Beta ensembles, stochastic Airy spectrum, and a diffusion

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Abstract

We prove that the largest eigenvalues of the beta ensembles of random matrix theory converge in distribution to the low-lying eigenvalues of the random Schrödinger operator $-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x$ restricted to the positive half-line, where $b'_x$ is white noise. In doing so we extend the definition of the Tracy-Widom($\beta$) distributions to all $\beta > 0$, and also analyze their tails. Last, in a parallel development, we provide a second characterization of these laws in terms of a one-dimensional diffusion. The proofs rely on the associated tridiagonal matrix models and a universality result showing that the spectrum of such models converge to that of their continuum operator limit. In particular, we show how Tracy-Widom laws arise from a functional central limit theorem.

1 Introduction

For any $\beta > 0$, consider the probability density function of $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{R}$ given by

$$P_{\beta}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta \sum_{k=1}^{n} \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta,$$

in which $Z_{n,\beta}$ is a normalizing constant. When $\beta = 1, 2$ or 4 this is the joint density of eigenvalues for the Gaussian orthogonal, unitary, or symplectic ensembles, G(O/U/S)E, of random matrix theory. The law (1.1) also describes a one-dimensional Coulomb gas at inverse temperature $\beta$, and thus is of physical interest. Still, the local statistics in these ensembles have previously only been well understood for the three special $\beta$’s.

It was therefore a major leap forward when, based on [24], Dumitriu and Edelman [9] discovered the following family of matrix models for all $\beta$. Let $g_1, g_2, \ldots, g_n$ be independent Gaussians with mean 0 and variance 2. Let also $\chi_1, \chi_2, \ldots, \chi_{(n-1)\beta}$ be independent $\chi$ random variables indexed by the shape parameter. Then the $n$ eigenvalues of the tridiagonal matrix
ensemble

\[ H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} \\ & \ddots & \ddots & \ddots \\ & \chi_{2\beta} & g_{n-1} & \chi_{\beta} \\ & & \chi_{\beta} & g_n \end{bmatrix} \]  \hspace{1cm} (1.2)

have joint law given by (1.1). These are more specifically referred to as the \( \beta \)-Hermite ensembles, on account of the fact that in the solvable \( \beta = 1, 2, \) and 4 cases the spectral correlation functions are given explicitly in terms of Hermite polynomials.

We focus on the implications of this discovery to the point process limits of the spectral edge in the general \( \beta \)-ensembles. The distributional limits of the largest eigenvalues in \( G(O/U/S)E \) comprise some of the most celebrated results in random matrix theory due to their surprising importance in physics, combinatorics, multivariate statistics, engineering, and applied probability: [1], [3], [14], [17], [18], and [20] mark a few highlights. The basic result is: for \( \beta = 1, 2, \) or 4 and \( n \uparrow \infty \), centered by \( 2\sqrt{n} \) and scaled by \( n^{1/6} \), the largest eigenvalue converges in law to the Tracy-Widom(\( \beta \)) distribution (see [22] and [23]), which is given explicitly in terms of the second Painlevé transcendent. There are allied results for second, third, etc. eigenvalues; see again [22] as well as [7].

The wide array of models for which Tracy-Widom describes the limit statistics identifies these laws as important new probability distributions. Still, our understanding of these laws is in its infancy. It remains desirable to obtain a set of characterizing conditions, like those classically known for say the Gaussian or Poisson laws. A description of the limit distribution of the largest eigenvalues in the general \( \beta \)-Hermite ensembles is a first step, providing additional information on the structure of the three Tracy-Widom laws via the structure of a one parameter family of distributions in which they naturally reside.

Towards an edge limit theorem at general \( \beta \), Sutton [21] and Edelman and Sutton [10] present a promising heuristic argument that the rescaled operators

\[ \tilde{H}_n^\beta = n^{1/6} \left( 2\sqrt{n} I - H_n^\beta \right), \]  \hspace{1cm} (1.3)

where \( I \) is the \( n \times n \) identity, should correspond to

\[ \mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x \]  \hspace{1cm} (1.4)

in the \( n \uparrow \infty \), or continuum, limit. Here \( b' \) indicates a white noise, and the proposed scaling of the matrix ensembles follows the edge scaling in the known cases. Thus, were it to hold,
the above correspondence would entail that the low-lying eigenvalues of $\tilde{H}_n^\beta$ converge in law to the those of $\mathcal{H}_\beta$. Our first result is a proof of this heuristic.

In Section 2 we give a precise definition of this limiting “stochastic Airy operator” (SAE$_\beta$). For now, let $L^* \subset C^0$ denote the space of functions $f$ satisfying $f(0) = 0$ and $\int_0^\infty (f')^2 + (1 + x)f^2 \, dx < \infty$. Then we say $(\psi, \lambda) \in L^* \times \mathbb{R}$ is an eigenfunction/eigenvalue pair for $\mathcal{H}_\beta$ if $\|\psi\|_2 = 1$ and

$$\psi''(x) = \frac{2}{\sqrt{\beta}} \psi(x) b_x' + (x - \lambda) \psi(x),$$

holds in the following integration-by-parts sense,

$$\psi'(x) = \frac{2}{\sqrt{\beta}} \psi(x) b_x + \int_0^x -\frac{2}{\sqrt{\beta}} b_y \psi'(y) \, dy + \int_0^x (y - \lambda) \psi(y) \, dy,$$

where all integrands are products of locally $L^2$ functions. The set of eigenvalues is then a deterministic function of the random Brownian path $b$.

**Theorem 1.1.** With probability one, for each $k \geq 0$ the set of eigenvalues of $\mathcal{H}_\beta$ has a well-defined $(k + 1)$st lowest element $\Lambda_k$. Moreover, let $\lambda_1 \geq \lambda_2 \geq \cdots$ denote the eigenvalues of the Hermite $\beta$-ensemble $H_n^\beta$. Then the vector

$$\left( \frac{n^{1/6}}{2} (2\sqrt{n} - \lambda_{\beta, \ell}) \right)_{\ell=1,\ldots,k}$$

converges in distribution as $n \to \infty$ to $(\Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1})$.

The proof relies on a universality result showing that the spectrum of tridiagonal models converges to that of their continuum operator limit. In particular, one can get Tracy-Widom limits by considering tridiagonal matrix versions of (1.4) that are simpler than (1.3).

Though the $n^{1/6}$ scaling across all beta was anticipated from the Tracy-Widom results at $\beta = 1, 2, 4$, previous authors have only obtained bounds on this rate. Dumitriu [8] proved that the fluctuations of $\lambda_{\beta,1}/\sqrt{n}$ are no greater than $n^{-1/2}$ while, for even integer values of beta, [6] shows the expected fluctuation to be $O(n^{-2/3})$ by way of Jack polynomials.

Later, we will also provide an equivalent variational formulation of the eigenvalue problem. Our next theorem gives yet another description of the limiting spectrum in terms of the explosion probability of the one-dimensional diffusion $x \mapsto p(x)$ defined by the Itô equation

$$dp(x) = \frac{2}{\sqrt{\beta}} db_x + (x - p^2(x)) \, dx.$$  

(1.8)

**Theorem 1.2.** Let $\kappa(x, \cdot)$ be the distribution of the first passage time to $-\infty$ of the diffusion $p(x)$ when started from $+\infty$ at time $x$. Then

$$P(\Lambda_0 > \lambda) = \kappa(-\lambda, \{\infty\}),$$

and, for $k \geq 0$,

$$P(\Lambda_k < \lambda) = \int_{\mathbb{R}^{k+1}} \kappa(-\lambda, dx_1) \kappa(x_1, dx_2) \cdots \kappa(x_k, dx_{k+1}).$$
Once again, these expressions are novel even for $\beta = 1, 2, 4$. A variant of (1.8) is shown to describe joint laws in Proposition 3.4.

Theorem 1.2 and the variational characterization of eigenvalues leads to our final result on the shape of the general beta laws, so far only known for $\beta = 1, 2, 4$.

**Theorem 1.3.** With the obvious notation $TW_\beta = -\Lambda_0(\beta)$, for $a \uparrow \infty$ it holds

\[
P\left(TW_\beta > a\right) = \exp\left(-\frac{2}{3}\beta a^{3/2}(1 + o(1))\right), \quad \text{and}
\]

\[
P\left(TW_\beta < -a\right) = \exp\left(-\frac{1}{24}\beta a^{3}(1 + o(1))\right).
\]

The above discussion carries over to a large class of $\beta$-ensembles. For example, for $\kappa \geq n$ let $W$ be an $n \times \kappa$ matrix comprised of independent standard real, complex, or quaternion Gaussians, and think of $W$ as $\kappa$ random vectors. The sample covariance matrix $WW^\dagger$, the Laguerre, or Wishart ensemble for $\beta = 1, 2, 4$ plays an important role in mathematical statistics.

Johansson in the complex case [17] and Johnstone in the real case [18], showed that the largest eigenvalues tend to the $\beta = 1$ and $\beta = 2$ Tracy-Widom laws whenever $\kappa/n \to \tau \in (0, \infty)$. Later, El Karoui [11] showed the same result holds even if $\kappa/n \to \infty$ or 0 if both $n, \kappa \to \infty$, this regime being important in applications. Our proof handles all these regimes together for all $\beta$. In the appropriate scaling, the joint density of eigenvalues is given by

\[
P_{n,\kappa}^\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\kappa}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}(\kappa-n+1)-1} e^{-\frac{\beta}{2} \lambda_k}.
\]  

(1.9)

This formula generalizes the Laguerre eigenvalues for non-integer $\kappa > n - 1$ and $\beta > 0$. As an application of our universal limit theorem, we show:

**Theorem 1.4.** Let $\lambda_1 \geq \lambda_2 \ldots$ denote the ordered Laguerre “eigenvalues” (1.9), and set

\[
\mu_{n,\kappa} = (\sqrt{n} + \sqrt{\kappa})^2, \quad \text{and} \quad \sigma_{n,\kappa} = \frac{(\sqrt{n\kappa})^{1/3}}{(\sqrt{n} + \sqrt{\kappa})^{4/3}}.
\]

(1.10)

Then for any $k$, as $n \to \infty$ with arbitrary $\kappa = \kappa_n > n - 1$ we have

\[
\left(\sigma_{n,\kappa}(\mu_{n,\kappa} - \lambda_\ell)\right)_{\ell=1,\ldots,k} \Rightarrow \left(\Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1}\right).
\]

Once again, Section 2 discusses the definition and basic properties of SAE$_\beta$. The diffusion connection and Theorem 1.2 are detailed in Section 3. Section 4 establishes, the tail bounds, Theorem 1.3. In Section 5 we prove a general result, Theorem 5.1, which provides weak
conditions under which the lowest eigenvalues of tridiagonal random matrices of type discrete Laplacian plus potential converge to the corresponding eigenvalues of continuum operator limit. It is anticipated that a result of this type will be of future importance in investigations of universality in random matrix theory, and in Section 5 it is employed to prove Theorems 1.1 and 1.4.

