such that $L(s')$ satisfies the same length bound, then $M(s)$ and $M(s')$ cannot be orientation-preservingly homeomorphic.

To prove Theorem 1.3, we control the lengths of γ_s and of a putative competitor curve through a one-parameter family of singular hyperbolic metrics connecting M to $M(s)$. This type of control implies a number of other quantitative results, such as information about Margulis numbers in the filled manifold $M(s)$.

The "furthermore" statement in Theorem 1.3 represents progress on the Cosmetic Surgery Conjecture. Gordon conjectured that distinct fillings of M always yield distinct oriented 3–manifolds [56]. Theorem 1.3 proves this conjecture for all fillings longer than an explicitly quantified cutoff. This leaves finitely many short fillings to check by computer, which is a manageable practical task for any given M. For instance, this method verifies the Cosmetic Surgery Conjecture for all knots in S^3 up to 13 crossings.

1.3. **Length spectra and commensurability.** Given that hyperbolic structures provide a rich source of topological invariants, it is natural to ask how good a job they do at distinguishing manifolds. Is a hyperbolic manifold M determined up to isometry by its volume or by the lengths of its geodesics?

In its most naive sense, this question has a negative answer. The number of manifolds sharing the same volume can be arbitrarily large [90, 74], although it is necessarily finite. The number of manifolds that share exactly the same *length spectrum* (lengths of geodesics, counted with multiplicity) can similarly be arbitrarily large [72]. However, every construction of isospectral examples works by taking finite covers of the same base manifold [84]. Manifolds that share a common cover (including manifolds that share a common quotient) are called *commensurable*.

This common feature led Reid [82] to ask whether all isospectral hyperbolic manifolds (that is, manifolds that share the same length spectrum) must also share a common cover. This is known to be true for arithmetic hyperbolic manifolds of dimension $d \neq 1 \mod 4$; see [20, 80, 81]. By contrast, my joint work with Millichap (a former PhD student) suggests that the answer to Reid's question might be negative for non-arithmetic manifolds.

Theorem 1.4 ([46])**.** *For every sufficiently large integer n, there is a pair of non-commensurable hyperbolic 3–manifolds M_n and M_n^μ whose length spectra agree on all lengths up to n. The volumes of these manifolds grow coarsely linearly with n.*

The proof of this result uses deep structural theorems in Kleinian groups, including bilipschitz models for 3–manifolds of the form $S \times \mathbb{R}$ developed by Brock, Canary, and Minsky [18, 75], to build examples containing a very thick collar about a surface S. Cutting M_n along this surface and re-gluing along a rigid involution μ produces M_n^μ. We had to develop new tools to show that M_n and M_n^μ are not commensurable; in fact, each is the minimal orbifold in its commensurability class.

2. Quantum invariants and geometric topology

In the mid-1980s, Jones invigorated low-dimensional topology by introducing the celebrated Jones polynomial [64]. This invariant of knots and links was originally constructed in terms of operator algebras, and later reformulated using the combinatorics of link projections. Since its introduction, the Jones polynomial has led to the resolution of several longstanding conjectures, for instance that the crossing number of an alternating knot is realized by an alternating diagram [67, 77]. However, it has long been an open and tantalizing question to decode what exactly the Jones polynomial and its relatives say about the geometry of the link complement.

This question has produced fairly extensive conjectures. Based on physical intuition, Witten has conjectured that the Jones polynomial can be interpreted in terms of geometric structures, in particular hyperbolic structures [92]. More recently, in the late 1990s, Kashaev and Murakami–Murakami conjectured that the volume of a hyperbolic link complement can be computed as an asymptotic limit of colored Jones polynomials [66, 76]. This conjecture is still wide open.

We recall that for each link $K \subset S^3$, and for each positive integer n, the colored Jones function is a Laurent polynomial invariant of the form

$$J_K^n(t) = \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \ldots + \beta_n' t^{r_n+1} + \alpha_n' t^{r_n}. \tag{2.1}$$

Here, $J_K(t) = J_K^2(t)$ is the classical Jones polynomial.

