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ALTERNATING SUM FORMULAE FOR THE DETERMINANT AND OTHER LINK INVARIANTS

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ABSTRACT

A classical result states that the determinant of an alternating link is equal to the number of spanning trees in a checkerboard graph of an alternating connected projection of the link. We generalize this result to show that the determinant is the alternating sum of the number of quasi-trees of genus j of the dessin of a non-alternating link.

*We regretfully inform you that Xiao-Song Lin passed away on the 14th of January, 2007.

Furthermore, we obtain formulas for coefficients of the Jones polynomial by counting quantities on dessins. In particular, we will show that the jth coefficient of the Jones polynomial is given by sub-dessins of genus less or equal to j.

Keywords: Knots; knot determinant; Jones polynomial; Turaev genus; Bollobás–Riordan–Tutte polynomial; dessin d'enfant.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

A classical result in knot theory states that the determinant of an alternating link is given by the number of spanning trees in a checkerboard graph of an alternating, connected link projection (see e.g. [6]). For non-alternating links, one has to assign signs to the trees and count the trees with signs, where the geometric meaning of the signs is not apparent. Ultimately, these theorems are reflected in Kauffman's spanning tree expansion for the Alexander polynomial (see [14, 18]) as well as Thistlethwaite's spanning tree expansion for the Jones polynomial [20]; the determinant is the absolute value of the Alexander polynomial as well as of the Jones polynomial at -1.

The first purpose of this paper is to show that the determinant theorem for alternating links has a very natural, topological/geometrical generalization to non-alternating links, using the framework that we developed in [8]: Every link diagram induces an embedding of the link into the neighborhood of an orientable surface, its Turaev surface, such that the projection is alternating on that surface. Now the two checkerboard graphs are graphs embedded on surfaces, i.e. dessins d'enfant (aka. combinatorial maps or ribbon graphs), and these two graphs are dual to each other. The minimal genus of all surfaces coming from that construction is the Turaev genus of the link. However, as in [8] one does not need the reference to the surface to construct the dessin directly from the diagram and to compute its genus. The Jones polynomial can then be considered as an evaluation of the Bollobás–Riordan–Tutte polynomial [4] of the dessin [8]. Alternating non-split links are precisely the links of Turaev genus zero. Our determinant formula recovers the classical determinant formula in that case.

For a connected link projection of higher Turaev genus, we will show that the determinant is given as the alternating sum of the number of spanning quasi-trees of genus j, as defined below, in the dessin of the link projection. Thus, the sign has a topological/geometrical interpretation in terms of the genus of sub-dessins. In particular, we will show that for Turaev genus 1 projections the determinant is the difference between the number of spanning trees in the dessin and the number of spanning trees in the dual of the dessin. The class of Turaev genus one knots and links includes for example all non-alternating pretzel knots.

Every link can be represented as a dessin with one vertex, and we will show that with this representation the numbers of j-quasi-trees arise as coefficients of the characteristic polynomial of a certain matrix assigned to the dessin. In particular, we will obtain a new determinant formula for the determinant of a link which comes solely from the Jones polynomial. Recall that the Alexander polynomial — and thus every evaluation of it — can be expressed as a determinant in various ways. The Jones polynomial, however, is not defined as a determinant.

The second purpose of the paper is to develop dessin formulas for coefficients of the Jones polynomial. We will show that the jth coefficient is completely determined by sub-dessins of genus less or equal to j and we will give formulas for the coefficients. Again, we will discuss the simplifications in the formulas if the dessin has one vertex. Starting with the work of the first and fourth author [10], the coefficients of the Jones polynomial have recently gained a new significance because of their relationship to the hyperbolic volume of the link complement. Under certain conditions, the coefficients near the head and the tail of the polynomial give linear upper and lower bounds for the volume. In [9,10] this was done for alternating links and in [11,12] it was generalized to a larger class of links.

The paper is organized as follows: Section 2 recalls the pertinent results of [8]. In Sec. 3, we develop the alternating sum formula for the determinant of the link. Section 4 shows a duality result for quasi-trees and its application to knots of Turaev genus one. In Sec. 5, we look at the situation when the dessin has one vertex. Section 6 shows results on the coefficients of the Jones polynomial within the framework of dessins.

2. The Dessin D'enfant Coming from a Link Diagram

We recall the basic definitions of [8]:

A *dessin d'enfant* (combinatorial map, oriented ribbon graph) can be viewed as a multi-graph (i.e. loops and multiple edges are allowed) equipped with a cyclic order on the edges at every vertex. Isomorphisms between dessins are graph isomorphisms that preserve the given cyclic order of the edges.

Equivalently, dessins correspond to graphs embedded on an orientable surface such that every region in the complement of the graph is a disk. We call the regions the *faces* of the dessins. Thus, the genus $g(\mathbb{D})$ of a dessin $\mathbb{D}$ with $v(\mathbb{D})$ vertices, $e(\mathbb{D})$ edges, $f(\mathbb{D})$ faces and k components is determined by its Euler characteristic:

$$\chi(\mathbb{D}) = v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) = 2k - 2g(\mathbb{D}).$$

For each Kauffman state of a (connected) link diagram, a dessin is constructed as follows: Given a link diagram $P(K)$ of a link K we have, as in Fig. 1, an A-splicing and a B-splicing at every crossing. For any state assignment of an A or B at each crossing we obtain a collection of non-intersecting circles in the plane, together with embedded arcs that record the crossing splice. Again, Fig. 1 shows this situation locally. In particular, we will consider the state where all splicings are A-splicings. The collection of circles will be the set of vertices of the dessin.

