P.5 Powers and Roots.
Powers:

"Exponents" are a special notation for repeated multiplication.

They will provide us with a shorthand for denoting expressions such as: $x.x.x.x.x.xy.y.y.y$

Namely, we will write this expression as: $x^6.y^4$.

In this example, we call 6 the "exponent" of the "base term x," and 4 the "exponent" of the "base term y."

In other words, an exponent simply tells us how many times the base must be repeated.

Special Note: We will keep different bases separate from each other.

Definition: For any positive integer n,

$$x^n = x.x.x.x.x.......x$$ **"n factors"**

The number x is the **base** and the number n is the **exponent.**

Rules of Exponents:

The basic rules of exponents are:

Rule 1: $x^m x^n = x^{m+n}$

Example: $x^2 x^3 = x^{2+3} = x^5$

That is, **when you multiply like bases, you add the exponents.**

Rule 2: $\dfrac{x^m}{x^n} = x^{m-n}$

Example: $\dfrac{x^9}{x^6} = x^{9-6} = x^3$

That is, **when you divide like bases, you subtract the exponents.**

Rule 3: $\left(x^m\right)^n = x^{mn}$

Example: $\left(x^3\right)^4 = x^{3.4} = x^{12}$

That is, **when you raise a power to a power, you multiply the exponents.**

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Rule 4: $x^{-m} = \dfrac{1}{x^m}$

Example: $x^{-4} = \dfrac{1}{x^4}$

That is, **negative exponents result in "flipping fractions over."**

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Rule 5: $(xy)^n = x^n y^n$

That is, the *n*th power of a product is equal to the product of the *n*th powers of the factors.

Example: $(2y)^3 = 2^3 y^3 = 8y^3$

Note that many of these properties were given with only two terms/factors but they can be extended out to as many terms/factors as we need. For example, rule 5 can be extended as follows.

$$(xyz)^n = x^n y^n z^n$$

Warning: DO NOT distribute exponents across addition or subtraction!

For example, $(x+y)^2$ **DOES NOT EQUAL** $x^2 + y^2$!

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Rule 6: $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

That is, the *n*th power of a quotient is equal to the quotient of the *n*th powers.

Example: $\left(\dfrac{x}{2}\right)^6 = \dfrac{x^6}{2^6} = \dfrac{x^6}{64}$

Warning: DO NOT distribute exponents across addition or subtraction!

For example, $\left(\dfrac{2x+1}{3x+4}\right)^2$ **DOES NOT EQUAL** $\dfrac{2x^2+1^2}{3x^2+4^2}$!

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*Zero exponents:

Rule 7: $x^0 = 1$

Example: $(2y)^0 = 1$ **and** $\left(\dfrac{2x+1}{3x+4}\right)^0 = 1$

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*** Roots:** If *n* is a positive integer that is greater than 1 and *a* is a real number then,

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

where *n* is called the **index**, *a* is called the **radicand**, and the symbol $\sqrt{}$ is called the **radical**. The left side of this equation is often called the radical form and the right side is often called the exponent form.

When there is no index number, *n*, it is understood to be a 2 or square root.
For example: $\sqrt{x}$ = square root of *x*.

CAUTION: If *n* is even and *a* is negative, then the root is not a real number.

Rules: The basic rules which govern radical expressions are:
Rule 1: $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ **'Product Rule'**
Note that if you have different index numbers, you CANNOT multiply them together. Also, note that **you can use this rule in either direction** depending on what your problem is asking you to do.

Rule 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ **'Quotient Rule'**

This rule can also work in either direction.

Warning: Note that while we can "break up" products and quotients under a radical,we can't do the same thing for sums or differences. In other words,

$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ AND $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

If we "break up" the root into the sum (or difference) of the two pieces we clearly get different answers! So, be careful to not make this very common mistake!

Examples: Evaluate, if possible, or simplify each of the following:

a. $\sqrt{20} = \sqrt{4.5} = \sqrt{4}.\sqrt{5} = 2\sqrt{5}$

b. $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

c. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$

d. $\sqrt[4]{16} = \sqrt[4]{2^4} = 2$

e. $\sqrt[4]{-16}$ is undefined; does not exist.

We are going to be simplifying more radicals, so we should next define **simplified radical form**. A radical is said to be in simplified radical form (or just simplified form) if each of the following are true.

1. All exponents in the radicand must be less than the index.
2. Any exponents in the radicand can have no factors in common with the index.
3. No radicals appear in the denominator of a fraction.

Examples:

Simplify each of the following.