My recent work with Kalfagianni and Purcell has found that under mild diagrammatic hypotheses, both the degrees and the outer coefficients of these polynomials have clear connections to the geometric topology of surfaces in $S^3 \smallsetminus K$. Theorem 2.1 says that the growth rate of the degree of $J_K^n(t)$, as $n \to \infty$, is actually a boundary slope of an essential surface for K. Theorem 2.2 says that the next-to-last coefficient β_n' is precisely the obstruction to this surface being a fiber in $S^3 \smallsetminus K$. Furthermore, the magnitude $|\beta_n'|$ provides a quantitative measurement of *how far* this surface is from being a fiber. This leads to tight, two-sided estimates on volume: for certain families of links, the coefficient β_n' determines the volume of $S^3 \smallsetminus K$ up to a factor of 4.15. This result, stated as Theorem 2.3, provides certain coarse evidence for the volume conjecture of Kashaev and Murakami–Murakami. The glue that holds these theorems together is a combinatorial object called the *state graph* associated to a link diagram and its Kauffman states.

2.1. State graphs and polynomials.

At each crossing of a diagram $D(K)$, there are two possible smoothings, or *resolutions* of the crossing, as depicted in Figure 2. A *Kauffman state* is a choice of A– or B–resolution at each crossing. Kauffman's insight [67] was that the Jones polynomial $J_K(t)$ can be expressed as a sum of terms corresponding to all the states of $D(K)$.

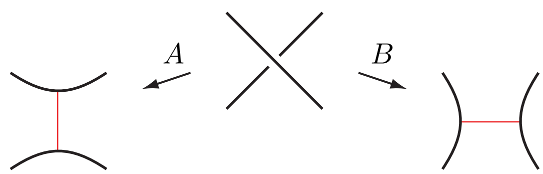

FIGURE 2. A– and B–resolutions at a crossing of D.

The choices that lead to a Kauffman state σ can be conveniently encoded in a *state graph* $\mathbb{G}_\sigma$. A state σ gives rise to a crossing–free collection of circles in the projection plane; each such circle gives rise to a vertex of $\mathbb{G}_\sigma$. Each crossing x of D gives rise to an edge between the state circles at the resolution of x. See Figure 2 for a local picture and Figure 3 for a full example. A state σ is called *adequate* if $\mathbb{G}_\sigma$ has no two-edge loops.

For A–adequate diagrams (meaning, the all–A state is adequate), one can easily read off the degree and certain outer coefficients of the colored Jones polynomial $J_K^n(t)$. For instance, the last coefficient always satisfies $|\alpha_n'| = 1$, while the next-to-last coefficient satisfies $|\beta_n'| = 1 - \chi(\mathbb{G}_A')$. Here, $\mathbb{G}_A'$ is the *reduced graph* obtained by removing duplicate edges in $\mathbb{G}_A$. These equalities are independent of n. See [27, 83], as well as an independent proof with my collaborators [26].

My work with Kalfagianni and Purcell [40, 41, 43] uses the combinatorics of state graphs to relate degrees and coefficients of the colored Jones polynomials to the topology of surfaces in $S^3 \smallsetminus K$. In the following theorem, an *A–adequate knot* is one admitting an A–adequate diagram. (The class of A–adequate knots is quite large. For instance, all alternating knots and 99% of knots up to 15 crossings admit such a diagram, after possibly taking a mirror image. A–adequate knots are typically hyperbolic [44].)

Theorem 2.1 ([40]). *Let $K \subset S^3$ be a knot with an A–adequate diagram. Then there is an essential surface $S_A \subset S^3 \smallsetminus K$, whose boundary curve ∂S_A winds once around the longitude and*