2 Basic properties of the Stochastic Airy Equation

Definition of the Stochastic Airy Operator

We use the classical Schwarz distribution theory. Recall that the space of distributions $D = D(\mathbb{R}^+)$ is the continuous dual of the space $C^\infty_0$ of all smooth compactly supported test functions under the topology of uniform-on-compact convergence of all derivatives. Recall as well that all continuous functions $f$ and their formal derivatives are distributions. They act on $C^\infty_0$ via integration by parts,

$$\langle \varphi, f^{(k)} \rangle := (-1)^k \int f(x) \varphi^{(k)}(x) dx,$$

where the latter is clearly defined. For instance, $b'$, the formal derivative of Brownian motion, is a random distribution, as $b$ is a random continuous function. The notation $\langle \cdot, \cdot \rangle$ distinguishes the above from an $L^2$ inner product $\langle \cdot, \cdot \rangle$.

Introduce $H^1_{loc}$, the space of functions $f : \mathbb{R}^+ \to \mathbb{R}$ for which $f' 1_I \in L^2$ for any compact set $I$. $SAE_\beta$ is then well defined as a random linear map $H^1_{loc} \to D$, sending $f$ to the distribution

$$\mathcal{H}_\beta f = -f'' + fb' + xf.$$

As $D$ is only closed under multiplication by smooth functions, one must make sense $fb'$ as an element of that space. Stieltjes integration by parts prompts

$$\int_0^y f'b' dx := -\int_0^y bf'dx + f(y)b_y - f(0)b_0.$$

The latter is a continuous function of $y$, and we define $fb'$ as its derivative.

Eigenvalues and eigenfunctions

We will consider the eigenvalues from two points of view,

(i) as the solutions of $\mathcal{H}_\beta f = \lambda f$ with given “boundary conditions”, or

(ii) as solutions of the usual variational problem.
The first approach is intimately tied to the Riccati transformation, which will be our main tool for analyzing solutions. The second will be useful in obtaining bounds on the eigenvalue distributions for SAE$_\beta$. Their equivalence for SAE$_\beta$ is established over the course of the next two subsections.

Recall the Hilbert space $L^*$ defined via the norm

$$
\|f\|_*^2 = \int_0^{\infty} (f')^2 + (1 + x) f^2 \, dx,
\quad L^* = \{ f : f(0) = 0, \|f\|_* < \infty \}.
$$

The framework just introduced makes it self-evident how to define the $L^*$ eigenfunctions and eigenvalues of SAE$_\beta$ in the sense of (i).

**Definition 2.1.** The eigenvalues and eigenfunctions of $H_\beta$ are the pairs $(f, \lambda) \in L^* \times \mathbb{R}$ satisfying

$$
H_\beta f = \lambda f,
$$

where both sides are interpreted as distributions ($L^* \subset C^0(\mathbb{R}^+) \subset D$). The $(k + 1)$st smallest point in this set, if exists, will be denoted $\Lambda_k$.

The $f \in L^*$ requirement allows us to avoid technicalities. Proposition 3.7 shows that it can be relaxed to $f \in C^0 \cap L^2$, or even further.

If we rewrite $H_\beta f = \lambda f$ as

$$
f'' = (b' + x - \lambda) f, \tag{2.1}
$$

the right hand side is a derivative of a continuous function, and thus $f'$ can be taken to be continuous (it is defined in an a.e. sense). In this way we arrive at two equivalent formulations of the eigenvalue problem. One is the coincidence of the integrated versions of both sides of (2.1). The other is the equivalence of the two sides as distributions. The first reproduces the definition (1.5) from the introduction

$$
f'(x) = \int_0^x -b_y f'(y) + (y - \lambda) f \, dy + f(x) b_x,
$$

showing at once that $f'$ inherits the Hölder($1/2$)− continuity properties of $b$, i.e., that it is Hölder($1/2 - \varepsilon$) continuous for all $\varepsilon > 0$. Thus $f \in$ Hölder($3/2$)−. The second, weak definition, will be useful in the variational analysis. It reads

$$
\int \varphi'' f \, dx = \int (x - \lambda) \varphi f \, dx + \int \left[ \int_0^x b_y f'(y) dy - b_x f(x) \right] \varphi' \, dx, \tag{2.2}
$$

and takes the form

$$
\int \varphi'' f \, dx = \int (x - \lambda) f \varphi - b f' \varphi - b f \varphi' \, dx \tag{2.3}
$$
after an integration by parts.

Before proceeding to the variational approach, we register some simple facts about $L^*$. 

**Fact 2.2.** Any $L^*$ bounded sequence has a subsequence $f_n$ that converges to some $f \in L^*$ in each of the following ways: (i) $f_n \to_{L^2} f$ (ii) $f_n' \to f'$ weakly in $L^2$, (iii) $f_n \to f$ uniformly-on-compacts and (iv) $f_n \to f$ weakly in $L^*$. 

**Proof.** Convergence modes (ii) and (iv) are simple applications of the Banach-Alaoglu theorem. (iii) stems from the familiar estimates $f_n^2(x) \leq 2\|f_n\|_2\|f_n'\|_2$ and $|f_n(y) - f_n(x)| = |\int f_n'1_{[x,y]}dz| \leq \|f_n'\|_2|x - y|^{1/2}$, showing that the sequence is uniformly equicontinuous on compacts. Last, (iii) implies $L^2$ convergence locally, while the bound $\sup_n \int xf_n^2 dx < \infty$ produces the uniform integrability required for (i). 

The quadratic form $\langle f, \mathcal{H}_\beta f \rangle$

The quadratic form $\langle f, \mathcal{H}_\beta f \rangle$ typically associated with self-adjoint operators is already defined for test functions $f \in C_0^\infty$. As a path to generalization, we proceed by decomposing the fluctuation term via

$$b = \bar{b} + (b - \bar{b}), \quad \bar{b}_x = \int_x^{x+1} b_y dy, \quad \text{and so} \quad \bar{b}'_x = b_{x+1} - b_x.$$ 

The idea is to smooth out the noise and then later control the difference between the noise and its mollification. We have: for any $f \in C_0^\infty$,

$$\langle f, \mathcal{H}_\beta f \rangle = \int_0^\infty (f')^2 dx + \int_0^\infty x f'^2 dx$$

$$+ \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) \bar{b}'_x dx + \frac{4}{\sqrt{\beta}} \int_0^\infty f'(x)f(x)(\bar{b}_x - b_x) dx. \quad (2.4)$$

Line two is just the introduced way to rewrite $\langle f^2, b' \rangle$, and the necessary extension requires this object to be finite over $L^*$. Step one is a simple bound on the Brownian paths.

**Lemma 2.3.** For $b_x, x > 0$ a Brownian motion, there is a random constant $C < \infty$ so that

$$\sup_{x > 0} \sup_{0 < y \leq 1} \frac{|b_{x+y} - b_x|}{\sqrt{\log (2 + x)}} \leq C \quad a.s. \quad (2.5)$$

As a consequence, $|\bar{b}'(x)| \vee |\bar{b}_x - b_x| \leq C \sqrt{\log (2 + x)}$.

**Proof.** By the triangle inequality $|b_{x+y} - b_x| \leq |b_{x+y} - b_{[x+y]}| + |b_{[x+y]} - b_{[x]}| + |b_x - b_{[x]}|$ it is enough to show that

$$\sup_{n > 1} \frac{X_n}{\sqrt{\log n}} \leq C,$$
where the $X_n = \sup_{0 < y \leq 1} |b_{n+y} - b_n|$ are independent and identically distributed. Further, $P(X_n > a) = 2\sqrt{2/\pi} \int_a^\infty e^{-m^2/2} dm \leq 2a^{-1} e^{-a^2/2}$, courtesy of the reflection principle. A Borel-Cantelli argument completes the proof.

Using the lemma, we may bound $|\bar{b}_x| \leq C(1 + x)$ and $|\bar{b}_x - b_x|$ by $\sqrt{C(1 + x)}$ in (2.4). An application of Cauchy-Schwarz yields

**Proposition 2.4.** The is the bound $|\langle f, \mathcal{H}_\beta f \rangle| \leq C\|f\|_2^2$. In particular $\langle f, \mathcal{H}_\beta f \rangle$ is defined for all $f \in L^*$; $\langle f, \mathcal{H}_\beta g \rangle$ can be defined by polarization, and is a continuous symmetric bilinear form $(L^*)^2 \to \mathbb{R}$.

**Variational characterization**

We are at last in position to define the ground state energy $\tilde{\Lambda}_0 = \tilde{\Lambda}_0(\beta, \omega)$ of $\mathcal{H}_\beta$ variationally. As anticipated, we will show that $\tilde{\Lambda}_0 = \Lambda_0$, the lowest point in the set of eigenvalues of $\mathcal{H}_\beta$.

Setting

$$\tilde{\Lambda}_0 := \inf \{ \langle f, \mathcal{H}_\beta f \rangle : f \in L^* \text{ with } f(0) = 0, \|f\|_2 = 1 \},$$

we begin by proving that $\tilde{\Lambda}_0 > C_2(\omega) > -\infty$ a.s. Random constants are denoted throughout by $C_\cdot$, and deterministic ones by $c_\cdot$.

**Lemma 2.5.** There are constants $c_1, C_2, C_3$ so that a.s. for all $f \in L^*$

$$c_1\|f\|_*^2 - C_2\|f\|_2^2 \leq \langle f, \mathcal{H}_\beta f \rangle \leq C_3\|f\|_*^2.$$  

(2.7)

**Proof.** The upper bound is from Proposition 2.4. For the lower bound, we repeat the definition

$$\langle f, \mathcal{H}_\beta f \rangle = \int_0^\infty (f')^2 \, dx + \int_0^\infty xf^2 \, dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f(x) \bar{b}_x \, dx + \frac{4}{\sqrt{\beta}} \int_0^\infty f'(x)f(x)(\bar{b}_x - b_x) \, dx.$$ 

The first line on the right amounts to $\|f\|_*^2 - \|f\|_2^2$, and it suffices to show that the terms $B_1, B_2$ in the second line do not ruin the picture and satisfy $B_1 + B_2 \geq -c_1\|f\|_*^2 - C\|f\|_2^2$ with $c_1 < 1$.

Lemma 2.3 provides the bounds $\frac{2}{\sqrt{\beta}}|\bar{b}_x| \leq c_2(C + x)$ and $\frac{2}{\sqrt{\beta}}|\bar{b}_x - b_x| \leq c_2\sqrt{C + x}$ for arbitrarily small $c_2$ and some random $C = C(c_2, b)$. Thus, $|B_1| \leq c_2\|f\|_*^2 + c_2C\|f\|_2^2$, and for the second term we have

$$|B_2| \leq \int_0^\infty |f'(x)f(x)|c_2\sqrt{C + x} \, dx \leq c_2\|f'(x)\|_2^2 + c_2 \int_0^\infty (C + x)f^2(x) \, dx.$$ 

In particular, $|B| \leq 3c_2\|f\|_*^2 + C'\|f\|_2^2$, where again $c_2$ may be taken as small as needed. □
Corollary 2.6. The infimum in the variational problem (2.6) is attained at an eigenfunction $f_0$ of $\mathcal{H}_\beta$ with eigenvalue $\Lambda_0$.