(a) $\sqrt{b^7}$

(b) $\sqrt[4]{32x^9y^5z^{12}}$

(c) $\sqrt{18x^6y^{11}}$

(d) $\sqrt[5]{32x^{12}y^7}$

Solution:

(a) $\sqrt{b^7}$

To simplify the given problem, we need to break down the radicand into perfect squares since the index is 2. Apply the product rule, then simplify.

So, $\sqrt{b^7} = \sqrt{b^2.b^2.b^2.b} = \sqrt{b^2}.\sqrt{b^2}.\sqrt{b^2}.\sqrt{b} = b.b.b.\sqrt{b}$

(b) $\sqrt[4]{32x^9y^5z^{12}}$

To simplify the given problem, we need to break down the radicand into perfect 4rth power factors since the index is 4. Then apply the product rule.

So, $\sqrt[4]{32x^9y^5z^{12}} = \sqrt[4]{2^4.2.x^4x^4.x.y^4.y.z^4.z^4.z^4} = 2.x^2.y.z^3.\sqrt[4]{2xy}$

Now try these.

(c) $\sqrt{18x^6y^{11}}$

(d) $\sqrt[5]{32x^{12}y^7}$

Examples:

Simplify each of the following.

(a) $\sqrt{\dfrac{4}{9}}$

(b) $\sqrt{\dfrac{x}{16}}$

(c) $\sqrt[3]{\dfrac{8x^4}{81y^{12}}}$

(d) $\sqrt[5]{\dfrac{a^6b^{10}}{c^5d^6}}$

Solution:

(a) $\sqrt{\dfrac{4}{9}}$

To simplify the given problem, we need to break down the radicand into perfect squares since the index is 2. Apply the quotient rule, then simplify.

So, $\sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{2^2}}{\sqrt{3^2}} = \dfrac{2}{3}$

(b) $\sqrt{\dfrac{x}{16}} = \dfrac{\sqrt{x}}{\sqrt{4^2}} = \dfrac{\sqrt{x}}{4}$

(c) $\sqrt[3]{\dfrac{8x^4}{81y^{12}}}$

(d) $\sqrt[5]{\dfrac{a^6b^{10}}{c^5d^6}}$

More Practice problems:

1. Simplify the following. Express answers in terms of positive exponents.

(a) $\left(2a^{-3}b^{2}\right)^{-2}$ (b) $\left(\dfrac{x^{2}}{y^{4}}\right)^{-3}$ (c) $\dfrac{4x^{-3}y^{-5}}{6x^{-4}y^{3}}$ (d) $\left(\dfrac{m^{-3}m^{3}}{n^{-2}}\right)^{-2}$

(e) $\left(\dfrac{x^{4}y^{-1}}{x^{-2}y^{3}}\right)^{2}$ (f) $\dfrac{x^{-2}y^{2}z}{x^{-3}y^{-3}z^{2}}$ (g) $\left(x^{2}y^{-1}\right)^{3}\left(x^{-1}y^{4}\right)^{2}$ (h) $\left(-3x^{4}y^{5}\right)\left(5x^{7}y^{3}\right)$

(i) $\left(\dfrac{a^{5}b^{-2}}{c^{4}}\right)^{-1}$ (j) $-15x^{0}$ (k) $\dfrac{\left(24x^{3}\right)\left(x^{-5}\right)}{\left(6x^{-20}\right)}$ (l) $\dfrac{-3x^{4}y^{7}}{-9x\ \ y^{3}}$

2. Simplify the following radicals:

(a) $\sqrt{12x^{3}y^{5}z^{2}}$ (b) $\sqrt[3]{\dfrac{8a^{7}}{27b^{3}}}$ (c) $\sqrt[3]{\dfrac{8x^{4}z^{5}}{81y^{12}}}$ (d) $\sqrt[5]{\dfrac{a^{6}b^{10}}{c^{5}d^{6}}}$

(e) $\sqrt{18a^{6}b^{4}}$ (f) $-\sqrt[3]{81x^{10}y^{5}}$ (g) $\sqrt[5]{32x^{12}y^{7}}$

Please check homework problems as well.

P.4 Fractional Expressions:
1.3 Operations with Rational Expressions:

*** Equivalent fractions:**

*** Rational Expressions:**

A **rational expression** is the quotient of two polynomials.

Ex: $\frac{2}{x-3}$, $\frac{x^2-4}{x^2-4x+5}$

*** Simplifying Rational Expressions:**

NOTE: To simplify a rational expression factor completely the numerator and denominator the cancel common factors.

A rational expression is **reduced to lowest terms** if all common factors from the numerator and denominator are canceled.