$$b(S_A) \;=\; \lim_{n \to \infty} \frac{4 r_n}{n^2}$$

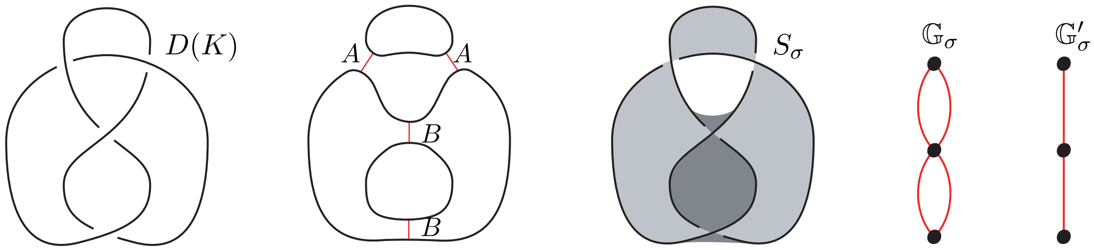

FIGURE 3. Left to right: a diagram $D(K)$. An adequate, homogeneous state σ. The state surface S_σ corresponding to σ. The graph $\mathbb{G}_\sigma$ embeds into S_σ as a spine. The reduced graph $\mathbb{G}'_\sigma$, in which duplicate edges of $\mathbb{G}_\sigma$ are removed.

times around the meridian. Here, r_n is the lowest degree of $J^n_K(t)$, as in equation (2.1)*. In words, the degrees of colored Jones polynomials determine a boundary slope of K.*

Theorem 2.1 establishes a large special case of a conjecture of Garoufalidis [55]: for any knot K, any limit of a subsequence of $\{4r_n/n^2\}$ should be a boundary slope of K. See Dunfield and Garoufalidis [31, 55] for other known cases of the conjecture.

Prior to this work, the main method of detecting boundary slopes relied on the character varieties of Culler and Shalen [24], which are closely connected to hyperbolic geometry. Thus it is exciting to see the same sort of geometric application arising out of quantum topology.

The surface S_A that appears in Theorem 2.1 is a *state surface* constructed from the all–A state of a diagram. That is: every state circle bounds a disk, and these disks can be joined by half-twisted bands to form a surface. This construction works for any state σ; see Figure 3. When σ is an adequate state, we know that S_σ is an essential surface [41, 79].

2.2. **The topology and geometry of state surfaces.** Because the graph $\mathbb{G}_\sigma$ embeds into S_σ as a spine, both $\mathbb{G}_\sigma$ and the reduced graph $\mathbb{G}'_\sigma$ turn out to be closely connected to the topology of the surface complement $S^3 \setminus S_\sigma$.

Theorem 2.2 ([34, 41, 43])**.** *For an link K with an A–adequate diagram $D(K)$, the following are equivalent:*

(a) The next-to-last coefficient of the Jones polynomial of K is $\beta' = 0$.
(b) The reduced state graph $\mathbb{G}'_A$ is a tree.
(c) The state surface S_A is a fiber in a fibration of $S^3 \setminus K$ over S^1.
(d) (If $S^3 \setminus K$ is hyperbolic) the state surface S_A is not quasi-Fuchsian.

Taken together, Theorems 2.1 and 2.2 offer striking evidence that the degrees and coefficients of colored Jones polynomials contain quite a lot of geometric information. In particular, the next-to-last coefficient β' is precisely the obstruction to a particular surface being a fiber. These theorems lay some of the first concrete bridges between quantum invariants and geometric topology.

In fact, there is a stronger version of Theorem 2.2 saying that when the coefficient β' is far from 0, the surface S_A is correspondingly far from being a fiber. For any essential surface $S \subset M^3$, one may cut the surface complement $M \setminus S$ along annuli into two kinds of pieces: I–bundles and complicated pieces called *guts*. This is exactly the annulus version of the JSJ decomposition. When S is a fiber, $M \setminus S$ is an I–bundle, and the guts are empty. Otherwise, the Euler characteristic $\chi(\mathrm{guts}(M \setminus S))$ measures the "distance" between S and a fiber.

For many families of A–adequate diagrams, as well as other diagrams with an adequate and homogeneous state σ, we proved that the Euler characteristic $\chi(\mathrm{guts}(S^3 \setminus S_\sigma))$ is equal to $\chi(\mathbb{G}'_\sigma)$, the Euler characteristic of the reduced graph. This result meshes nicely with a theorem of Agol, Storm, and Thurston [4] that says the hyperbolic volume of $S^3 \setminus K$ is bounded below by a constant times $\chi(\mathrm{guts}(S^3 \setminus S_\sigma))$. As a consequence, we obtain volume estimates in terms of state graphs, hence in terms of Jones polynomial coefficients. The following sample theorem should be compared with Theorem 1.1.