Proof. Again by Lemma 2.3, for any $\varepsilon > 0$ the bounds $|\overline{b}'_x| \leq \varepsilon (1 + x)$ and $|\overline{b}_x - b_x| \leq \varepsilon \sqrt{1 + x}$ hold for all $x > X(b, c)$. As in the proof of Lemma 2.5, it follows

$$\langle f, \mathcal{H}_\beta f \rangle = \|f\|^2_2 - \|f\|^2_2 + \frac{2}{\sqrt{\beta}} \int_0^X f^2(x) \overline{b}'_x dx + \frac{4}{\sqrt{\beta}} \int_0^X f'(x)f(x)(\overline{b}_x - b_x) dx + \mathcal{E},$$

(2.8)

where the error term satisfies $|\mathcal{E}| \leq \varepsilon \|f\|^2_2$.

Next choose a minimizing sequence $f_n \in L^*: \|f_n\|_2 = 1$ with $\langle f_n, \mathcal{H}_\beta f_n \rangle \to \tilde{\Lambda}_0$. By Lemma 2.5, $\|f_n\|_s < B$ for some $B = B(b)$, and by Fact 2.2, we can find a subsequence along which $f_n \to f$ uniformly on compacts, in $L^2$, and also weakly in $H^1$. Terms 2, 3, 4 on the right hand side of (2.8) then converge to their evaluations at $f$, while term 1 plainly satisfies $\|f\|^2_s \leq \lim \inf \|f_n\|^2_2$. Letting $\varepsilon \to 0$ yields $\langle f, \mathcal{H}_\beta f \rangle \leq \tilde{\Lambda}_0$, with the opposite inequality holding by definition.

To complete the picture, taking the functional derivative $\frac{d}{d\varepsilon} \langle f + \varepsilon \varphi, \mathcal{H}_\beta(f + \varepsilon \varphi) \rangle|_{\varepsilon = 0}$ of the variational problem (2.6) in directions $\varphi \in C_0^\infty$ shows that the minimizer satisfies $\mathcal{H}_\beta f = \lambda f$ in the sense of distributions. It is therefore an eigenfunction and $\tilde{\Lambda}_0 = \Lambda_0$. \qed

The above formulation may now be used to define higher eigenvalues in the expected manner: $\tilde{\Lambda}_1$ arising from restricting the class of potential minimizers to be perpendicular to $f_0$, and so on. This defines a sequence of eigenvalues $\tilde{\Lambda}_k$, $k \geq 0$, each of which is finite. That $\cdots < \tilde{\Lambda}_k < \tilde{\Lambda}_{k+1} < \cdots$ will come later; the standard trick with the bilinear form defined in Lemma 2.4 yields the $L^2$-orthogonality of the corresponding eigenfunctions. In short:

Lemma 2.7. The $(k+1)$st lowest element $\Lambda_k$ in the set of eigenvalues of $\mathcal{H}_\beta$ exists and in fact $\Lambda_k = \tilde{\Lambda}_k$.

Remark 2.8. Lemma 2.3, as used in Lemma 2.5, is related to the second condition of our general convergence result, Theorem 5.1, below. In addition, a discrete version of this estimate is employed in the context of the $\beta$-ensembles converging to SAE$_\beta$.

### 3 Riccati transform and diffusion formulas

The Riccati map is a classical tool in the study one-dimensional random Schrödinger spectra. Its use dates back to Halperin [16] who computed the density of states for $-d^2/dx^2 + b'_x$, though see also [15]. For earlier applications to local statistics such as the ground state energy, see [4], [5], and [19].
Return to the eigenvalue problem (1.5)

\[\psi''(x) = \frac{2}{\sqrt{\beta}} \psi'(x)b'_x + (x - \lambda)\psi(x),\]  

(3.1)

understood in the integration-by-parts sense. The Riccati transform is simply the logarithmic derivative \(p(x) = \psi'(x)/\psi(x)\). This turns (3.1) into a first order differential equation: \(p(0) = \infty\), and

\[p'(x) = x - \lambda - p^2(x) - \frac{2}{\sqrt{\beta}} b'(x),\]  

(3.2)

understood in the same way.

Solutions of (3.2) may blow up (to \(-\infty\)) at finite times, as will happen whenever \(\psi\) vanishes. In this case \(p\) is immediately restarted at \(+\infty\) at that time-point in order to continue the solution corresponding to (3.1). It is convenient to think of \(p\) as taking values in the disjoint union of countable copies of the reals, \(\mathbb{R}_0, \mathbb{R}_{-1}, \mathbb{R}_{-2} \ldots\). Points \((n, x)\) in this space are ordered lexicographically, though we sometimes refer to these points by their second coordinate \(x\). A natural topology on this space is provided by the two-point compactification of each copy of the reals, glued together at the endpoints so as to respect the lexicographic ordering. (This ordering and topology can also be defined by considering the evolution of \(\arg(\psi' + i\psi)\) as a continuous, real-valued function, and applying the tangent map.)

It is not hard to verify the following.

**Fact 3.1.** The solution \(p_\lambda(x) = p(x, \lambda)\) of (3.2) is unique and increasing in \(\lambda\) for each \(x\). It is also decreasing in \(x\) at each blowup (or “explosion”). Moreover, the function \(p\) is continuous when the image space is considered in the topology discussed above.

Next consider the truncation \(H^L\) of \(H_\beta\), defined on the finite interval \([0, L]\) with Dirichlet \((\psi = 0)\) boundary conditions at both endpoints.

**Lemma 3.2.** Fix \(\lambda\), and denote \((-n, y) = p(L, \lambda)\). Then the number \(n\) of blowups of \(p(x, \lambda)\) to \(-\infty\) on \([0, L]\) equals the number of eigenvalues of \(H^L\) at most \(\lambda\).

**Proof.** Being classical, we provide just a sketch. First, \(\lambda\) is an eigenvalue of \(H^L\) if and only if \(p_\lambda\) blows up to \(-\infty\) at the endpoint \(L\). For large negative \(\lambda\), there are no blowups for any given noise path. As \(\lambda\) increases, continuity and monotonicity implies that existing blowups move towards the beginning of the interval — new ones can only appear at the endpoint \(L\). At those \(\lambda\) we have a new eigenvalue and the claim follows.

To extend the picture to the full line we need the following.
Lemma 3.3. As $L \to \infty$ the first $k$ eigenvalues $\Lambda_{L,0}, \ldots, \Lambda_{L,k-1}$ of $H_L$ converge to the first $k$ eigenvalues of $H_\beta$.

Proof. A trivial modification of the proof of Lemma 5.9 together with Fact 2.2 shows that $\lim \inf \Lambda_{L,k} \geq \Lambda_k$. Next, for an inductive proof, assume that the $\Lambda_{L,\ell} \to \Lambda_\ell$ for $\ell < k$.

Let $f_k^\varepsilon$ be a function of compact support $\varepsilon$-close to $f_k$ in $L^*$. Let also

$$g_L = g_{L,k} = f_k^\varepsilon - \sum_{\ell=0}^{k} \langle f_k^\varepsilon, f_{L,\ell} \rangle f_{L,\ell}.$$

The hypothesis entails that $f_{L,\ell} \to \lambda_{L,\ell}^2$. So, for large $L$, each coefficient in the above sum is bounded by $2\varepsilon$, and $g_L$ will be $c\varepsilon$-close to $f_k$ in $L^*$. Then, by the variational characterization we have

$$\lim \sup_{L \to \infty} \Lambda_{L,k} \leq \lim \sup_{L \to \infty} \frac{\langle g_L, H_\beta g_L \rangle}{\langle g_L, g_L \rangle},$$

since $f_k^\varepsilon$ is eventually supported on $[0, L]$. Then, as $\varepsilon \to 0$, the right hand side converges to $\langle f_k, H_\beta f_k \rangle / \langle f_k, f_k \rangle = \Lambda_k$.

Taking the $L \to \infty$ limit of the claims of Lemma 3.2 then yields

Proposition 3.4. Let $N(\lambda)$ be the number of blowups of the equation (3.2) to $-\infty$. Then for almost all $\lambda$, $N(\lambda)$ equals the number of eigenvalues of $H_\beta$ at most $\lambda$. In other words, the cadlag version of $N(\lambda)$ is $H_\beta$'s eigenvalue counting function.

Of course, for any fixed $\lambda$, the Riccati equation may be taken in the Itô sense,

$$dp(x) = -\frac{2}{\sqrt{\beta}} db_x + (x - \lambda - p^2(x)) dx,$$

which is to say that $p = p_\lambda = (\log \psi)'$ performs the indicated diffusion, restarted at $+\infty$ instantaneously after each explosion to $-\infty$. The content of the above is that the total explosion count equals the count of eigenvalues $\leq \lambda$.

Proof of Theorem 1.2. The strong Markov property of the motion (3.3) implies that the sequence of explosion times, $m_0 = 0, m_1, m_2, \ldots$ is itself a Markov process. Let $\kappa(x, \cdot)$ be the distribution of the first such time of $p_\lambda$ under $P_{(\infty,x)}$, that is, when started from $\infty$ at time $x$. This law is supported on $(x, \infty]$ with a point mass at $\infty$. By the preceding,

$$P(\Lambda_{k-1} < \lambda) = P_{(\infty,0)}(p_\lambda(x) \text{ has at least } k \text{ explosions})$$

$$= P_{(\infty,-\lambda)}(p_0(x) \text{ has at least } k \text{ explosions})$$

$$= \int_{\mathbb{R}^k} \kappa(-\lambda, dx_1) \kappa(x_1, dx_2) \ldots \kappa(x_{k-1}, dx_k).$$

The second equality uses the obvious translation equivariance of $p$. \qed
Immediate is the fact that the $\mathcal{H}_\beta$ eigenvalues are a.s. distinct. It also follows that this set has no accumulation point, or that the procedure of the previous section exhausts the full $\mathcal{H}_\beta$ eigenvalue set. In particular, we have

**Proposition 3.5.** A.s., $\Lambda_k \to \infty$ as $k \to \infty$.

*Proof.* Starting at any time $x$, there is positive probability $\kappa(x, \{\infty\})$ that $p_0$ will converge to $\infty$ without blowing up to $-\infty$. Monotonicity implies that $\kappa(x, \{\infty\})$ is increasing in $x$. Thus the number of eigenvalues below $\lambda$ is dominated by a geometric random variable with parameter $\kappa(-\lambda, \{\infty\})$, whence it is finite. The claim follows. \qed

We mention that Proposition 3.5 may also be established by making sense of the resolvent operator of $\mathcal{H}_\beta$ and showing it maps the unit ball in $L^*$ onto Hölder($3/2$) functions vanishing at infinity. The simplicity of the above proof demonstrates the advantages of the diffusion picture.