Example 1a: Reduce $\frac{4}{12}$ to lowest terms

Not reduced to lowest terms $\Rightarrow \frac{4}{12} = \frac{(4)(1)}{(4)(3)} = \frac{1}{3} \Leftarrow$ reduced to lowest terms

With rational expressions it works exactly the same way.

Not reduced to lowest terms $\Rightarrow \frac{(x+3)(x-1)}{x(x+3)} = \frac{x-1}{x} \Leftarrow$ reduced to lowest terms

** We have to be careful with canceling. There are some common mistakes that students often make with these problems. Remember that in order to cancel a factor it must multiply the whole numerator and the whole denominator. So, the $x+3$ above could cancel since it multiplied the whole numerator and the whole denominator. However, the x's in the reduced form can not be cancelled, since the x in the numerator is not times the whole numerator.

To see why the x's don't cancel in the reduced form above put a number in and see what happens. Let's plug in $x=4$.

$\frac{4-1}{4} = \frac{3}{4}$ $\frac{4-1}{4} = -1$ (If 4 gets canceled)

Clearly the two answers are not the same number!

Note: Only COMMON FACTORS of the numerator and denominator can be canceled.

Example 1b: Simplify the following:

$$\frac{-6axy}{8y}, \quad \frac{4-x}{x-4}, \quad \frac{2ax-2x}{2ax+2x}, \quad \frac{2x-3y+4z}{4ax-6ay+8az}, \quad \frac{ax-ay+bx-by}{x-y}$$

*** Multiplying Rational Expressions:**

1) Completely factor each numerator and denominator.
2) Multiply the numerators and multiply the denominators.
3) Simplify the result as far as possible by **cancelling common factors**.

Example 2:

$$\frac{3x+1}{2x}\cdot\frac{2x-4}{3x^2-2x-1}$$

$$=\frac{3x+1}{2x}\cdot\frac{2(x-2)}{(3x+1)(x-1)}$$

$$=\frac{(3x+1)2(x-2)}{2x(3x+1)(x-1)}$$

$$=\frac{x-2}{x(x-1)}$$

Example 3: $\frac{x^2-2xy}{3x+3y}\cdot\frac{x^2-y^2}{xy-2y^2}$

- **Dividing Rational Expressions:**

1) Completely factor each numerator and denominator.
2) Change to multiplication.
3) Invert (flip) the second fraction and proceed as in multiplication.

Example 4:

$$\frac{x^2-16}{x^2+5x+4}\div\frac{x-4}{x^2-3x-4}$$

$$=\frac{(x-4)(x+4)}{(x+4)(x+1)}\div\frac{x-4}{(x-4)(x+1)}$$

$$=\frac{(x-4)(x+4)}{(x+4)(x+1)}\cdot\frac{(x+1)(x-4)}{x-4}=\frac{(x-4)(x+4)(x+1)(x-4)}{(x+4)(x+1)(x-4)}=x-4$$

*** Finding the Least Common Denominator (LCD):**

1) Completely factor each **denominator.**
2) The LCD is the product of all unique factors each raised to the greatest power that appears in any factored denominator.

Example 5:

1) $\dfrac{2}{3x^5y^2}, \dfrac{3z}{5xy^3}$

2) $\dfrac{z}{z-1}, \dfrac{7}{z+1}$

3) $\dfrac{7}{m^2-10m+25}, \dfrac{2m}{2m^2-9m-5}, \dfrac{m-1}{m^2-25}$

4) $\dfrac{x}{x^2-4}, \dfrac{x}{6-3x}$

Hint: If **opposite factors** occur, do not use both in the LCD. Instead, factor -1 from one of the opposite factors so that the factors are then identical.

Ex: If you have factors like *x-2* and *2-x*, these are called **opposite factors.** Notice that you can factor a *-1 from 2-x* so that the factors are identical.

- **Adding or Subtracting Rational Expressions:**

1) With the same denominator:

If the denominators are the same, keep the same denominator just add numerators. Then simplify if possible.

Example 6:

$$\frac{x}{x^2+5x+4}+\frac{2x+3}{x^2+5x+4}$$

$$=\frac{x+2x+3}{x^2+5x+4}=\frac{3x+3}{x^2+5x+4}=\frac{3(x+1)}{(x+4)(x+1)}=\frac{3}{x+4}$$

Example 7:

$$\frac{x^2}{x+7}-\frac{49}{x+7}$$

$$=\frac{x^2-49}{x+7}=\frac{(x+7)(x-7)}{x+7}=x-7$$

2) With different denominators: (LCD is NEEDED)

a. Factor completely each denominator.
b. Find the LCD of the rational expression.
c. Write each rational expression as an equivalent rational expression whose denominator is the LCD found in step (b)
d. Add or subtract numerators, and write the result over the common denominator.
e. Simplify the resulting rational expression if possible.