To define the desired dessin associated to a link diagram, we need to define an orientation on each of the circles resulting from the A or B splicings, according to a given state assignment. We orient the set of circles in the plane by orienting each

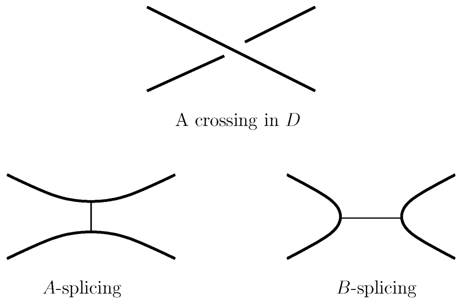

Fig. 1. Splicings of a crossing, A-graph and B-graph.

component clockwise or anti-clockwise according to whether the circle is inside an odd or even number of circles, respectively. Given a state assignment $s : E \to \{A, B\}$ on the crossings (the eventual edge set $E(\mathbb{D})$ of the dessin), the associated dessin is constructed by first resolving all the crossings according to the assigned states and then orienting the resulting circles according to a given orientation of the plane.

The vertices of the dessin correspond to the collection of circles and the edges of the dessin correspond to the crossings. The orientation of the circles defines the orientation of the edges around the vertices. We will denote the dessin associated to state s by $\mathbb{D}(s)$. Of particular interest for us will be the dessins $\mathbb{D}(A)$ and $\mathbb{D}(B)$ coming from the states with all-A splicings and all-B splicings. For alternating projections of alternating links $\mathbb{D}(A)$ and $\mathbb{D}(B)$ are the two checkerboard graphs of the link projection. In general, we showed in [8] that $\mathbb{D}(A)$ and $\mathbb{D}(B)$ are dual to each other.

We will need several different combinatorial measurements of the dessin:

Definition 2.1. Denote by $v(\mathbb{D}), e(\mathbb{D})$ and $f(\mathbb{D})$ the number of vertices, edges and faces of a dessin $\mathbb{D}$. Furthermore, we define the following quantities:

$$\begin{aligned} k(\mathbb{D}) &= \text{the number of connected components of } \mathbb{D}, \\ g(\mathbb{D}) &= \frac{2k(\mathbb{D}) - v(\mathbb{D}) + e(\mathbb{D}) - f(\mathbb{D})}{2}, \text{the } \textit{genus} \text{ of } \mathbb{D}, \\ n(\mathbb{D}) &= e(\mathbb{D}) - v(\mathbb{D}) + k(\mathbb{D}), \text{the } \textit{nullity} \text{ of } \mathbb{D}. \end{aligned}$$

The following spanning sub-dessin expansion was obtained in [8] by using results of [5]. A *spanning sub-dessin* is obtained from the dessin by deleting edges. Thus, it has the same vertex set as the dessin.

Theorem 2.2. *Let $\langle P \rangle \in \mathbb{Z}[A, A^{-1}]$ be the Kauffman bracket of a connected link projection diagram P and $\mathbb{D} := \mathbb{D}(A)$ be the dessin of P associated to the all-A-splicing. The Kauffman bracket can be computed by the following spanning sub-dessin $\mathbb{H}$ expansion*:

$$A^{-e(\mathbb{D})}\langle P \rangle = A^{2-2v(\mathbb{D})}(X-1)^{-k(\mathbb{D})} \sum_{\mathbb{H} \subset \mathbb{D}} (X-1)^{k(\mathbb{H})} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}$$

under the following specialization: $\{X \to -A^4, Y \to A^{-2}\delta, Z \to \delta^{-2}\}$ *where* $\delta := (-A^2 - A^{-2})$.

Theorem 2.2, after substitution, yields the following sub-dessin expansion for the Kauffman bracket of P.

Corollary 2.3.

$$\langle P \rangle = \sum_{\mathbb{H} \subset \mathbb{D}} A^{e(\mathbb{D})-2e(\mathbb{H})} \left(-A^2 - A^{-2}\right)^{f(\mathbb{H})-1}.$$

3. Dessins Determine the Determinant

The determinant of a link is ubiquitous in knot theory. It is the absolute value of the Alexander polynomial at -1 as well as the Jones polynomial at -1. Furthermore, it is the order of the first homology group of the double branched cover of the link complement. For other interpretations, see e.g. [6].

We find the following definition helpful:

Definition 3.1. Let $\mathbb{D}$ be a connected dessin that embeds into a surface S. A spanning quasi-tree of genus j or spanning j-quasi-tree in $\mathbb{D}$ is a sub-dessin $\mathbb{H}$ of $\mathbb{D}$ with $v(\mathbb{H})$ vertices and $e(\mathbb{H})$ edges such that $\mathbb{H}$ is connected and spanning and

(1) $\mathbb{H}$ is of genus j.
(2) $S - \mathbb{H}$ has one component, i.e. $f(\mathbb{H}) = 1$.
(3) $\mathbb{H}$ has $e(\mathbb{H}) = v(\mathbb{H}) - 1 + 2j$ edges.

In particular, the spanning 0-quasi-trees are the regular spanning trees of the graph. Note that by Definition 2.1 either two of the three conditions in Definition 3.1 imply the third one.

Theorem 2.2 now leads to the following formula for the determinant $\det(K)$ of a link K.

Theorem 3.2. *Let P be a connected projection of the link K and $\mathbb{D} := \mathbb{D}(A)$ be the dessin of P associated to the all-A splicing. Suppose $\mathbb{D}$ is of genus $g(\mathbb{D})$.*

Furthermore, let $s(j, \mathbb{D})$ be the number of spanning j-quasi-trees of $\mathbb{D}$.

Then

$$\det(K) = \left| \sum_{j=0}^{g(\mathbb{D})} (-1)^j \, s(j, \mathbb{D}) \right|.$$

Proof. Recall that the Jones polynomial $J_K(t)$ can be obtained from the Kauffman bracket, up to a sign and a power of t, by the substitution $t := A^{-4}$.