This connection between the limiting top eigenvalues of the random matrix ensembles and the explosion time of a simple, one-dimensional diffusion is new even in the deeply studied cases of $\beta = 1, 2$ or 4. We now recall the formulas of Tracy-Widom which, in conjunction with our result, produce the identities

$$P_{(\infty, \lambda)}(m_1 = \infty) = \begin{cases} \exp\left(-\frac{1}{2} \int_\lambda^\infty (s - \lambda)u^2(s) \, ds\right) \exp\left(-\frac{1}{2} \int_\lambda^\infty u(s) \, ds\right), & \beta = 1, \\ \exp\left(-\int_\lambda^\infty (s - \lambda)u^2(s) \, ds\right), & \beta = 2, \\ \exp\left(-\frac{1}{2} \int_{\lambda'}^\infty (s - \lambda')u^2(s) \, ds\right) \cosh(\int_{\lambda'}^\infty u(s) \, ds), & \beta = 4. \end{cases} \quad (3.4)$$

Here, $u(s)$ is the solution of $u'' = su + 2u^3$ (Painlevé II) subject to $u(s) \sim \text{Ai}(s)$ as $s \to +\infty$, and $\lambda' = 2^{2/3}\lambda$ in the $\beta = 4$ distribution. An important problem for the future is to obtain the equivalent of the closed Tracy-Widom formulas for general $\beta$. Even a direct verification of (3.4) would be interesting.

**Remark 3.6.** The diffusion (3.3) seems efficient for simulating Tracy-Widom distributions as well as distributions of higher eigenvalues. First note that for $x \ll 1$, $p(x)$ comes down from $+\infty$ like $1/x$. Also, the more time accumulated inside the parabola $\rho_\lambda = \{p^2(x) \leq x + \lambda\}$, the less likely explosion becomes. That is, the typical path which hits $-\infty$ does so by tunneling through the narrow part of $\rho_\lambda$. In line with these heuristics good simulations of the general TW$_\beta$ distributions may be obtained by tracking the explosion probability for $p(x)$ begun at say $p(0) = O(10^3)$ and run only for $O(1)$ time.
Moving to applications of the Riccati map to the eigenfunctions, a detailed but standard analysis of the diffusion (3.3) using domination arguments shows that for each \( \lambda \), with probability one, \( p(x)/\sqrt{x} \to 1 \) (after a finite number of initial blowups and restarts). By Fubini, this holds for almost all \( \lambda \). By monotonicity, this happens for all \( \lambda \) except for eigenvalues. Thus we get

**Proposition 3.7.** A.s. for all solutions \( f \in H^1_{\text{loc}} \) of \( \mathcal{H}_\beta f = \lambda f \), with \( f(0) = 0 \) we have the following. If \( f \) is not an eigenfunctions then \( f'(x)/(f(x)\sqrt{x}) \to 1 \). In particular, if \( f \) grows slower than \( \exp((2/3 - \varepsilon)x^{3/2}) \) then \( f \) is an eigenfunction.

**Remark 3.8.** Note that the results of this section, and indeed, the entire paper, easily extend to general initial boundary conditions.

We conclude this section with a decay bound on the \( \mathcal{H}_\beta \) eigenfunctions. Compare this to the noiseless (\( \beta = \infty \)) limit in which case all eigenfunctions are simply shifts of the Airy function \( \text{Ai}(\cdot) \).

**Proposition 3.9.** There are random constants where \( C_{f,\varepsilon} \) so that a.s. for \( \varepsilon > 0 \) and all eigenfunctions \( f \) of \( \mathcal{H}_\beta \) we have

\[
|f(x)| \leq C_{f,\varepsilon} \exp(-(2/3 - \varepsilon)x^{3/2}).
\]

**Proof.** Let \( p, q \) be the solutions of (3.2) corresponding to Dirichlet and Neumann boundary conditions at 0 (i.e. \( q(0) = 0 \)), and let \( p \) correspond to the specified eigenfunction \( f \). Then \( q \) cannot correspond to an eigenfunction of the Neumann problem and so by Proposition 3.7 (in conjunction with Remark 3.8) \( q(x)/\sqrt{x} \to 1 \). Just from the differential equation (3.2) we have

\[
\frac{d}{dx}(q - p) = -(q - p)(q + p),
\]

and so

\[
(q - p)(x) = C \exp \left( \int_{m}^{x} -(q + p)(y) \, dy \right) \tag{3.5}
\]

for \( m < \infty \) some random time past the final explosion of \( q \) with \( p(m) \) finite. With the notation \( Q(x) = \int_{m}^{x} q(y) \, dy, P(x) = \int_{m}^{x} p(y) \, dy, \) and \( R(x) = Q(x) - P(x) \) we have

\[
R'(x) = C \exp(R(x) - 2Q(x)). \tag{3.6}
\]

Now \( Q(x) = 2/3x^{3/2}(1 + o(1)) \), and \( P(x) = C + \log |f(x)| \to -\infty \), which implies that

\[
R(x) - x^{3/2} \to \infty. \tag{3.7}
\]
It now suffices to show that for all \( \varepsilon > 0 \) and \( x > x_0(\varepsilon) \) we have \( R(x) \geq 4/3(1 - \varepsilon)x^{3/2} \).
Assume the contrary. Then we can find \( x_0 \) large so that the right hand side of (3.6) at \( x = x_0 \) is at most 1 and for \( x > x_0 \) we have \( q(x) > 1/2 \). We now claim that the solutions of the ODE (3.6), started at \( x = x_0 \) are dominated by the solutions of \( \hat{R}'(x) = 1 \). Indeed, when \( R(x) \leq \hat{R}(x) = C + x \) for all large \( x \), contradicting (3.7).

\[ C \exp(R(x) - 2Q(x)) \leq C \exp(\hat{R}(x) - 2Q(x)) = C \exp \left( R(x_0) - 2Q(x_0) + \int_{x_0}^{x} 1 - 2q(y) \, dy \right) \leq 1 \]

so that the monotonicity can be maintained. Thus \( R(x) \leq \hat{R}(x) = C + x \) for all large \( x \), contradicting (3.7).

4 Tracy-Widom tail bounds: an application of SAE\( \beta \)

This section contains the proof of Theorem 1.3.

With \( \Lambda_0(\beta) = -TW_\beta \) now defined by (2.6), both the upper bound on \( P(TW_\beta < -a) \) and the lower bound on \( P(TW_\beta > a) \) follow from suitable choices of test function \( f \) in

\[ \langle f, \mathcal{H}_\beta f \rangle = \int_0^\infty \left[ f'(x)^2 + xf^2(x) \right] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) \, db_x \geq \Lambda_0 \| f \|_2^2. \]

The other two bounds run through the Riccati correspondence.

**Lower bound, right tail.** Begin with the observation that

\[ P(TW_\beta > a) = P(\Lambda_0(\beta) < -a) \geq P\left( \langle f, \mathcal{H}_\beta f \rangle < -a \langle f, f \rangle \right) \]

\[ = P\left( \frac{2}{\sqrt{\beta}} \| f \|_4^2 \, g < -a \| f \|_2^2 - \| f' \|_2^2 - \| \sqrt{x} f \|_2^2 \right) \]

for any choice of \( f \in L^* \), and \( g \) a standard Gaussian variable. Here we have used the rule that for \( h \) deterministic, \( \int_0^\infty h \, db \) is a centered Gaussian with variance \( \| h \|_2^2 \).

Wishing to maximize this probability, the observation is: for there to be a large negative eigenvalue the random potential, and then so also the eigenfunction \( f \), should be localized. This leads one to neglect the \( \int_0^\infty xf^2 \) term and look for the maximizer \( f \) of the expression entering the Gaussian tail, namely

\[ \frac{\int_0^\infty (af^2 + f'^2)}{(\int_0^\infty f^2)^{1/2}}. \]
Viewing this problem on the whole line and neglecting boundary conditions, the optimizers can be computed exactly: \( f(x) = c_1 \text{sech}(\sqrt{a} x + c_2) \).

Note that \( \int_{-\infty}^{\infty} \text{sech}^2(x)dx = 2, \int_{-\infty}^{\infty} (\text{sech}'(x))^2dx = 2/3, \) and \( \int_{-\infty}^{\infty} \text{sech}^4(x)dx = 4/3. \) Let \( f(x) = \text{sech}(\sqrt{a}(x - 1)). \) Then, on \( \mathbb{R}^+ \), with \( \sim \) denoting asymptotics as \( a \uparrow \infty \) we have

\[
a \|f\|^2_2 \sim 2\sqrt{a}, \quad \|f'\|^2_2 \sim \frac{2}{3} \sqrt{a}, \quad \|\sqrt{x}f\|^2_2 = O\left(\frac{1}{\sqrt{a}}\right), \quad \|f^4\|_4 \sim \frac{4}{3\sqrt{a}}.
\]

Further, while \( f(0) \neq 0 \), it decays exponentially there as \( a \to \infty \), allowing for an admissible modification which shares the above evaluations. Returning to (4.1) we find that

\[
P(TW_\beta > a) \geq P\left(\frac{2}{\sqrt{\beta}} \times \frac{2}{\sqrt{3}} a^{-1/4} g < -a^{1/2} \left(2 + \frac{2}{3} + o(1)\right)\right),
\]

producing the desired bound from the simple Gaussian tail estimate \( P(g > c) = e^{-c^2(1/2+o(1))}. \)

**Upper bound, left tail.** The reasoning is the same as that just employed, though in minimizing the right hand side of \( P(TW_\beta < -a) \leq P(\langle f, H_\beta f \rangle > a\langle f, f \rangle) \) one expects the optimal \( f \) to be relatively “flat”. Neglecting the \( \int (f')^2 \) term leads to the choice

\[
f(x) = (x \sqrt{a}) \wedge (a-x)^+ \wedge (a-x)^+.
\]

The middle term is dominant, while the others control \( \|f'\|_2 \). Then

\[
a \|f\|^2_2 \sim \frac{a^3}{2}, \quad \|f'\|^2_2 = O(a), \quad \|\sqrt{x}f\|^2_2 \sim \frac{a^3}{6}, \quad \|f^4\|_4 \sim \frac{a^3}{3}.
\]

The proof is completed by substitution

\[
P(TW_\beta < -a) \leq P\left(\frac{2}{\sqrt{\beta}} \times \frac{1}{\sqrt{3}} a^{3/2} g > a^3 \left(\frac{1}{2} - \frac{1}{6} + o(1)\right)\right) = \exp\left(- \frac{\beta}{24} a^3 (1 + o(1))\right).
\]