Theorem 2.3 ([41]). *Let $D(K)$ be a diagram of a hyperbolic link K, obtained as the closure of a positive braid with at least 3 crossings in each twist region. Then the state surface S_A satisfies*

$$\chi(\mathrm{guts}(S^3 \smallsetminus S_A)) \;=\; \chi(\mathbb{G}'_A) \;=\; 1 - |\beta'|.$$

As a consequence, the volume of $S^3 \smallsetminus K$ is bounded in terms of the Jones coefficient $|\beta'|$:

$$v_8\left(|\beta'_K| - 1\right) \;\leq\; \mathrm{vol}(S^3 \smallsetminus K) \;<\; 15v_3\left(|\beta'_K| - 1\right) - 10v_3,$$

where $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

Similar volume estimates were proved for alternating links by Dasbach and Lin [28], and for several other families of links by Kalfagianni, Purcell, and myself [36, 37, 38, 41]. However, while previous estimates were all somewhat indirect, the lower bound on volume in Theorem 2.3 works by finding a direct and intrinsic relationship between the reduced state graph $\mathbb{G}'_A$ (hence, Jones coefficients) and the geometric topology of $S^3 \smallsetminus K$. As a consequence of this direct argument, the upper and lower volume bounds differ by a rather small factor of about 4.15.

Theorems 2.2 and 2.3 are proved by constructing a detailed polyhedral decomposition of the surface complement $S^3 \smallsetminus S_A$, whose combinatorial structure is closely linked to the combinatorics of the ribbon graph $\mathbb{G}_A$. In particular, all the I–bundles in the complement of S_A are visible in the individual polyhedra. As a result, we can detect when all of $S^3 \smallsetminus S_A$ is an I–bundle (hence S_A is a fiber), as well as compute the guts.

See also Section 3.3 for an application of state surfaces to virtual questions in 3–manifolds.

3. Geometry for fibering and virtual fibering

Fibered 3–manifolds arise from a natural way to construct a manifold from lower-dimensional pieces. Given an (orientable) surface S, and an (orientation-preserving) homeomorphism $\varphi : S \to S$, the *mapping torus* of φ is

$$M_\varphi = (S \times [0,1])/(x,0) \sim (\varphi(x),1).$$

The manifold M_φ fibers over the circle with fiber S, and the map φ is called the *monodromy* of the fibration. Thurston showed that M_φ is hyperbolic if and only if $\chi(S) < 0$ and φ is *pseudo-Anosov*, which means that $\varphi^n(\gamma)$ is not homotopic to γ for any $n \neq 0$ and any essential simple closed curve $\gamma \subset S$ [85, 87]. So this construction produces a great variety of hyperbolic manifolds.

A major recent breakthrough, combining the work of Agol [1, 3], Kahn–Markovic [65], and Wise [91], is that every finite-volume hyperbolic 3–manifold has a finite cover that fibers over the circle. Because many geometric properties are essentially preserved in finite covers, this result makes it possible to study the geometry of any hyperbolic 3–manifold using the structural data of a fibration. (Nevertheless, proving effective results using virtual fibering requires information about the degree of the cover. This motivates the investigations of Section 3.3.)

3.1. Geometric estimates from combinatorics.

When a fibered 3–manifold is hyperbolic, the work of Minsky, Brock, and Canary on Kleinian surface groups provides a combinatorial, bilipschitz model of the hyperbolic metric [75]. The bilipschitz constants depend only on the fiber surface [18]. However, the existence of these constants is proved using compactness arguments; as a result the constants are unknown.