By Theorem 2.2, we have for some power $u = u(\mathbb{D})$:

$$\pm J_K(A^{-4}) = A^u \sum_{\mathbb{H}\subset\mathbb{D}} (X-1)^{k(\mathbb{H})-1} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}$$

$$= A^u \sum_{\mathbb{H}\subset\mathbb{D}} A^{-2-2e(\mathbb{H})+2v(\mathbb{H})} \delta^{f(\mathbb{H})-1}. \tag{3.1}$$

We are interested in the absolute value of $J_K(-1)$. Thus, $\delta = 0$ and, since $k(\mathbb{H}) \leq f(\mathbb{H})$:

$$|J_K(-1)| = \left| \sum_{\mathbb{H}\subset\mathbb{D}, f(\mathbb{H})=1} A^{-2-2e(\mathbb{H})+2v(\mathbb{H})} \right| \tag{3.2}$$

$$= \left| \sum_{\mathbb{H}\subset\mathbb{D}, f(\mathbb{H})=1} A^{-4g(\mathbb{H})} \right|. \tag{3.3}$$

Collecting the terms of the same genus and setting $A^{-4} := -1$ proves the claim. □

Remark 3.3. For genus $j = 0$ we have $s(0, \mathbb{D})$ is the number of spanning trees in the dessin $\mathbb{D}$. Recall that a link has Turaev genus zero if and only if it is alternating. Thus, in particular, we recover the well-known theorem that for alternating links the determinant of a link is the number of spanning trees in a checkerboard graph of an alternating connected projection.

Theorem 3.2 is a natural generalization of this theorem for non-alternating link projections.

Example 3.4. Figure 2 shows the non-alternating 8-crossing knot 8_{21}, as drawn by Knotscape (http://www.math.utk.edu/∼ morwen/knotscape.html), and Fig. 3 the all-A associated dessin.

The dessin in Fig. 3 is of genus 1. Thus, the only quasi-trees are of genus 0 and of genus 1. The quasi-trees of genus 0 are the spanning trees of the dessin. The dessin in the example contains 9 spanning trees, i.e. $s(0, \mathbb{D}) = 9$. A quasi-tree of genus 1 must have 4 edges. Furthermore, it must contain either of the two loops, otherwise it would not be of genus 1. Two of the remaining three edges must form a cycle which interlinks with that loop. A simple count yields 24 of these and thus the determinant of the knot is $24 - 9 = 15$.

4. Duality

The following theorem is a generalization of the result that for planar graphs the spanning trees are in one-one correspondence to the spanning trees of the dual graphs:

Theorem 4.1. *Let $\mathbb{D} = \mathbb{D}(A)$ be the dessin of all-A splicings of a connected link projection of a link L. Suppose $\mathbb{D}$ is of genus $g(\mathbb{D})$ and $\mathbb{D}^*$ is the dual of $\mathbb{D}$.*

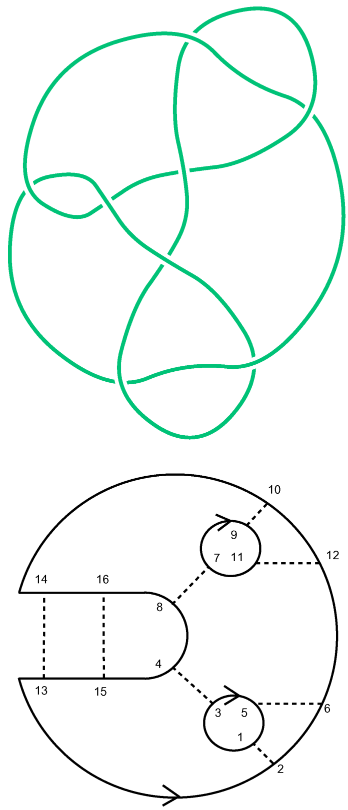

Fig. 2. The eight-crossing knot 8_{21} with its all-A splicing projection diagram.

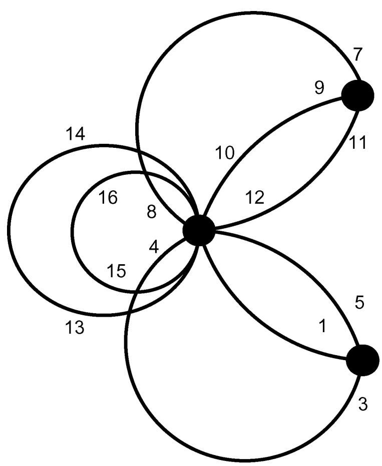

Fig. 3. All-A splicing dessin for 8_{21}.

We have: *The j-quasi-trees of $\mathbb{D}$ are in one-one correspondence to the $(g(\mathbb{D})-j)$-quasi-trees of $\mathbb{D}^*$. Thus*

$$s(j,\mathbb{D}) = s(g(\mathbb{D}) - j, \mathbb{D}^*).$$

Proof. Let $\mathbb{H}$ be a spanning j-quasi-tree in $\mathbb{D}$. Denote by $\mathbb{D} - \mathbb{H}$ the sub-dessin of $\mathbb{D}$ obtained by removing the edges of $\mathbb{H}$ from $\mathbb{D}$, and by $(\mathbb{D} - \mathbb{H})^*$ the sub-dessin of the dual $\mathbb{D}^*$ obtained by removing the edges dual to the edges in $\mathbb{H}$. From $f(\mathbb{H}) = 1$, it follows that $(\mathbb{D} - \mathbb{H})^*$ is connected and spanning. Furthermore, $f((\mathbb{D} - \mathbb{H})^*) = 1$.