**Lower bound, left tail.** For this we use the diffusion description of Theorem 1.2, namely

\( P(TW_\beta < -a) = P_{(\infty, -a)}(p \text{ never explodes}), \) where \( p \) is the diffusion (1.8), and the subscript indicates the space-time starting point. By monotonicity,

\[
P_{(\infty, -a)}(p \text{ never explodes}) \geq P_{(1, -a)}(p \text{ never explodes}) \geq P_{(0, -a)}(p(x) \in [0, 2] \text{ for all } x \in [-a, 0]) P_{(0, 0)}(p \text{ never explodes}).
\]

The last factor in line two is some positive number not depending on \( a \). To bound the first probability from below, employ the Cameron-Martin formula

\[
P_{(1, -a)}(p(x) \in [0, 2] \text{ for all } x \in [-a, 0]) = E_{(1, -a)} \left[ \exp\left(-\frac{\beta}{4} \int_{-a}^{0} (x - b_x^2)db_x - \frac{\beta}{8} \int_{-a}^{0} (x - b_x^2)^2dx\right) \right] ; b_x \in [0, 2] \text{ for all } x \leq 0.
\]
where $b_x$ is a Brownian motion with diffusion coefficient $2/\sqrt{\beta}$. On the event in question,

$$ \frac{\beta}{8} \int_{-a}^{0} (x - b_x^2)^2 \, dx = \frac{\beta a^3}{24} + O(a^2), $$

and

$$ \int_{-a}^{0} (x - b_x^2) \, db_x = -ab_a + \frac{1}{3} (b_{-a}^3 - b_0^3) = O(a). $$

To finish, note that $P(-a,0)(b_x \in [0,2] \text{ for } x \leq 0) \geq e^{-ca}$, and so does not interfere with the leading order.

**Upper bound, right tail.** For the final bound write

$$ P(TW_\beta > a) = P_\infty(m < \infty), $$

where $m$ is the passage time to $-\infty$ of the diffusion

$$ dp(x) = \frac{2}{\sqrt{\beta}} db_x + (a + x - p^2(x)) \, dx, $$

started at $p(0) = \infty$. Next, introduce

$$ M(p(x),x) = \exp \left( \frac{\beta}{2} \left[ \frac{1}{3} p^3(x) - (a + x)p(x) \right] \right), $$

which is a local martingale — an application of Itô’s rule produces a zero drift term. Introduce as well the critical parabolas

$$ \rho_+ = \left\{ p, x : p = \sqrt{a + x} \right\}, \quad \rho_- = \left\{ p, x : p = -\sqrt{a + x} \right\}, $$

so named as

$$ M(p,x) = \exp \left[ \pm \frac{\beta}{3} (a + x)^{3/2} \right] \text{ on } \rho_\pm. $$

On the event $m < \infty$, the diffusion $p$ must first hit $\rho_+$ and then $\rho_-$ at times $m_+$ and $m_-$. Plainly $P_\infty(m < \infty) \leq P_\infty(m_- < \infty)$, and by the strong Markov property

$$ P_\infty(m_- < \infty) = E_{\infty} \left[ P_{(p(m_+),m_+)}(m_- < \infty) \right] \leq \sup_{v_+ \in \rho_+} P_{v_+}(m_- < \infty). $$

Now let $\tau$ be the first exit time for the process $(M,p,x)$ from some compact set. $M(p(x \wedge \tau), x \wedge \tau)$ is then a martingale, as the diffusion coefficient in its Itô expansion is bounded. The optional stopping theorem and the fact that $M \geq 0$ gives, for any $v_+ \in \rho_+$,

$$ e^{-\beta a^{3/2}/3} \geq M(v_+) = E_{v_+} \left[ M(p(m_- \wedge \tau), m_- \wedge \tau) \right] \geq E_{v_+} \left[ M(p(m_-), m_-) 1(m_- < \tau) \right] \geq P_{v_+}(m_- < \tau) \inf_{v_- \in \rho_-} M(v_-). $$
Taking $\tau \uparrow \infty$ by increasing the compact set, the monotone convergence theorem produces
\[ e^{-\beta a^{3/2}/3} \geq P_{v_+}(m_- < \infty) \inf_{x > 0} e^{\beta(a+x)^{3/2}/3}, \]
or that $P_{v_+}(m_- < \infty) \leq e^{-(2/3)\beta a^{3/2}}$, as required.

5 Convergence of discrete models and universality

This section establishes a general and rather weak set of conditions under which the bottom eigenvalues of random symmetric tridiagonal matrices converge to the bottom eigenvalues of a corresponding stochastic differential operator. In many ways this is the central result of the paper.

To explain the setup, consider a sequence of discrete-time $\mathbb{R}^2$-valued random sequences $((y_{n,1,k}, y_{n,2,k}); 1 \leq k \leq n)$. Let $m_n = o(n)$ be a scaling parameter. (In the particular case of the Hermite $\beta$-ensembles we have $m_n = n^{1/3}$.) For each $n$, we build an $n \times n$ tridiagonal matrix $H_n$.

Let $T_n$ denote the shift operator $(T_nv)_k = v_{k+1}$ acting on $\mathbb{R}_1 \times \mathbb{R}_2 \ldots$. Let $(T^*_n v)_k = v_{k-1} 1_{k \geq 1}$ be its adjoint, and let $R_n$ denote the restriction operator $(R_nv)_k = v_{k} 1_{k \leq n}$. Let also $\Delta_n = m_n(I - T^*_n)$ be the difference quotient operator, and finally set
\[ H_n = R_n \left( -\Delta_n \Delta^*_n + (\Delta_n y_{n,1}) \times + (\Delta_n y_{n,2}) \times \frac{1}{2}(T_n + T^*_n) \right), \] (5.1)
where the subscript $\times$ denotes element-wise multiplication by the corresponding vector. Then $H_n$ maps the coordinate subspace $\mathbb{R}^n \to \mathbb{R}^n$, and its matrix with respect to the coordinate basis in $\mathbb{R}^n$ is symmetric tridiagonal with $(2m_n^2 + m_n(y_{n,1,k} - y_{n,1,k-1}), k \geq 1)$ on the diagonal and $(-m_n^2 + m_n(y_{n,2,k} - y_{n,2,k-1})/2, k \geq 1)$ below and above the diagonal. Roughly speaking, $H_n$ is the discrete Laplacian plus integrated potential $y_{n,1} + y_{n,2}$.

Additionally, define $y_{n,i}(x) = y_{n,i,\lfloor xm_n \rfloor} 1_{xm_n \in [0,n]}$. By choice, $(\sqrt{m_n} \times y_{n,i,k}, k \geq 0)$ is on the scale of a simple random walk, so no additional spatial scaling will be required.

Our basic convergence result rests on two sets of assumptions on the processes $y_{n,i=1,2}$.

Assumption 1 (Tightness/Convergence) There exists a continuous process $x \mapsto y(x)$ such that
\[ (y_{n,i}(x); x \geq 0) \quad i = 1, 2 \quad \text{are tight in law,} \]
\[ (y_{n,1}(x) + y_{n,2}(x); x \geq 0) \Rightarrow (y(x); x \geq 0) \quad \text{in law,} \] (5.2)
with respect to the Skorokhod topology of paths, see [13] for definitions.

**Assumption 2 (Growth/Oscillation bound)** There is a decomposition

\[ y_{n,i,k} = m_n^{-1} \sum_{\ell=1}^{k} \eta_{n,i,\ell} + w_{n,i,k} \tag{5.3} \]

for \( \eta_{n,i,k} \geq 0 \), deterministic, unbounded nondecreasing continuous functions \( \bar{\eta}(x) > 0, \zeta(x) \geq 1 \), and random constants \( \kappa_n(\omega) \geq 1 \) defined on the same probability space which satisfy the following. The \( \kappa_n \) are tight in distribution, and, almost surely,

\[ \bar{\eta}(x)/\kappa_n - \kappa_n \leq \eta_{n,1}(x) \leq \kappa_n(1 + \bar{\eta}(x)), \tag{5.4} \]

\[ \eta_{n,2}(x) \leq 2m_n^2 \tag{5.5} \]

\[ |w_{n,1}(\xi) - w_{n,1}(x)|^2 + |w_{n,2}(\xi) - w_{n,2}(x)|^2 \leq \kappa_n(1 + \bar{\eta}(x)/\zeta(x)). \tag{5.6} \]

for all \( n \) and \( x, \xi \in [0, n/m] \) with \( |x - \xi| \leq 1 \).

We may now define the limiting operator. Just as in Section 2 we note that

\[ H = -\frac{d^2}{dx^2} + y(x) \tag{5.7} \]

maps \( H_{1,loc}^1 \) to the space \( D \) of distributions via integration by parts. Without changing the notation, we generalize the Hilbert space \( L^* \subset L^2(\mathbb{R}^+) \) introduced there. This consists of functions with \( f(0) = 0 \) and

\[ \|f\|_*^2 = \int_0^\infty f'(x)^2 + (1 + \bar{\eta}(x)) f^2(x) dx < \infty. \]

The eigenvalues and eigenfunctions are defined again as \( (\lambda, f) \in \mathbb{R} \times L^* \setminus \{0\} \) with \( \|f\|_2 = 1 \) satisfying (5.7). Recall from Section 2 that this means

\[ f'(x) = \int_0^x -y(z)f'(z) - \lambda f \, dz + f(x)y(x), \]

or, equivalently, for all \( \varphi \in C_0^\infty \) it holds

\[ \int f \varphi'' dx = \int -\lambda f \varphi - y f' \varphi - y f \varphi' dx. \]

With the picture laid out, a few words on Assumptions 1 and 2 are in order. The former simply asks for correspondence between \( H_n \) and \( H \) at the level of integrated potentials. The latter, more technical condition, will imply the compactness necessary to maintain discrete spectrum as \( n \uparrow \infty \).
Theorem 5.1 (Convergence in law). Given Assumption 1 and 2 above and any fixed $k$, the bottom $k$ eigenvalues of the matrices $H_n$ converge in law to the bottom $k$ eigenvalues of the operator $H$.

We will also show that, after a natural embedding, the eigenfunctions also converge in $L^2$.

Proof: Reduction to the deterministic case. It will be convenient to find subsequences along which we have limits for all desired quantities.

The upper bound (5.4) shows that $(\int_0^x \eta_{n,i}(t); x \geq 0)$ is tight in distribution for $i = 1, 2$. For any subsequence we can extract a further subsequence so that we have joint distributional convergence

\[
\left( \int_0^x \eta_{n,i}(t); \ x \geq 0 \right) \Rightarrow \left( \eta_i(t); \ x \geq 0 \right),
\]

\[
(y_{n,i}(x); \ x \geq 0) \Rightarrow (y_i(x); \ x \geq 0),
\]

where the convergence in the first lines is in the uniform-on-compacts topology, and the second, in the Skorokhod topology. Then by Skorokhod’s representation theorem (see Theorem 1.8, Chapter 2 of [13]) we can realize this convergence as a.s. convergence on some probability space so that the conditions of Proposition 5.2 below are satisfied with probability one.