The combinatorial input for these bilipschitz models comes from certain combinatorial complexes associated to S. One of these is the *curve complex* $\mathcal{C}(S)$, whose vertices are isotopy classes of essential closed curves on S, and whose edges represent disjoint curves. In a similar manner, one may define the *pants graph* $\mathcal{P}(S)$ (whose vertices are pants decompositions) and the *arc complex*

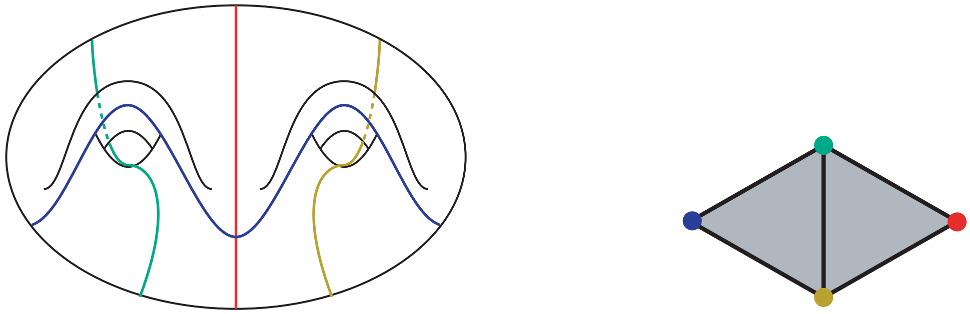

FIGURE 4. Left: essential arcs in a punctured surface S. Right: the corresponding subcomplex of the arc complex $\mathcal{A}(S)$. Disjoint arcs are connected by edges.

$\mathcal{A}(S)$ of a surface with punctures (whose vertices are essential arcs from puncture to puncture). A small piece of the arc complex is illustrated in Figure 4.

The geometry of a fibered manifold M_φ tends to be controlled by the distance by which the map φ moves vertices in these complexes. More precisely, define the *stable translation distance* of φ to be

$$\overline{d}_{\mathcal{A}}(\varphi) = \lim_{n\to\infty} \frac{d_{\mathcal{A}}(v, \varphi^n(v))}{n}, \quad \text{for an arbitrary vertex } v \in \mathcal{A}(S),$$

and similarly for $\mathcal{C}(S)$ and $\mathcal{P}(S)$. With this definition, Brock showed that the hyperbolic volume of a mapping torus M_φ is bounded above and below by constants times the stable translation distance $\overline{d}_{\mathcal{P}}(\varphi)$ [15, 16]. Schleimer and I proved that when the fiber S has punctures, the translation distance $\overline{d}_{\mathcal{A}}(\varphi)$ in the arc complex $\mathcal{A}(S)$ leads to explicit estimates on the geometry of cusps in M_φ.

Theorem 3.1 ([50])**.** *Let S be a surface with one puncture. Then, for any pseudo-Anosov $\varphi\colon S \to S$, the mapping torus M_φ contains a maximal cusp C, which satisfies*

$$\frac{\overline{d}_{\mathcal{A}}(\varphi)}{450\,\chi(S)^4} \;<\; \operatorname{area}(\partial C) \;\leq\; 9\,\chi(S)^2\,\overline{d}_{\mathcal{A}}(\varphi).$$

There is a closely analogous estimate for surfaces with multiple punctures, in terms of the complex $\mathcal{A}(S,p)$ of arcs into a particular puncture.

The explicit control of Theorem 3.1 is achieved by sweeping out the manifold M_φ with a one-parameter family of geometrically controlled surfaces homotopic to the fiber, and keeping track of the shortest arc as the time parameter changes. This result, as well as the sweep-out methods that we developed, have recently been used by Aougab, Patel, Taylor, and Webb to give explicit estimates on other quantities in a fibered 3–manifold [6, 7].

3.2. **Surfaces for cubulation and virtual fibering.** In the last several years, spectacular advances in geometric group theory have been employed to simultaneously resolve several long-standing virtual conjectures in low-dimensional topology. A manifold M is said to *virtually* have a property P if the property holds in some finite cover $\hat{M} \to M$. Following the work of Agol [3], Kahn–Markovic [65], and Wise [91], we know that all aspherical 3–manifolds are virtually Haken, meaning that their finite covers have embedded incompressible surfaces. We know that non-positively curved 3–manifolds are virtually fibered. We also know that π_1–injective surfaces in a hyperbolic manifold M are virtually embedded. Finally, we know that M virtually has Betti number $\geq k$ and that $\pi_1(M)$ is linear over the integers.