We have:

$$\begin{aligned} v(\mathbb{H}) - e(\mathbb{H}) + f(\mathbb{H}) &= v(\mathbb{D}) - e(\mathbb{H}) + 1 = 2 - 2j \\ v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) &= 2 - 2g(\mathbb{D}). \end{aligned}$$

Thus,

$$\begin{aligned} v((\mathbb{D} - \mathbb{H})^*) - e((\mathbb{D} - \mathbb{H})^*) + f((\mathbb{D} - \mathbb{H})^*) &= f(\mathbb{D}) - (e(\mathbb{D}) - e(\mathbb{H})) + 1 \\ &= 2 - 2g(\mathbb{D}) - v(\mathbb{D}) + e(\mathbb{H}) + 1 \\ &= 2 - 2(g(\mathbb{D}) - j). \end{aligned}$$

Hence, $(\mathbb{D} - \mathbb{H})^*$ is a $(g(\mathbb{D}) - j)$-quasi-tree in $\mathbb{D}^*$. □

Recall that the Turaev genus zero links are precisely the alternating links. The following corollary generalizes to the class of Turaev genus one links the aforementioned, classical interpretation of the determinant of connected alternating links as the number of spanning trees in its checkerboard graph.

Corollary 4.2. *Let $\mathbb{D} = \mathbb{D}(A)$ be the all-A dessin of a connected link projection of a link L and $\mathbb{D}^*$ its dual. Suppose $\mathbb{D}$ is of Turaev genus one. Then*

$$\det(L) = |\#\{\textit{spanning trees in } \mathbb{D}\} - \#\{\textit{spanning trees in } \mathbb{D}^*\}|.$$

We apply Corollary 4.2 to compute the determinants of non-alternating pretzel links. The Alexander polynomial as well as the Jones polynomial, and consequently the determinant is invariant under mutations (see, e.g. [15]). Hence, it is sufficient to consider the case of $K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$ pretzel links, as depicted in Fig. 4. We assume that the links are non-alternating, i.e. $n \geq 1$ and $m \geq 1$.

Example 4.3. Consider the pretzel link $K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$, where $n \geq 1$, $m \geq 1$ and $p_i, q_i > 0$ for all i. The determinant of $K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$, is

$$\det(K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)) = \left| \prod_{i=1}^{n} p_i \prod_{j=1}^{m} q_j \left(\sum_{i=1}^{n} \frac{1}{p_i} - \sum_{j=1}^{m} \frac{1}{q_j} \right) \right|.$$

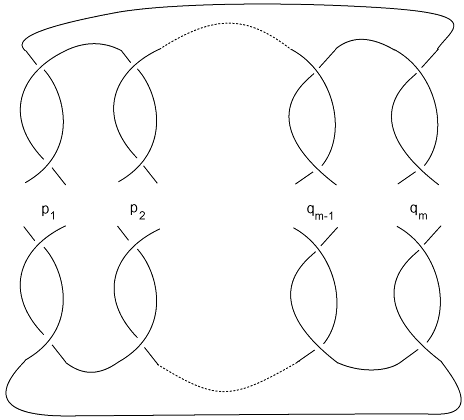

Fig. 4. The $K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$ pretzel link.

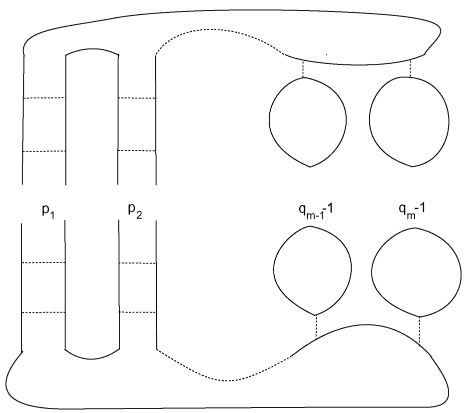

Fig. 5. The all-A splicings of the $K(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$ pretzel link.

Proof. Figure 5 shows the all-A splicing diagram of these links. The all-A dessin $\mathbb{D} = \mathbb{D}(A)$ has

$$v(\mathbb{D}) = n + \sum_{j=1}^{m}(q_j - 1) = n - m + \sum_{j=1}^{m} q_j$$

vertices and $e(\mathbb{D}) = \sum_{i=1}^{n} p_i + \sum_{j=1}^{m} q_j$ edges. For the numbers of faces, we have to count the vertices in the all-B dessin. We compute:

$$f(\mathbb{D}) = m - n + \sum_{i=1}^{n} p_i.$$

Now, we get for the Euler characteristic:

$$\chi(\mathbb{D}) = v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) = 0$$

and thus the Turaev genus is one.

It remains to compute the difference between the number of spanning trees in the dessin and the number of spanning trees in its dual. This is a simple counting argument. □

Remark 4.4. The class of Turaev genus one knots and links is quite rich. For example, it contains all non-alternating Montesinos links. It also contains all semi-alternating links (whose diagrams are constructed by joining together two alternating tangles, and thus have exactly two over-over crossing arcs and two under-under arcs).

5. Dessins with One Vertex

5.1. *Link projection modifications*

Here we show that every knot/link admits a projection with respect to which the all-A dessin has one vertex. Such dessins are useful for computations.

Lemma 5.1. *Let $\tilde{P}$ be a projection of a link L with corresponding all-A dessin $\tilde{\mathbb{D}}$. Then $\tilde{P}$ can be modified by Reidemeister moves to new a projection P such that the corresponding dessin $\mathbb{D} = \mathbb{D}(A)$ has one vertex. Furthermore, we have*:

(1) $e(\tilde{\mathbb{D}}) + 2v(\tilde{D}) - 2 = e(\mathbb{D})$,
(2) $g(\tilde{\mathbb{D}}) + v(\tilde{D}) - 1 = g(\mathbb{D})$.