Note that (5.4-5.5) are local Lipschitz bounds on the $\int \eta_{n,i}$, and so they are inherited by their limit $\eta_i$. Thus $\eta_i = (\eta_i)^\dagger$ is defined almost everywhere, and satisfies (5.4-5.5). Further, $\eta_i$ can be defined everywhere so that (5.4-5.5) continues to hold.

It also follows that each $w_{n,i} = y_{n,i} - \sum \eta_{n,i}$ must have a limit which we denote $w_i$. We further denote $w = w_1 + w_2$ and $\eta = \eta_1 + \eta_2$. The claim now follows from Proposition 5.2 below.

\[
\kappa_n \Rightarrow \kappa,
\]

Proposition 5.2 (Deterministic convergence). Assume that the each of the convergence statements in (5.8) hold deterministically and that the bounds (5.4-5.6) hold with some deterministic constant $\kappa$ replacing $\kappa_n$. Then, for any $k$, the lowest $k$ eigenvalues of the matrices $H_n$ converge to the lowest $k$ eigenvalues of $H$.

In the next subsection, we establish some properties of the limiting operator. Afterwards, we prove Proposition 5.2.
Properties of the limiting operator

Just as in Section 2, we extend the bilinear form $\langle \cdot, H \cdot \rangle$ from $C_0^\infty \times L^*$ to $L^* \times L^*$. We want to define the extension as $\langle f, Hf \rangle := \int f'^2 + \eta f^2 dx + \int f^2 dw$ (with $\langle f, Hg \rangle$ then defined by polarization), but we still need to define and control the last term. By the next lemma, this can be done via the integration by parts already employed in (2.4):

$$\int_0^\infty f^2(x) dw_x = \int_0^\infty f^2(x) (w_{x+1} - w_x) dx + 2 \int_0^\infty f'(x) f(x) (\int_x^{x+1} w_t dt - w_x) dx. \quad (5.9)$$

**Lemma 5.3.** The integrals on the right of (5.9) are defined and finite for $f \in L^*$. Moreover there exist $c_8, c_9, c_{10} > 0$ so that

$$c_8 \| f \|_s^2 - c_9 \| f \|_2^2 \leq \langle f, Hf \rangle \leq c_{10} \| f \|_s^2.$$

**Proof.** By taking limits of the inequalities (5.4-5.6) on $\eta_{h,n}$ and $w_n$ we get bounds for $\eta$ and $w$. In particular $\max(|w_{x+1} - w_x|, |w_{x+1} - w_x|^2)$ $\leq c_\varepsilon + \varepsilon \bar{\eta}$ where $\varepsilon$ can be made small. Now we write $\langle f, Hf \rangle = A + B$ where $B$ is the fluctuation term (5.9), and the potential term satisfies $\frac{1}{\varepsilon} \| f \|_s^2 - c \| f \|_2^2 \leq A \leq c \| f \|_s^2$. To bound $B$, first write

$$\int_0^\infty f^2(x) |w_{x+1} - w_x| dx \leq \langle f, (c_\varepsilon + \varepsilon \bar{\eta}) f \rangle.$$

For the second term, we average the inequality $\sup_{|x-\xi| \leq 1} |w_\xi - w_x| \leq |c_\varepsilon + \varepsilon \bar{\eta}(x)|^{1/2}$ and use an inequality of means:

$$2 \int_0^\infty \langle f'(x) f(x) (\int_x^{x+1} w_t dt - w_x) \rangle dx \leq \sqrt{\varepsilon} \langle f'(x) \rangle^2 + \langle f, \frac{1}{\sqrt{\varepsilon}} (c_\varepsilon + \varepsilon \bar{\eta}) f \rangle.$$

The above inequalities give $|B| \leq 2\sqrt{\varepsilon} \| f \|_s^2 + c'_\varepsilon \| f \|_2^2$. Setting $\varepsilon$ small we get the results. □

The bounds immediately imply the following.

**Corollary 5.4.** (i) The bilinear form $\langle \cdot, H \cdot \rangle : L^* \times L^* \to \mathbb{R}$ is continuous. (ii) It does not depend on the decomposition $y = w + \int \eta$. (iii) The eigenvalues and eigenfunctions $(\lambda, f)$ of $H$ satisfy $\langle g, Hf \rangle = \lambda \langle g, f \rangle$ for all $g \in L^*$. (iv) In particular, $\langle f, Hf \rangle = \lambda \langle f, f \rangle$.

**Proof of the Corollary.** Since $L^* \subset L^2$ is a continuous embedding, it suffices to prove that $\langle \cdot, H \cdot \rangle + c_9 \langle \cdot, \cdot \rangle$ is continuous. This form is nonnegative definite by the lemma. Continuity (i) now follows from Cauchy-Schwarz applied to the form and the bounds of the Lemma. For (ii) and (iii) approximate $g$ by smooth compactly supported functions and use continuity. □
Together, these two statements provide an analogue of Lemma 2.5 in a more general context. In particular, the (discrete) eigenvalues of $H$ may now be defined variationally. The arguments used in Section 2 in conjunction with the limiting bounds of (5.4-5.6) give the following.

**Lemma 5.5.** The lowest $k$ elements of the set of eigenvalues of $H$ exist admit the variational characterization via the bilinear form $\langle f, H f \rangle$.

**Tightness**

Next, we define the discrete analogue of the norm $\| \cdot \|$. For $v \in \mathbb{R}^n$ let $\|v\|_2^2 = m_n^{-1} \sum_{k=1}^n v_k^2$ with scaling to match the continuum norm. Let $\bar{\eta}_{n,k} = \bar{\eta}(k/m_n)$ and let

$$
\|v\|_n^2 = \|\Delta_n v\|_2^2 + \|\bar{\eta}_{n,v}\|_2^2 + \|v\|_2^2,
$$

and note that $\|v\|_2 \leq \|v\|_n$. We continue with a bound on $H_n$.

**Lemma 5.6.** Assume the bounds (5.4-5.6). Then there are constants $c_{11}, c_{12}, c_{13} > 0$ so that for all $n$ and all $v$ we have

$$
c_{11}\|v\|_n^2 - c_{12}\|v\|_2^2 \leq \langle v, H_n v \rangle \leq c_{13}\|v\|_n^2.
$$

**Proof.** Drop the subscript $n$, and recall the definition of the difference quotient $\Delta v_k = m(v_{k+1} - v_k)$. We recall the following consequence of the discretized bounds (5.4-5.6):

$$
\begin{align*}
\bar{\eta}_k / \kappa - \kappa &\leq \eta_{1,k} + \eta_{2,k} \leq \kappa \bar{\eta}_k + \kappa \\
\eta_{2,k} &\leq 2m^2, \\
|w_{i,\ell} - w_{i,k}| &\leq \varepsilon \bar{\eta}_k + c_{\varepsilon}, \quad k \leq \ell \leq k + m, \quad i = 1, 2.
\end{align*}
$$

Here $\varepsilon$ can be arbitrarily small at the expense of $c_{\varepsilon}$. Let $w_k = w_{1,k}$, $u_k = w_{2,k}$. By definition of $H_n$,

$$
m\langle v, H_n v \rangle = \sum_{k=0}^n \left( (\Delta v_k)^2 + \eta_{2,k} v_k v_{k+1} + \eta_{1,k} v_k^2 \right) + \sum_{k=0}^n \left( \Delta u_k v_k^2 + \Delta u_k v_{k+1}^2 \right). \quad (5.11)
$$

Let $A, B$ denote the two sums. Using the inequality $ab \geq -(b - a)^2/3 + a^2/4$, we obtain the following lower bound for the second summand in $A$:

$$
\eta_{2,k} v_k v_{k+1} \geq -\frac{\eta_{2,k}}{3} (v_{k+1} - v_k)^2 + \frac{\eta_{2,k} v_k^2}{4} \geq -\frac{2}{3} \langle \Delta v_k \rangle^2 + \frac{\bar{\eta}_{2,k} v_k^2}{4 \kappa} - \frac{\kappa v_k^2}{4},
$$

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thus we get $A \geq \frac{mn}{2} \|v\|_r^2 - cm\|v\|^2$. We also clearly have $|A| \leq cm\|v\|^2$. To bound $|B|$, set

$$\delta w_k = (w_{k+m} - w_k).$$

Summation by parts gives

$$\sum_{k=0}^{n} \Delta w_k v_k^2 = \sum_{k=0}^{n} (\Delta w_k - \delta w_k) v_k^2 + \sum_{k=0}^{n} \delta w_k v_k^2$$

$$= \sum_{k=0}^{n} \left( \sum_{\ell=k+1}^{k+m} (w_{\ell} - w_k) \right) (v_{k+1}^2 - v_k^2) + \sum_{k=0}^{n} \delta w_k v_k^2. \quad (5.12)$$

By the bound on $w$ the absolute value of the first summand in (5.12) is not more than

$$m (\varepsilon \bar{\eta}_k + c_\varepsilon)^{1/2} |v_{k+1}^2 - v_k^2| \leq \frac{1}{\sqrt{\varepsilon}} (\varepsilon \bar{\eta}_k + c_\varepsilon) (v_k + v_{k+1})^2 + \sqrt{\varepsilon} m^2 (v_{k+1} - v_k)^2.$$ 

Together with the bound on the second sum in (5.12) this yields

$$m \left| \sum_{k=0}^{n} \Delta w_k v_k^2 \right| \leq \left( \frac{1}{\sqrt{\varepsilon}} + 1 \right) \sum_{k=0}^{n} (\varepsilon \bar{\eta}_k + c_\varepsilon) v_k^2 + \sqrt{\varepsilon} m^2 \sum_{k=0}^{n} (v_{k+1} - v_k)^2. \quad (5.13)$$

The argument for $u$ starts with the same summation by parts, with $v_k v_{k+1}$ playing the part of $v_k^2$. After bounding the $u$ terms, we use the inequalities $2|v_k u_{k+1}| \leq v_k^2 + u_{k+1}^2$ and

$$|v_k u_{k-1} - v_{k+1} u_{k+1}| \leq |v_k| |u_{k-1}| + |v_{k+1}| |u_{k+1} - u_k|$$

together with Cauchy-Schwarz to get an estimate of the form (5.13) for $\sum_{k=0}^{n} \Delta u_k v_k v_{k+1}$. Thus we find that $|B| \leq c\sqrt{\varepsilon} m \|v\|_r^2 + c_\varepsilon m \|v\|_2^2$. For $\varepsilon$ sufficiently small the claims follow.