Although the details took enormous ingenuity, the proof (for a closed hyperbolic manifold M) follows a conceptually simple outline:

1. Kahn and Markovic show that a closed hyperbolic 3–manifold M contains a *ubiquitous collection* of immersed essential surfaces – enough surfaces to separate any pair of points on the sphere at

infinity [65]. Consequently, by a theorem of Bergeron and Wise [12], M is homotopy equivalent to a non-positively curved cube complex X dual to these surfaces.

2. Agol [3], building on the work of Wise [91], shows that some finite cover of X is *special* and has embedded hyperplanes, meaning that the corresponding surfaces are also embedded.
3. Fundamental groups of special cube complexes embed into right-angled Artin groups [60]. Subgroups of these groups are sufficiently well-understood that the other properties follow in further finite covers [1, 5, 60].

For non-closed hyperbolic manifolds, such as knot complements, the proof of virtual specialness and virtual fibering is contained in Wise's manuscript [91], and is much more involved. Many mathematicians have asked for a streamlined proof following the above outline. My work with Cooper [21] contributes to such a proof.

Theorem 3.2 ([21])**.** *Let M be a cusped hyperbolic 3–manifold. Then M contains a ubiquitous collection of immersed quasifuchsian surfaces. These surfaces separate any pair of points on the sphere at infinity of $\widetilde{M}$, and intersect each cusp of M in exactly two slopes. Consequently, M is homotopy equivalent to a compact non-positively curved cube complex X dual to these surfaces.*

This theorem resolves a previously-open question posed by Agol [30, Problem 3.5]. The proof involves performing a long Dehn filling on M, as in Section 1.2, applying the Kahn–Markovic theorem to find almost-geodesic surfaces in the manifold $M(s)$, and then carefully controlling the geometry as the surfaces are pulled back to M. We also extend and apply the geometric combination theorems of Baker and Cooper [10] to build a ubiquitous collection of closed surfaces.

After our paper [21] was first distributed, Groves and Manning completed the program outlined above, showing that the cube complex X constructed in our work has a special finite cover [58]. As a consequence, Agol's criterion [1] shows that M is virtually fibered. This completes a direct second-generation proof of Wise's theorem.

3.3. **Special covers of alternating links.** There is one particular context where the above outline for the virtual theorems takes a particularly simple form. This is the setting of prime, alternating links in S^3. Given an alternating link diagram $D = D(K)$, a classical construction introduced by Dehn [29] produces a square complex X_D that is homotopy equivalent to $S^3 \smallsetminus K$. This complex has two vertices (one on each side of the projection plane), one edge through each region of $D(K)$, and one square for each crossing of $D(K)$. See Figure 5 for an example.

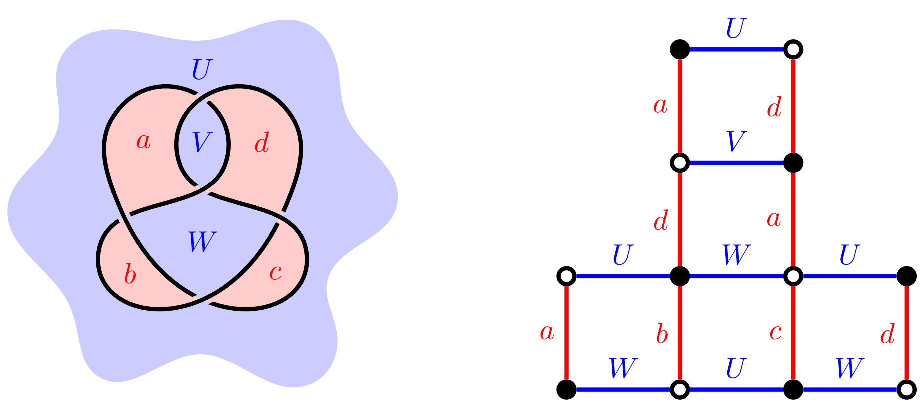

FIGURE 5. Left: an alternating diagram $D = D(K)$. Right: the Dehn complex X_D corresponding to D. Labels on the edges of X_D correspond to regions of $D(K)$. Edges with the same label are identified. The hyperplanes of this complex correspond to the checkerboard surfaces shown on the left.