Proof. For a connected projection of the link L, consider the collection of circles that we obtain by an all-A splicing of the crossings. If there is only one circle we are done. Otherwise, one can perform a Reidemeister move II near a crossing on two arcs that lie on two neighbor circles as in Fig. 6.

The new projection will have one circle less in its all-A splicing diagram. Also two crossings were added and a new face was created. If the link projection is non-connected, one can transform it by Reidemeister II moves into a connected link projection. It is easy to check that the genus behaves as predicted. The claim follows. □

Remark 5.2. Dessins with one vertex are equivalent to Manturov's "d-diagrams" [17]. Note that the procedure of using just Reidemeister moves of type II is similar in spirit to Vogel's proof of the Alexander theorem [21, 3].

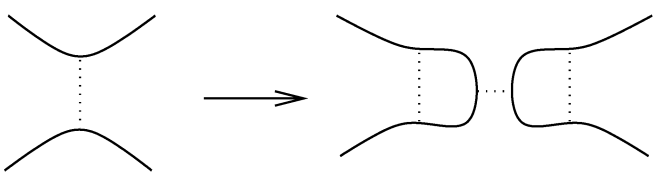

Fig. 6. Reduction of the number of vertices by a Reidemeister II move.

5.2. *The determinant of dessins with one vertex*

Dessins with one vertex can also described as chord diagrams. The circle of the chord diagram corresponds to the vertex and the chords correspond to the edges. In our construction, the circle of the chord diagram is the unique circle of the state resolution, and the chords correspond to the crossings. The cyclic orientation at the vertex induces the order of the chords around the circle. For each chord diagram $\mathbb{D}$, one can assign an intersection matrix [2,7] as follows: Fix a base point on the circle, disjoint from the chords and number the chords consecutively.

The intersection matrix is given by:

$$\mathrm{IM}(\mathbb{D})_{ij} = \begin{cases} \mathrm{sign}(i-j), & \text{if the } i\text{th chord and the } j\text{th chord intersect,} \\ 0, & \text{else.} \end{cases}$$

Recall that the number of spanning j-quasi trees in $\mathbb{D}$ was denoted by $s(j, D)$. Now

Theorem 5.3. *For a dessin* $\mathbb{D}$ *with one vertex, the characteristic polynomial of* $\mathrm{IM}(\mathbb{D})$ *satisfies*:

$$\det(\mathrm{IM}(\mathbb{D}) - xI) = (-1)^m \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} s(j, \mathbb{D}) x^{m-2j},$$

where $m = e(\mathbb{D})$ *is the number of chords, i.e. the number of edges in the dessin.*

In particular

$$\det(\mathbb{D}) = |\det(\mathrm{IM}(\mathbb{D}) - \sqrt{-1}I)|.$$

Proof. The result follows from combining Theorem 3.2 and a result of Bar-Natan and Garoufalidis [2]. Bar-Natan and Garoufalidis use chord diagrams to study weight systems coming from Vassiliev invariant theory, thus in a different setting than we do. However, by [2] for a chord diagram $\mathbb{D}$ the determinant $\det(\mathrm{IM}(\mathbb{D}))$ is either 0 or 1 and, translated in our language, it is 1 precisely if $f(\mathbb{D}) = 1$.

Furthermore, since $f(\mathbb{D}) - 1$ and the number of edges have the same parity, we know that $\det(\mathrm{IM}(\mathbb{D})) = 0$ for an odd number of edges.

The matrix $\mathrm{IM}(\mathbb{D})$ has zeroes on the diagonal. Thus, the coefficient of x^{m-j} in $\det(\mathrm{IM}(\mathbb{D}) - xI)$ is $(-1)^{m-j}$ times the sum over the determinants of all $j \times j$ submatrices that are obtained by deleting $m-j$ rows and the $m-j$ corresponding columns in the matrix $\mathrm{IM}(\mathbb{D})$. Those submatrices are precisely $\mathrm{IM}(\mathbb{H})$ for $\mathbb{H}$ a sub-dessin of $\mathbb{D}$ with j edges. In particular, the determinant of $\mathrm{IM}(\mathbb{H})$ is zero for j odd. For j even we know that $\det(\mathrm{IM}(\mathbb{H})) = 1$ if $f(\mathbb{H}) = 1$ and 0 otherwise. Since for 1-vertex dessins D the genus $2g(\mathbb{D}) = e(\mathbb{D}) - f(\mathbb{D}) + 1$ those $\mathbb{H}$ with $f(\mathbb{H}) = 1$ are precisely the $j/2$-quasi-trees. This, together with Theorem 3.2 implies the claim. □

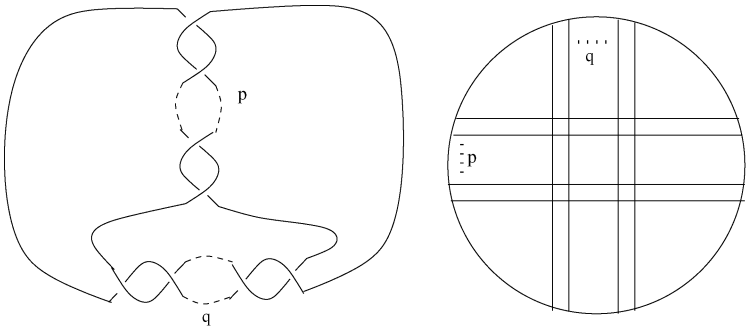

Fig. 7. The (p, q)-twist knot and its all-A splicing dessin in chord diagram form.

Example 5.4. The (p, q)-twist knots as in Fig. 7 have an all-A-dessin with one vertex.

The figure-8 knot is given as the $(2, 3)$-twist knot. Its intersection matrix is

$$\mathrm{IM}(\mathbb{D}) = \begin{pmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of $\mathrm{IM}(\mathbb{D})$ is $-6x^3 - x^5$. In particular, the determinant of the figure-8 knot is $6 - 1 = 5$.