**Operator convergence**

**Embedding vector spaces.** We embed the domain $\mathbb{R}^n$ of $H_n$ in $L^2(\mathbb{R}^+)$ in an isometric way, identifying $v \in \mathbb{R}^n$ with the step function $v(x) = v_{[m_n x]}$ supported on $[0, n/m_n]$. Let $L_n^*$ denote the space of such step functions, and let $P_n$ denote the $L^2$-projection to this space. Let $(T_n f)(x) = f(x + m_n^{-1})$ denote the shift operator, and let $R_n(f) = f 1_{[0,n/m_n]}$ denote the restriction. Let $\Delta_n = m_n(I - T_n^i)$. These operators are simply extensions of the already defined action of $T_n$ and $\Delta_n$ on $L_n^*$. Thus the formula (5.1) extends the definition of $H_n$ to $L^2$.

It is easy to check the following: (i) $P_n$ and $T_n$, and so $\Delta_n$ commute; (ii) for $f \in L^2$ we have $P_n f \to f$ in $L^2$; and (iii) when $f' \in L^2$ and $f(0) = 0$ we have $\Delta_n f \to f'$ in $L^2$.

**Lemma 5.7.** Assume that $f_n \in L_n^*$ and $f_n \to f$ weakly in $L^2$ and $\Delta_n f_n \to f'$ weakly in $L^2$. Then for all $\varphi \in C_0^\infty$ we have $\langle \varphi, H_n f_n \rangle \to \langle \varphi, H f \rangle$. In particular

$$\langle P_n \varphi, H_n P_n \varphi \rangle = \langle \varphi, H_n P_n \varphi \rangle \to \langle \varphi, H \varphi \rangle. \quad (5.14)$$
Proof. Because we are dealing with \( \varphi \) of compact support, may drop the restriction part \( R_n \) for \( H_n \). The convergence

\[
\langle \varphi, \Delta_n \Delta_n^t f \rangle = \langle \Delta_n \Delta_n^t \varphi, f \rangle \to \langle \varphi', f \rangle = \langle \varphi, f'' \rangle
\]

is clear, so it remains to check the potential term. First note that if \( I \) is a finite interval, and \( g_n \to_{L^2} g \) and \( h_n \to h \) is \( L^2(I) \)-bounded and converges weakly in \( L^2(I) \) then

\[
\langle g_n, h_n 1_I \rangle \to \langle g, h 1_I \rangle.
\]  

(5.15)

Let \( I \) be a finite closed interval supporting \( \Delta_n \varphi, \varphi' \) and \( \varphi \). The potential term is

\[
\langle \varphi, (\Delta_n y_{n,1})x + (\Delta_n y_{n,2})x \frac{1}{2}(T_n + T_n^t) f \rangle.
\]

Setting \( y_n = y_{n,1} + y_{n,2} \), we first approximate the right hand side by

\[
\langle \varphi, (\Delta_n y_n)x f_n \rangle = \langle \Delta_n^t (\varphi f_n), y_n \rangle = \langle \varphi \Delta_n^t f_n + f_n \Delta_n^t \varphi + m_n^{-1} \Delta_n^t f_n \Delta_n^t \varphi, y_n \rangle = \langle \Delta_n^t f_n, \varphi y_n \rangle + \langle f_n, y_n \Delta_n^t \varphi \rangle + m_n^{-1} \langle \Delta_n^t f_n, y_n \Delta_n^t \varphi \rangle.
\]

The first two terms in the above converge to the desired limits by (5.15), and the last one converges to 0 because it is bounded without the extra scaling term. The error term in the above approximation comes as a sum of \( T_n \) and \( T_n^t \) terms; we consider twice the \( T_n^t \) part:

\[
|\langle \varphi, (\Delta_n y_{n,2})x (I - T_n^t f_n) \rangle| = |\langle \varphi, m_n^{-1} \Delta_n y_{n,2}, \Delta_n f_n \rangle| \leq \|m_n^{-1} \Delta_n y_{n,2} 1_I \|_2 \|\Delta_n f_n \|_2 \sup_{x \in \mathbb{R}} |\varphi(x)|.
\]  

(5.16)

Now \( m_n^{-1} \Delta_n y_{n,2} 1_I \) is the restriction to \( I \) of the difference \( y_{n,2} - T_n^t y_{n,2} \), in which both terms converge to \( y_2 \) in the Skorokhod topology. In particular, they converge a.e., and since they are locally bounded, their difference converges locally in \( L^2 \) to 0. This shows that (5.16) vanishes in the limit. We handle the \( T_n \) term similarly. \( \square \)

Lemma 5.8. Recall the discrete \( \| \cdot \|_s \) norm from (5.10). Assume that \( f_n \in L^*_n \), \( \|f_n\|_s \leq c \), and \( \|f_n\|_2 = 1 \). Then there exists \( f \in L^* \) and a subsequence \( n_k \) so that \( f_{n_k} \to_{L^2} f \) and for all \( \varphi \in C_0^\infty \) we have \( \langle \varphi, H_{n_k} f_{n_k} \rangle \to \langle \varphi, H f \rangle \).

Proof. Since \( f_n \) and \( \Delta_n f_n \) are bounded in \( L^2 \), we can find a subsequence along which \( f_n \to f \in L^2 \) and \( \Delta_n f_n \to \tilde{f} \in L^2 \) weakly. Considering \( \langle \Delta_n f_n, 1_{[0,t]} \rangle \) we get that \( \int \tilde{f} = f \), that is \( f \) has a differentiable version and \( \tilde{f} = f' \). The bounded nature of the \( \bar{\eta} \) terms in the \( L^*_n \)-norm gives sufficient tightness so that we have \( f \in L^* \) and \( f_n \to_{L^2} f \). The last part then follows from Lemma 5.7. \( \square \)
We break up the proof of Proposition 5.2 into two Lemmas. Let \((\lambda_{n,k}, v_{n,k}), k \geq 0\) be the lowest eigenvalues and the embedded normalized eigenfunctions of \(H_n\), and let \((\Lambda_k, f_k)\) be the same for \(H\).

**Lemma 5.9.** For \(k \geq 0\) we have \(\lambda_k = \lim \inf \lambda_{k,n} \geq \Lambda_k\).

**Proof.** Assume \(\lambda_k < \infty\). Since the eigenvalues of \(H_n\) are uniformly bounded below, we can find a subsequence so that \((\lambda_{n,1}, \ldots, \lambda_{n,k}) \to (\xi_1, \ldots, \xi_k = \lambda_k)\). By Lemma 5.6, the corresponding eigenfunctions have \(L_n^*\)-norm uniformly bounded, and Lemma 5.8 now implies that for a further subsequence, their \(L^2\) limit exists. Moreover, by the same lemma this limit must consist of orthonormal eigenfunctions of \(H\) with eigenvalues at most \(\lambda_k\). The orthonormality of the eigenfunction set shows that they correspond to \(k\) distinct states and the proof is finished. \(\square\)

**Lemma 5.10.** For \(k \geq 0\) we have \(\lambda_{k,n} \to \Lambda_k\) and \(v_{n,k} \to_{L^2} f_k\).

**Proof.** For an inductive proof, we assume the claim holds up to \(k-1\). First, we find \(f_{k}^\varepsilon \in C_0^\infty\) \(\varepsilon\)-close to \(f_k\) in \(L^*\). Consider the vector

\[
f_{n,k} = P_n f_{k}^\varepsilon - \sum_{\ell=1}^{k-1} \langle v_{n,\ell}, P_n f_{k}^\varepsilon \rangle v_{n,\ell}.
\]

We have a uniform bound on the \(L_n^*\) norm of \(v_{n,\ell}\) by Lemma 5.6, and \(\|P_n f_{k}^\varepsilon - v_{n,k}\|_2 \leq \|P_n f_{k}^\varepsilon - f_{k}^\varepsilon\|_2 + \|v_{n,k} - f_{k}^\varepsilon\|_2\), is, for large \(n\) bounded by \(2\varepsilon\). Thus the \(L_n^*\)-norm of the sum is bounded by \(c\varepsilon\). By the uniform bound \(\langle v, H_n v \rangle \leq c\|v\|_{L_n^*}\) of Lemma 5.6 and the variational characterization in finite dimensions we also have that

\[
\limsup_{n \to \infty} \lambda_{n,k} \leq \limsup_{n \to \infty} \frac{\langle f_{n,k}, H_n f_{n,k} \rangle}{\langle f_{n,k}, f_{n,k} \rangle} = \limsup_{n \to \infty} \frac{\langle P_n f_{k}^\varepsilon, H_n P_n f_{k}^\varepsilon \rangle}{\langle P_n f_{k}^\varepsilon, P_n f_{k}^\varepsilon \rangle} + o_\varepsilon(1),
\]

where \(o_\varepsilon(1) \to 0\) as \(\varepsilon \to 0\). Then (5.14) of Lemma 5.7 provides

\[
\lim_{n \to \infty} \langle P_n f_{k}^\varepsilon, H_n P_n f_{k}^\varepsilon \rangle = \langle f_{k}^\varepsilon, H f_{k}^\varepsilon \rangle,
\]

and therefore the right hand side of (5.18) equals

\[
\frac{\langle f_{k}^\varepsilon, H f_{k}^\varepsilon \rangle}{\langle f_{k}^\varepsilon, f_{k}^\varepsilon \rangle} + o_\varepsilon(1) = \frac{\langle f_k, H f_k \rangle}{\langle f_k, f_k \rangle} + o_\varepsilon(1).
\]

Now letting \(\varepsilon \to 0\) the right hand side converges to \(\langle f_k, H f_k \rangle / \langle f_k, f_k \rangle = \Lambda_k\). We have shown \(\lambda_{n,k} \to \Lambda_k\).

Lemma 5.8 implies that any subsequence of the \(v_{n,k}\) has a further subsequence converging in \(L^2\) to some \(g \in L^*\) satisfying \(H g = \Lambda_k g\). Thus \(g = f_k\), and so \(v_{n,k} \to_{L^2} f_k\). \(\square\)
6 CLT and tightness for tridiagonal $\beta$-ensembles

At last we verify that the $\beta$-Hermite and Laguerre ensembles satisfy the conditions (5.2)-(5.6) of Theorem 5.1, and so complete the proofs of Theorems 1.1 and 1.4.

The following theorem is what we need from the far more general Theorem 7.4.1 on page 354 in Ethier and Kurtz [13]. Denote $\Delta y_{n,k} = y_{n,k} - y_{n,k-1}$.

Corollary 6.1. Let $a \in \mathbb{R}$ and $h \in C_1(\mathbb{R}^+)$, and let $y_n$ be a sequence of processes with $y_{n,0} = 0$ and independent increments. Assume that

$$m_n E\Delta y_{n,k} = h'(k/m_n) + o(1), \quad m_n E(\Delta y_{n,k})^2 = a^2 + o(1), \quad m_n E(\Delta y_{n,k})^4 = o(1)$$

uniformly for $k/m_n$ on compact sets as $n \to \infty$. Then $y_n(t) = y_{n,\lfloor t m_n \rfloor}$ converges in law, with respect to the Skorokhod topology, to the process $h(t) + ab_t$, where $b$ is standard Brownian motion.