Weinbaum proved that Dehn complexes of prime, alternating diagrams are non-positively curved [89]. Besides surfaces, these complexes are some of the earliest and most natural examples of cubed spaces. While introducing the notion of specialness, Haglund and Wise conjectured that Dehn complexes are virtually special [60, Remark 4.10]. While this conjecture was settled by Wise's more general theorem [91], Bering and I give a direct and effective proof.

Theorem 3.3 ([13])**.** *Let $D = D(K)$ be a prime, alternating link diagram with c crossings. Then the Dehn complex X_D has a special cover $\widehat{X}_D$ of degree at most $12(c-1)!$.*

Beyond bounding the degree of the cover $\widehat{X}_D$, we actually *construct* the cover. As a consequence, we are able to gain structural understanding of surfaces in small-degree covers of the link complement $S^3 \smallsetminus K$. For instance, we get an effective version of the following theorem of Cooper, Long, and Reid [23]:

Corollary 3.4. *Let $D = D(K)$ be a prime, alternating link diagram with $c \geq 3$ crossings. Then $S^3 \smallsetminus K$ has a cover M of degree at most $12(c-1)!$, which contains four disjoint, orientable surfaces whose union does not separate M. Recording intersections of a loop with these surfaces yields a surjection $\pi_1(M) \to F_4$.*

Given that the fundamental group of the cover $\widehat{X}_D$ embeds into a right-angled Coxeter group with a simple description, we also learn something about linear representations of the original knot group:

Corollary 3.5. *Let $D = D(K)$ be a prime, alternating link diagram with c crossings. Then $S^3 \smallsetminus K$ embeds into $SL(m, \mathbb{Z})$, where $m \leq 288((c-1)!)^2$.*

Our next goal in this project is to carry out an effective version of Agol's virtual fibering argument [1]. Given that we already understand the Coxeter group into which $\widehat{X}_D$ embeds, this should be a plausible task.

4. Random manifolds and random groups

In addition to proving effective statements *all* manifolds or groups in a particular class, it is often helpful to see what properties are satisfied by a *generic* object produced using some well-defined random process. Paradigmatic examples in this vein include Gromov's results about random groups [57], as well as Maher's theorem that random 3–manifolds are hyperbolic [69]. Several of my current and recent results explore random phenomena.

4.1. **Triangulations of random mapping tori.**

Dunfield and Thurston introduced a useful model of random 3–manifolds [32]. The idea is to start with a closed surface S_g of genus g, choose a finite generating set for the mapping class group $MCG(S_g)$, and perform a random walk in these generators. The n-th step of the random walk is denoted φ_n. A *random Heegaard splitting* is constructed by gluing two genus-g handlebodies via the homeomorphism φ_n, for large n. The mapping torus of φ_n, denoted M_{φ_n}, is a *random mapping torus.* One can then ask what properties hold generically for large n.

Many results in this realm are known. With overwhelming probability, 3–manifolds constructed via random Heegaard splittings or random mapping tori are known to be hyperbolic [70, 69]. The volume of a manifold constructed in this grows asymptotically linearly with n [88].

In recent work with Taylor and Worden (a former PhD student), I investigated the fine-scale geometry of random mapping tori [51]. We showed that a random mapping torus M_{φ_n} contains a large region that looks like "your favorite" mapping torus M_ψ. That is, for each ψ, there are are large metric balls in M_{φ_n} that are almost-isometric to a large metric ball in M_ψ.