5.3. *The Jones polynomial at* $t = -2$

By work of Jaeger, Vertigan and Welsh [13], evaluating the Jones polynomial is $\#P$-hard at all points, except at eight points: All fourth and sixth roots of unity. In particular, the determinant arises as one of these exceptional points. However, letting computational complexity aside, Theorem 2.2 gives an interesting formula in terms of the genus for yet another point: $t = A^{-4} = -2$.

Lemma 5.5. *Let P be the projection of a link K with dessin $\mathbb{D} = \mathbb{D}(A)$ such that $\mathbb{D}$ has one vertex. Then, the Kauffman bracket at $t = A^{-4} := -2$ evaluates to*

$$\langle P \rangle = A^{e(\mathbb{D})} \sum_{\mathbb{H} \subset \mathbb{D}} (A^{-4})^{g(\mathbb{H})}.$$

Proof. By Corollary 2.3, we have the following sub-dessin expansion for the Kauffman bracket of P:

$$\langle P \rangle = \sum_{\mathbb{H} \subset \mathbb{D}} A^{e(\mathbb{D}) - 2e(\mathbb{H})} \left(-A^2 - A^{-2}\right)^{f(\mathbb{H}) - 1}.$$

The term

$$A^{-2}(-A^2 - A^{-2}) = (-1 - A^{-4})$$

is 1 at $t = A^{-4} = -2$ and, with $v(\mathbb{D}) = v(\mathbb{H})$ for all spanning sub-dessin $\mathbb{H}$ of $\mathbb{D}$, the claim follows. □

6. Dessins and the Coefficients of the Jones Polynomial

Let P be a connected projection of a link L, with corresponding all-A dessin $\mathbb{D} := \mathbb{D}(A)$ and let

$$\langle P \rangle = \sum_{\mathbb{H} \subset \mathbb{D}} A^{e(\mathbb{D})-2e(\mathbb{H})}(-A^2 - A^{-2})^{f(\mathbb{H})-1} \tag{6.1}$$

denote the spanning sub-dessin expansion of the Kauffman bracket of P as obtained earlier. Let $\mathbb{H}_0 \subset \mathbb{D}$ denote the spanning sub-dessin that contains no edges (so $v(\mathbb{H}_0) = v(\mathbb{D})$ and $e(\mathbb{H}_0) = 0$) and let $M := M(P)$ and $m := m(P)$ denote the maximum and minimum powers of A that occur in the terms that lead to $\langle P \rangle$. We have

$$M(P) \leq e(\mathbb{D}) + 2v(\mathbb{D}) - 2,$$

and the exponent $e(\mathbb{D}) + 2v(\mathbb{D}) - 2$ is realized by $\mathbb{H}_0$; see [8, Lemma 7.1]. Let a_M denote the coefficient of the extreme term $A^{e(\mathbb{D})+2v(\mathbb{D})-2}$ of $\langle P \rangle$. Below we will give formulae for a_M; similar formulae can be obtained for the lowest coefficient, say a_m, if one replaces the the all-A dessin with the all-B dessin in the statements below. We should note that a_M is not, in general, the first non-vanishing coefficient of the Jones polynomial of L. Indeed, the exponent $e(\mathbb{D}) + 2v(\mathbb{D}) - 2$ as well as the expression for a_M we obtain below, depends on the projection P and it is not, in general, an invariant of L. In particular, a_M might be zero and, for example, we will show that this is the case in Example 6.2.

The following theorem extends and recovers results of Bae and Morton, and Manchón [1, 16] within the dessin framework.

Theorem 6.1. *We have*

(1) *For $l \geq 0$, let a_{M-l} denote the coefficient of $A^{e(\mathbb{D})+2v(\mathbb{D})-2-4l}$ in the Kauffman bracket $\langle P \rangle$. Then, the term a_{M-l} only depends on spanning sub-dessins $\mathbb{H} \subset \mathbb{D}$ of genus $g(\mathbb{H}) \leq l$.*

(2) *The highest term is given by*

$$a_M = \sum_{\mathbb{H} \subset \mathbb{D},\, g(\mathbb{H})=0=k(\mathbb{H})-v(\mathbb{D})} (-1)^{v(\mathbb{D})+e(\mathbb{H})-1}. \tag{6.2}$$

In particular, if $\mathbb{D}$ does not contain any loops then $a_M = (-1)^{v(\mathbb{D})-1}$ and the only contribution comes from $\mathbb{H}_0$.

Proof. The contribution of a spanning $\mathbb{H} \subset \mathbb{D}$ to $\langle P \rangle$ is given by

$$X_{\mathbb{H}} := A^{e(\mathbb{D})-2e(\mathbb{H})}(-A^2 - A^{-2})^{f(\mathbb{H})-1}. \tag{6.3}$$

A typical monomial of $X_{\mathbb{H}}$ is of the form $A^{e(\mathbb{D})-2e(\mathbb{H})+2f(\mathbb{H})-2-4s}$, for

$$0 \leq s \leq f(\mathbb{H}) - 1.$$

For a monomial to contribute to a_{M-l}, we must have

$$e(\mathbb{D}) - 2e(\mathbb{H}) + 2f(\mathbb{H}) - 2 - 4s = e(\mathbb{D}) + 2v(\mathbb{D}) - 2 - 4l, \tag{6.4}$$

or

$$f(\mathbb{H}) = v(\mathbb{D}) + e(\mathbb{H}) + 2s - 2l. \tag{6.5}$$

Now we have

$$\begin{aligned} 2g(\mathbb{H}) &= 2k(\mathbb{H}) - v(\mathbb{D}) + e(\mathbb{H}) - f(\mathbb{H}) \\ &= 2k(\mathbb{H}) - 2v(\mathbb{D}) + 2l - 2s, \end{aligned}$$