Proof. The time-homogeneity required in the theorem can be replaced by introducing a space coordinate recording time. The supremum increment bound of the theorem follows by Markov’s inequality from the fourth moment bound here. \hfill \square

The $\beta$-Hermite case

Starting with the scaled Hermite matrix ensembles $H_n = \tilde{H}_n^\beta$, we identify $m_n = n^{1/3}$. After rearranging some terms we find

$$y_{n,1,k} = w_{n,1,k} = -n^{-1/6}(2/\beta)^{1/2} \sum_{\ell=1}^k g_\ell,$$

$$y_{n,2,k} = n^{-1/6} \sum_{\ell=1}^k 2\left(\sqrt{n} - \frac{1}{\sqrt{\beta}} \chi_{\beta(n-k)}\right).$$

Also, by choosing $\eta_{n,2,k} = 2\sqrt{n} - 2\beta^{-1/2} \chi_{\beta(n-k)}$ both $w_{n,1,k}$ and $w_{n,2,k}$ are independent-increment martingales. Using the notation and results of Corollary 6.1 and standard moment computations for the normal and gamma distributions, we get the following.

Lemma 6.2. As $n \to \infty$ for the Skorokhod topology we have, in law

$$y_{n,i}(\cdot) \Rightarrow (2/\beta)^{1/2} b_x + x^2(i - 1), \quad i = 1, 2.$$

Independence of the $i = 1, 2$ cases now implies (5.2) of Assumption 1.
Lemma 6.3. The bounds (5.4), (5.5) of Assumption 2 hold with $\bar{\eta}(x) = x$.

Proof. There is the estimate
\[
\sqrt{r}(1 - 4/r) \leq \mathbb{E} \chi_r = \sqrt{2} \frac{\Gamma((r + 1)/2)}{\Gamma(r/2)} \leq \sqrt{r},
\]
and, with again $\eta_{n,2,k} = 2\sqrt{n} - 2\beta^{-1/2} \mathbb{E} \chi_{\beta(n-k)}$, it follows that
\[
k^n - 1/2 - c \leq \eta_{n,2,k} \leq 2kn^{-1/2} + c,
\]
where $c$ depends on $\beta$ only. \qed

Lastly, for (5.6) of Assumption 2, it suffices to prove a tight random constant bound on
\[
\sup_{k=1,\ldots,n/m} \sup_{\ell=0,\ldots,m_n} |w_{n,i,km_n+\ell} - w_{n,i,km_n}|
\]
Squaring, replacing the first sup by a sum, and then taking expectations gives
\[
\sum_{k=1}^{n/m_n} \mathbb{E} \sup_{\ell=0,\ldots,m_n} |w_{n,i,km_n+\ell} - w_{n,i,km_n}|^2 \leq \sum_{k=1}^{n/m_n} \frac{2\mathbb{E} |w_{n,i,(k+1)m_n} - w_{n,i,km_n}|^2}{k^2 - 2\varepsilon}.
\]
Here we used the $L^2$ maximal inequality for martingales, see Section 2.2 of [13]. The expectation is now bounded by a constant independent of $n, k$, and so is the entire sum, as required.

The $\beta$-Laguerre case

Once again [9] provides a family of tridiagonal “$\beta$-Laguerre ensembles”, with explicit eigenvalue densities interpolating between those at $\beta = 1, 2, 4$. Take the $n \times n$ bidiagonal random matrix
\[
W_{n,\kappa}^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix}
\tilde{\chi}_\kappa \\
\chi_{\beta(n-1)} & \tilde{\chi}_{\beta(n-2)} \\
& \ddots & \ddots \\
& & \chi_{\beta2} & \tilde{\chi}_{\beta(n+1)} \\
& & & \chi_{\beta} & \tilde{\chi}_{\beta(n+1)}
\end{bmatrix}, \quad (6.1)
\]
where $\{\chi_{\beta(n-k)}\}_{k=1,\ldots,n-1}$ and $\{\tilde{\chi}_{\beta(n-k)}\}_{k=1,\ldots,n-1}$ are independent sequences of (independent) $\chi$ variables of the indicated parameter. Here $\kappa \in \mathbb{R}$ and necessarily $\kappa > n - 1$. Then, by [9], the eigenvalues of $(W_{n,\kappa}^\beta)(W_{n,\kappa}^\beta)^\dagger$ have joint density (1.9).

While the above puts $\kappa > n - 1$, the obvious duality reproduces all known real and complex ($\beta = 1, 2$) results for any limiting ratio of dimensions $\kappa \to \infty$ and $n \to \infty$. Finally,
it should be mentioned that this $\beta$ family generalizes the so-called “null” Wishart ensembles, distinguishing the important class of $W\Sigma W^\dagger$ type matrices with non-identity $\Sigma$. For progress on the spectral edge in the non-null case, consult [2] and [12].

We now proceed with the proof of Theorem 1.4. It suffices to prove the claim along a further subsequence of any given subsequence. This allows us to assume that $\kappa = \kappa(n)$ is an increasing function of $n$, and that $n/\kappa(n) \to \theta \in [0,1]$. Begin with the matrix (6.1), denoted now simply $W_n$. The “undressed” ensemble $\beta W_n W_n^\dagger$ has the processes

$$\tilde{\chi}_2^{\beta \kappa}, \tilde{\chi}_2^{\beta(\kappa-1)} + \chi_2^{\beta(n-1)}, \tilde{\chi}_2^{\beta(\kappa-2)} + \chi_2^{\beta(n-2)}, \ldots$$

$$\tilde{\chi}_2^{\beta \kappa} \chi_2^{\beta(n-1)}, \tilde{\chi}_2^{\beta(\kappa-1)} \chi_2^{\beta(n-2)}, \tilde{\chi}_2^{\beta(\kappa-2)} \chi_2^{\beta(n-3)}, \ldots$$

along the main and off-diagonals, respectively. Up to fist order, the top right corner of the matrix $W_n W_n^\dagger$ has $n + \kappa$ on the diagonal, and $\sqrt{n\kappa}$ off-diagonal. So the top right corner of

$$\frac{1}{\sqrt{n\kappa}}((\sqrt{n} + \sqrt{\kappa})^2 I_n - W_n W_n^\dagger)$$

is approximately a discrete Laplacian. If time is scaled by $m_n^{-1}$, then space will have to be scaled by $m_n^2$ for this to converge to the continuum Laplacian. Now the desired convergence of drift and noise terms determines, up to constant factors

$$m_n = \left(\frac{\sqrt{n\kappa}}{\sqrt{n} + \sqrt{\kappa}}\right)^{2/3}, \quad H_n = \frac{m_n^2}{\sqrt{n\kappa}}((\sqrt{n} + \sqrt{\kappa})^2 I_n - W_n W_n^\dagger).$$

Now the $y$’s are defined by formula (5.3), and are just partial sums of shifted and scaled versions of (6.2) and (6.3). That is,

$$\Delta y_{n,1,k} = \frac{m_n}{\sqrt{n\kappa}}\left(\n + \kappa - \beta^{-1}(\chi_2^{\beta(n-k)} + \tilde{\chi}_2^{\beta(\kappa-k)})\right),$$

$$\Delta y_{n,2,k} = \frac{m_n}{\sqrt{n\kappa}}2\left(\sqrt{n\kappa} - \beta^{-1}\chi_2^{\beta(n-k)}\tilde{\chi}_2^{\beta(\kappa-k+1)}\right).$$

As before, we set $\eta$ to be the expected increments and $w$ to be the centered $y$. The $y_{n,i}, i = 1, 2$ are again independent increment processes, though they are not independent of one another. We set $\gamma_n = 2\sqrt{n/\kappa}/(\sqrt{n/\kappa} + 1)^2$, noting that $\gamma_n \to \gamma \in [0,1/2]$. Then we have

$$m_n \mathbb{E}(\Delta y_{1,n,k}) = \frac{m_n^2}{\sqrt{n\kappa}} 2k = \frac{m_n^3}{\sqrt{n\kappa}} 2x = \gamma_n x = \gamma x + o(1),$$

$$m_n \mathbb{E}(\Delta y_{1,n,k})^2 = \frac{1 - \gamma}{2\beta} + o(1), \quad m_n \mathbb{E}(\Delta y_{1,n,k})^{4} = o(1),$$

uniformly for $k/m_n$ in compacts, so Corollary 6.1 shows that $y_{n,1}(x)$ converges to the process

$$\sqrt{\gamma}/\sqrt{2\beta} b_x + \gamma x^2/2,$$

whence it is tight. Similarly we get the process convergence $y_{n,2}(x) \Rightarrow \sqrt{1 - \gamma}/\sqrt{2\beta} b_x + (1 - \gamma)x^2/2$. 

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To get the convergence of the sum and (5.4), we instead consider the process defined by

\[ \Delta y_{n,k} = \frac{m_n}{\sqrt{n \kappa}} \left( (\sqrt{n} + \sqrt{\kappa})^2 - \beta^{-1}(\chi_{\beta(n-k)} + \tilde{\chi}_{\beta(k-k)})^2, \right) \]

noting that the process \( y_{n,k} - y_{n,1,k} - y_{n,2,k} \) is sub-scaling and hence converges to the 0 process in law by a fourth moment bound. Now \( y_{n,k} \) has independent increments, and the same brand of moment computations already considered along with Corollary 6.1 imply that it converges to \( 2/\sqrt{\beta} b_x + x^2/2 \).

Towards tightness, a bit of work shows that for \( \beta \) fixed and all \( k \geq 1, \kappa, n > 10 \) we have

\[ c_1 \frac{k}{m_n} \leq m_n(\eta_{n,1,k} + \eta_{n,2,k}) = \frac{m_n^2}{\sqrt{n \kappa}} \left( 2k + 2\sqrt{n \kappa} - \frac{2}{\beta} \mathbb{E}[\chi_{\beta(n-k)}] \mathbb{E}[\tilde{\chi}_{\beta(k-k)}] \right) \leq c_2 \frac{k}{m_n} \]

We also have the upper bound \( m_n(\eta_{n,2,k}) \leq 2m_n^2 \). This verifies (5.4) and (5.5) with \( \bar{\eta}(x) = x \).

The verification of the oscillation bounds (5.6) is identical to the \( \beta \)-Hermite case. Indeed, all we used there was that \( \sqrt{m_n} w_{n,i,k} \) are martingales whose increments are independent and have bounded second moments.

Acknowledgments We would like to thank A. Edelman and B. Sutton for making earlier versions of [10] available to us. Thanks as well to M. Krishnapur, H.P. McKean and B. Valkó for comments. J. R. thanks the Department of Mathematics of the University of Colorado at Boulder for their hospitality during the Spring 2006 semester when much of this work was completed. The work of B.R. was supported in part by NSF grant DMS-0505680, and that of B.V. by a Sloan Foundation fellowship, by the Canada Research Chair program, and by NSERC and Connaught research grants.

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