Theorem 4.1 ([51])**.** *Let S be a hyperbolic surface, and fix a finite generating set for $MCG(S)$. Let $\psi \in MCG(S)$ be any pseudo-Anosov mapping class in the principal stratum. Then, for almost* *every sample path (φ_n) of a simple random walk, the mapping tori M_{φ_n} contain basepoints x_n such that the sequence (M_{φ_n}, x_n) converges geometrically to the infinite cyclic cover of M_ψ.*

In particular, if the veering triangulation determined by ψ is non-geometric, then so is the veering triangulation determined by φ_n for large n.

The veering triangulation is a combinatorial invariant of pseudo-Anosov mapping classes, introduced by Agol [2]. It is known to have strong combinatorial properties [35, 63], and is useful in determining conjugacy in mapping class groups [11, 71]. Agol asked the natural question of whether the triangulation is always *geometric*, meaning realized by positively oriented tetrahedra. Our work gives a strong negative answer.

The proof of Theorem 4.1 combines dynamical arguments in Teichmüller space, using ideas of Gadre and Maher [54], with a parametrized version of the Ending Lamination Theorem proved by Brock, Canary, and Minsky [18, 75].

4.2. **Random quotients of cubulated hyperbolic groups.** In foundational work, Gromov introduced a density model for random groups [57]. The idea is to start with a free group F_n and pick a parameter $d \in (0,1)$, called a *density.* Then, for a large number $\ell \gg 0$, take the quotient of F_n by $(2n-1)^{\ell \cdot d}$ randomly chosen relators of length at most ℓ. The number of relators is a $(1/d)$-th root of all the available elements of length at most ℓ.

Gromov proved that a sharp phase transition occurs at density $1/2$. When $d > 1/2$, then with overwhelming probability, the resulting group is trivial or $\mathbb{Z}/2$. When $d < 1/2$, then with overwhelming probability, the resulting group is torsion-free, hyperbolic, and 2–dimensional [57]. Random groups at density $d < 1/2$ are known to contain surface groups [19], and at density $d < 1/6$ they are realized by non-positively curved cube complexes [78].

In current work with Wise [52], we extend some of these ideas from quotients of free groups (that is, fundamental groups of graphs) to quotients of other hyperbolic groups. Given a generating set for a hyperbolic group G, the number of elements of length at most ℓ is asymptotic to $e^{b\ell}$, where $b > 0$ is the *growth exponent* of G (depending on the generating set). Analogous to Gromov's definition, one can fix $d \in (0,1)$ and take a quotient of G by $e^{b\ell \cdot d}$ relations, which are randomly chosen from the set of all words of length at most ℓ. We prove the following.

Theorem 4.2 ([52])**.** *Let G is a hyperbolic group, where $G = \pi_1 X$ for a compact non-positively curved cube complex X. Let $b > 0$ be the growth exponent of G with respect to a generating set (or a geometric action on a metric space). For a sufficiently low density $d \in (0,1)$, the following holds. If $g_1, \ldots, g_k$ are relations sampled at random from the set of all elements of length at most ℓ, where the number of relations is $k \leq e^{b\ell \cdot d}$, then with overwhelming probability as $\ell \to \infty$, the quotient*

$$\overline{G} = G / \langle\langle g_1, \ldots, g_k \rangle\rangle$$

is hyperbolic and is the fundamental group of a compact, non-positively curved cube complex.

Knowing that the quotient group is hyperbolic and cubulated has many powerful properties. By Agol's theorem [3], this means $\overline{G}$ is virtually special. It follows that $\overline{G}$ is large and linear, and its quasiconvex subgroups are separable.

The density $d > 0$ sufficient for this conclusion depends only on $\widetilde{X}$. In particular, it depends on the gap in growth between the action of G and the action of a subgroup stabilizing a hyperplane of $\widetilde{X}$. That such a gap exists follows from my previous work with Dahmani and Wise [25]. We use the gap in growth to show that a natural presentation for $\overline{G} = G / \langle\langle g_1, \ldots, g_k \rangle\rangle$ satisfies the cubical small cancellation conditions in Wise's work [91]. In turn, the small cancellation conditions suffice to show the quotient $\overline{G}$ is hyperbolic and construct an action on a cube complex.

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