Example 8:

a) $\dfrac{5x}{x^2-4}-\dfrac{2}{x^2+x-2}$

b) $\dfrac{3}{x+2}+\dfrac{2x}{x-2}$

c) $\dfrac{2x-1}{2x^2-9x-5}+\dfrac{x+3}{6x^2-x-2}$

d) $\dfrac{3}{x^2-9}-\dfrac{x}{x^2-6x+9}+\dfrac{1}{x+3}$

e) $\dfrac{2x-6}{x-1}-\dfrac{4}{1-x}$

* **Complex Fractions:**

A complex fraction is a rational expression whose numerator, denominator, or both contain one or more rational expressions. Examples are

$$\frac{\frac{1}{a}}{\frac{b}{2}} \qquad \frac{\frac{x}{2y^2}}{\frac{6x-2}{9y}} \qquad \frac{x+\frac{1}{y}}{y+1} \qquad \frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$$

They can be simplified by treating the numerator and the denominator as separate problems.

Then we have a "division" problem.

For example, to simplify $\dfrac{x-\frac{9}{x}}{4x-12}$,

we first complete the subtraction problem contained in the numerator of the entire fraction: $x-\dfrac{9}{x}=\dfrac{x^2-9}{x}$.

Now we have the division problem: $\dfrac{\left(\dfrac{x^2-9}{x}\right)}{\left(\dfrac{4x-12}{1}\right)}$

We invert and multiply: $\left(\dfrac{x^2-9}{x}\right)\left(\dfrac{1}{4x-12}\right)$.

As before, we should now factor in order to reduce:

$$\left(\frac{(x+3)(x-3)}{x}\right)\left(\frac{1}{4(x-3)}\right)$$

Now cancel the common factor "$(x-3)$."

So our final answer in factored form, reduced to lowest terms is:

$$\frac{x+3}{4x}.$$

Remember, you do not need a common denominator when multiplying (or dividing) fractions.

Steps for simplifying complex fractions:

1) Simplify the numerator and the denominator of the complex fraction so that each is a single fraction.
2) Perform the indicated division by multiplying the numerator of the complex fraction by the reciprocal of the denominator of the complex fraction.
3) Simplify if possible.

Example 9: Simplify the following:

a) $\dfrac{\dfrac{x}{3y^2}}{\dfrac{6x-2}{9y}}$

b) $\dfrac{\dfrac{x}{y^2}+\dfrac{1}{y}}{\dfrac{y}{x^2}+\dfrac{1}{x}}$

c) $\dfrac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$

d) $\dfrac{x+\dfrac{1}{y}}{y+1}$

Solution:

a) $\dfrac{\dfrac{x}{3y^2}}{\dfrac{6x-2}{9y}}=\dfrac{x}{3y^2}\div\dfrac{6x-2}{9y}=\dfrac{x}{3y^2}\cdot\dfrac{9y}{6x-2}=\dfrac{3x}{y(6x-2)}$

b) First we simplify the numerator and denominator separately so that each is a single fraction.

$\dfrac{\dfrac{x}{y^2}+\dfrac{1}{y}}{\dfrac{y}{x^2}+\dfrac{1}{x}}=\dfrac{\dfrac{x}{y^2}+\dfrac{1.y}{y.y}}{\dfrac{y}{x^2}+\dfrac{1.x}{x..x}}$ Note the LCD of the fractions that are in the numerator is y^2 and the LCD of the fractions that are in the denominator is x^2.

$=\dfrac{\dfrac{x+y}{y^2}}{\dfrac{y+x}{x^2}}=\dfrac{x+y}{y^2}\cdot\dfrac{x^2}{y+x}=\dfrac{x^2}{y^2}$

Now, try the following problems:

1. Perform the indicated operations and simplify your answers.

(a) $\frac{x}{x-3}+\frac{3}{3-x}$

(b) $\frac{1}{4x^2}-\frac{2x+1}{3x^3}+\frac{3}{12x}$

(c) $\frac{y-3}{y^2-4}-\frac{y+2}{y^2-4y+4}-\frac{2}{2-y}$

(d) $\frac{4}{2x-1}\cdot\frac{10x-5}{16}$

(e) $\frac{x+1}{x-x^2}\cdot\frac{x^2-2x+1}{x^2-1}$

(f) $\frac{4x^2-4x+1}{2x^2+5x-3}\div\frac{2x^2-3x-2}{2x^2+7x+3}$

(g) $\frac{\frac{x^2}{y^2}-1}{\frac{x}{y}+1}$

(h) $\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

(i) $\frac{a+b^{-1}}{b+a^{-1}}$