or $g(\mathbb{H}) = k(\mathbb{H}) - v(\mathbb{D}) + l - s$. But since $v(\mathbb{D}) \geq k(\mathbb{H})$ (every component must have a vertex) and $s \geq 0$, we conclude that

$$l = g(\mathbb{H}) + v(\mathbb{D}) - k(\mathbb{H}) + s \geq g(\mathbb{H}),$$

as desired. Now to get the claims for a_M: Note that for a monomial of $X_{\mathbb{H}}$ to contribute to a_M, we must have

$$g(\mathbb{H}) = k(\mathbb{H}) - v(\mathbb{D}) - s \tag{6.6}$$

which implies that $s = g(\mathbb{H}) = 0$ and $v(\mathbb{D}) = k(\mathbb{H})$. It follows that $\mathbb{H}$ contributes to a_M if and only if all of the following conditions are satisfied:

(1) $f(\mathbb{H}) = v(\mathbb{D}) + e(\mathbb{H})$.
(2) $k(\mathbb{H}) = v(\mathbb{D})$. Thus, $\mathbb{H}$ consists of $k := k(\mathbb{H})$ components each of which has exactly one vertex and either $\mathbb{H}$ has no edges or every edge is a loop.
(3) $g(\mathbb{H}) = 0$.
(4) The contribution of $\mathbb{H}$ to a_M is $(-1)^{f(\mathbb{H})-1}$.

This finishes the proof of the theorem. □

Example 6.2. The all-A dessin of Fig. 3 contains one sub-dessin with no edges, two sub-dessins with exactly one loop and one sub-dessin of genus zero with two loops. Thus $a_M = 0$.

A connected link projection is called A-adequate if and only if the all-A dessin $\mathbb{D}(A)$ contains no loops; alternating links admit such projections. We consider two edges as equivalent if they connect the same two vertices. Let $e' = e'(\mathbb{D}(A))$ denote the number of equivalence classes of edges.

The following is an extension in [19] to the class of adequate links of a result in [10] for alternating links. We will give the dessin proof for completeness, since it shows a subtlety when dealing with dessins in our context: Not all dessins can occur as a dessin of a link diagram.

Corollary 6.3. *For A-adequate diagrams $a_{M-1} = (-1)^v(e' - v + 1)$.*

Proof. With the notation and setting of the proof of Theorem 6.1, we are looking to calculate the coefficient of the power $A^{e(\mathbb{D})+2v(\mathbb{D})-6}$. The analysis in the proof of Theorem 6.1 implies that a spanning sub-dessin $\mathbb{H} \subset \mathbb{D}$ contributes to a_{M-1} if it satisfies one of the following:

(1) $v(\mathbb{H}) = k(\mathbb{H})$ and $g(\mathbb{H}) = 1$.
(2) $v(\mathbb{H}) = k(\mathbb{H})$ and $g(\mathbb{H}) = 0$.
(3) $v(\mathbb{H}) = k(\mathbb{H}) + 1$ and $g(\mathbb{H}) = 0$.

Since the link is adequate $\mathbb{D}(A)$ contains no loops and we cannot have any $\mathbb{H}$ as in (1). Furthermore, the only $\mathbb{H}$ with the properties of (2) is the sub-dessin $\mathbb{H}_0$ that contains no edges. Finally, the only case that occurs in (3) consists of those sub-dessins $\mathbb{H}_1$ that are obtained from $\mathbb{H}_0$ by adding edges between a pair of vertices. The dessin is special since it comes from a link diagram. Each vertex in the dessin represents a circle in the all-A splicing diagram of the link and each edge represents an edge there. Because these edges do not intersect, $\mathbb{H}_1$ must have genus 0.

Note that any sub-dessin $\mathbb{H}' \subset \mathbb{H}_1$ is either $\mathbb{H}_0$ or is of the sort described in (3). We will call $\mathbb{H}_1$ maximal if it is not properly contained in one of the same type with more edges. Thus, there are e' maximal $\mathbb{H}_1$ for $\mathbb{D}(A)$. The contribution of $\mathbb{H}_1$ to a_{M-1} is $(-1)^{v(\mathbb{D})-3+e(\mathbb{H}_1)}$. Thus, the contribution of all $\mathbb{H}' \subset \mathbb{H}_1$ that are not $\mathbb{H}_0$ is

$$\sum_{j=1}^{e(\mathbb{H}_1)} \binom{e(\mathbb{H}_1)}{j} (-1)^{v(\mathbb{D})-3+j} = (-1)^{v(\mathbb{D})}.$$

Thus, the total contribution in a_{M-1} of all such terms is $(-1)^v e'$.

To finish the proof, observe that the contribution of $\mathbb{H}_0$ comes from the second term of the binomial expansion

$$X_{\mathbb{H}_0} := A^{e(\mathbb{D})}(-A^2 - A^{-2})^{f(\mathbb{H}_0)-1}. \tag{6.7}$$

Since $f(\mathbb{H}_0) = v$ this later contribution is $(-1)^{v-1}(v-1)$. □

The expression in Theorem 6.1 becomes simpler, and the lower order terms easier to express, if the dessin $\mathbb{D}$ has only one vertex. By Lemma 5.1 the projection P can always be chosen so that this is the case.

Corollary 6.4. *Suppose P is a connected link projection such that $\mathbb{D} = \mathbb{D}(A)$ has one vertex. Then,*

$$a_{M-l} = \sum_{\mathbb{H}\subset\mathbb{D},\, g(\mathbb{H})=0}^{g(\mathbb{H})=l} (-1)^{e(\mathbb{H})} \binom{e(\mathbb{H}) - 2g(\mathbb{H})}{l - g(\mathbb{H})}. \tag{6.8}$$

In particular,

$$a_M = \sum_{\mathbb{H}\subset\mathbb{D},\, g(\mathbb{H})=0} (-1)^{e(\mathbb{H})} \tag{6.9}$$

and

$$a_{M-1} = \sum_{\mathbb{H}\subset\mathbb{D},\, g(\mathbb{H})=1} (-1)^{e(\mathbb{H})} + \sum_{\mathbb{H}\subset\mathbb{D},\, g(\mathbb{H})=0} (-1)^{e(\mathbb{H})} e(\mathbb{H}). \tag{6.10}$$

Proof. For a 1-vertex dessin $\mathbb{D}$, we have

$$k(\mathbb{D}) = v(\mathbb{D}) = 1 \text{ and, thus } 2g(\mathbb{D}) = e(\mathbb{D}) - f(\mathbb{D}) + 1.$$

Now, Eq. (6.1) simplifies to

$$\begin{aligned} \langle P \rangle &= \sum_{\mathbb{H}\subset\mathbb{D}} A^{e(\mathbb{D})-2e(\mathbb{H})+2f(\mathbb{H})-2} \left(-1-A^{-4}\right)^{f(\mathbb{H})-1} \\ &= \sum_{\mathbb{H}\subset\mathbb{D}} A^{e(\mathbb{D})-4g(\mathbb{H})} \left(-1-A^{-4}\right)^{e(\mathbb{H})-2g(\mathbb{H})}. \end{aligned}$$

The claim follows from collecting the terms. □

Parallel edges, i.e. neighboring edges that are parallel in the chord diagram, in a dessin are special since they correspond to twists in the diagram. It is useful to introduce weighted dessins: Collect all edges, say $\mu(c) - 1$ edges parallel to a given edge c and replace this set by c weighted with weight $\mu(c)$. Note, that $\tilde{\mathbb{D}}$ has the same genus as $\mathbb{D}$.

Corollary 6.5. *For a knot projection with a* 1*-vertex dessin* $\mathbb{D}$ *and weighted dessin* $\tilde{D}$, *we have*:

$$\langle P \rangle = \sum_{\tilde{\mathbb{H}}\subset\tilde{\mathbb{D}}} A^{e(\mathbb{D})-4g(\tilde{\mathbb{H}})} (-1-A^{-4})^{-2g(\tilde{\mathbb{H}})} \prod_{c\in\tilde{\mathbb{H}}} (-1-A^{-4\mu(c)}).$$

Proof. For a given edge c collect in

$$\langle P \rangle = \sum_{\mathbb{H}\subset\mathbb{D}} A^{e(\mathbb{D})-4g(\mathbb{H})} \left(-1-A^{-4}\right)^{e(\mathbb{H})-2g(\mathbb{H})}$$

all terms where $\mathbb{H}$ contains an edge parallel to c. This sub-sum is

$$\begin{aligned} &\sum_{\mathbb{H}\subset\mathbb{D},\, \mathbb{H} \text{ contains edge parallel to } c} A^{e(\mathbb{D})-4g(\mathbb{H})} \left(-1-A^{-4}\right)^{e(\mathbb{H})-2g(\mathbb{H})} \\ &= \sum_{\mathbb{H}\subset\mathbb{D},\, \tilde{\mathbb{H}}=\mathbb{H}-\{\text{edges parallel to } c\}\cup c} \sum_{j=1}^{\mu(c)} \binom{\mu(c)}{j} A^{e(\mathbb{D})-4g(\mathbb{H})} \left(-1-A^{-4}\right)^{e(\tilde{\mathbb{H}})-1+j-2g(\mathbb{H})} \\ &= \sum_{\mathbb{H}\subset\mathbb{D},\, \tilde{\mathbb{H}}=\mathbb{H}-\{\text{edges parallel to } c\}\cup c} A^{e(\mathbb{D})-4g(\mathbb{H})} (-1-A^{-4})^{e(\tilde{\mathbb{H}})-1-2g(\mathbb{H})} (-1-A^{-4\mu(c)}) \end{aligned}$$

The claim follows by repeating this procedure for each edge c. □

Example 6.6. The (p, q)-twist knot is represented by the weighted, 1-vertex dessin with two intersecting edges, one with weight p and one with weight q. By Corollary 6.5, its Kauffman bracket is:

$$A^{-p-q}\langle P\rangle = 1 + (-1 - A^{-4p}) + (-1 - A^{-4q}) + A^{-4}(-1 - A^{-4})^{-2}(-1 - A^{-4p})(-1 - A^{-4q}).$$

Remark 6.7. Corollary 6.4 implies the following for the first coefficient a_M. Suppose $\mathbb{D}$ is a 1-vertex, genus 0 dessin with at least one edge. Then, every sub-dessin also has genus 0.

Thus,

$$\sum_{\mathbb{H}\subset\mathbb{D},\, g(\mathbb{H})=0} (-1)^{e(\mathbb{H})} = \sum_{j=0}^{e(\mathbb{D})} \binom{e(\mathbb{D})}{j}(-1)^j = 0.$$

For an arbitrary dessin, let $\mathbb{H}_1, \ldots, \mathbb{H}_n$ be the maximal genus 0 sub-dessins of $\mathbb{D}$. Define a function ϕ on dessins which is 1 if the dessin contains no edges and 0 otherwise.

Then

$$a_M = \sum_i \phi(\mathbb{H}_i) - \sum_{i,j,\, i<j} \phi(\mathbb{H}_i \cap \mathbb{H}_j) + \sum_{i,j,k,\, i<j<k} \phi(\mathbb{H}_i \cap \mathbb{H}_j \cap \mathbb{H}_k) - \cdots$$

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