3.1 SOLVING QUADRATIC EQUATIONS:

***** A QUADRATIC is a polynomial whose highest exponent is 2.

*** The "standard form" of a quadratic equation is:**

$$ax^2 + bx + c = 0$$

with "a", "b" and "c" representing real numbers, and "a" is not equal to zero.

* What do we mean by a *root* of a quadratic?

A solution to the quadratic equation.

* How many roots does a quadratic have?

Always two.

In order to solve such equations, we will need to employ one of the following methods:

1. Factoring (Equation must be written in standard form)

2. The Quadratic Formula (Equation must be written in standard form)

3. Square root principle

4. Completing the Square

1) **Factoring**: is the easiest method if you can factor the polynomial. (equation must be in standard form).

Example 1: Solve $x^2 - x - 12 = 0$

Solution: First we need to **make sure that the equation is written in standard form**. This is already done for us here.

$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$

Here, we have a product of two terms that is equal to zero. This means that at least one of the following must be true.

$x - 4 = 0$	OR	$x + 3 = 0$
$x = 4$	OR	$x = -3$

Note that each of these is a linear equation that is easy to solve. So we have two solutions to the equation, $x = 4$ and $x = -3$.

2) The Quadratic Formula: (equation must be in standard form).

if $ax^2 + bx + c = 0$ then

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be used to solve any quadratic equation.

Note: The letters "a," "b" and "c" represent real numbers, but "a" cannot equal zero.

Example 2: Solve: $x^2 + 2x = 7$

Solution:

Remember, that we need to write the equation in standard form.

$$x^2 + 2x - 7 = 0$$

Then identify the values to be plugged in the quadratic formula. For this equation we have.

$$a = 1 \qquad b = 2 \qquad c = -7$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{32}}{2}$$

There are the two solutions for this equation. We need to simplify the answer, however, we need to be careful . One of the COMMON mistakes at this point is to “cancel” to 2’s in the numerator and denominator.

In order to do any simplification here we will first need to simplify the square root. At that point we can do some canceling.

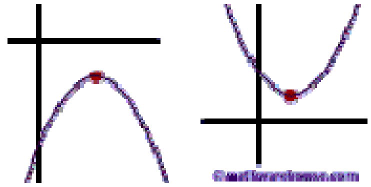
$$x = \frac{-2 \pm \sqrt{(16)2}}{2} = \frac{-2 \pm \sqrt{2}}{2} = \frac{2\left(-1 \pm 2\sqrt{2}\right)}{2} = -1 \pm 2\sqrt{2}$$

Notice that a part of the quadratic formula can be used to predict the type of solutions that the associated quadratic equation will have.

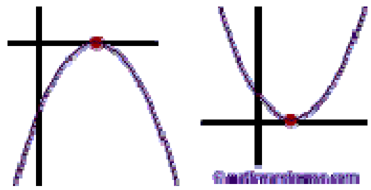
Namely, the expression under the square root, **"b^2 - 4ac."**

We call this expression the **"discriminant,"** since it "discriminates" between the following three cases:

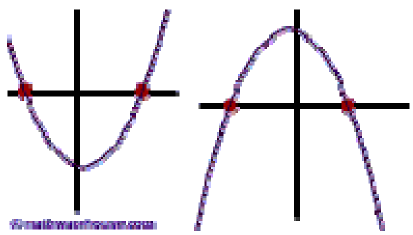
1. If **b^2 - 4ac** is **less than 0**, then the equation has **no real solutions** (because it means there is a negative number under the square root, sometimes called "**Complex roots**"). The graph does not cross the $x - axis$.



2. If **b^2 - 4ac** is **equal to 0**, then the equation has **exactly 1 real solution** (sometimes called a "**double root**"). The graph crosses the $x - axis$ at exactly one point.



3. If **b^2 - 4ac** is **greater than 0**, then the equation has **2 different real solutions** (sometimes called "**distinct roots**"). The graph crosses the $x - axis$ at two points.



Example 3: Use the discriminant to determine the type of solution for each of the following quadratic equations.

(a) $3x^2 + 1 = 5x$

(b) $2x^2 + 5x + 2 = 0$

(c) $4x^2 - 20x = -25$

Solution:
All we need to do here is make sure the equation is in standard form, determine the value of *a*, *b*, and *c*, then plug them into the discriminant.

a) $3x^2 + 1 = 5x$
First get the equation in standard form.

$3x^2 - 5x + 1 = 0$

We then have, $a = 13$ $b = -5$ $c = 1$

Plugging into the discriminant $b^2 - 4ac = (-5)^2 - 4(13)(1) = -27 < 0$
Since the discriminate is negative, we will have two complex solutions.
Also try, $x^2 - 4x + 7 = 0$

(b) $2x^2 + 5x + 2 = 0$

This equation is already in standard form. $a = 2$ $b = 5$ $c = 2$

The discriminant is $b^2 - 4ac = (5)^2 - 4(2)(2) = 9 > 0$

Since the discriminate is positive, we will have two real distinct solutions.

(c) $4x^2 - 20x = -25$

Again, we need the equation in standard form.

$4x^2 - 20x + 25 = 0$

We then have, $a = 4$ $b = -20$ $c = 25$

The discriminate is, $b^2 - 4ac = (-20)^2 - 4(4)(25) = 0$

In this case we will have a double root since the discriminant is zero.

3) Square root principle: if $x^2 = k$, then $\sqrt{x^2} = \pm\sqrt{k}$; $x = \pm\sqrt{k}$.

This tells us that we have two numbers here. One is $x = \sqrt{k}$ and the other is $x = -\sqrt{k}$.

Example 4: Solve**:** $x^2 - 100 = 0$

Solution:

Write the equation as $x^2 = k$

$x^2 = 100$ $x = \pm\sqrt{100} = \pm 10$

So, there are two solutions to this equation, $x = \pm 10$. Remember this means that there are two solutions here, $x = -10$ and $x = +10$.

Example 4: Solve $(x+3)^2 = 9$

Solution:

This one looks different from the previous example, however it works the same way. The square root principle can be used anytime we have *something* squared equals a number. Here the *something* that is squared is not a single variable it is something else.

When we apply the square root principle, look what happens:

$$\sqrt{(x+3)^2} = \pm\sqrt{9}$$

$x + 3 = \pm 3$ which means that $x = -3 \pm 3$,

so the two solutions here are: $x = -3 + 3 = 0$ and $x = -3 - 3 = -6$

4) Completing the square : Perform the following steps:

- **a.** Isolate the terms with the variables on one side of the equation and arrange them in descending order.
- **b.** Divide by the coefficient of the squared term if that coefficient is not 1.
- **c.** Complete the square by taking half the coefficient of x and adding its square to both sides of the equation.
- **d.** Express one side of the equation as the square of a binomial.
- **e.** Use the square root principle.
- **f.** Solve for the variable.

Example 5: Solve $x^2 - 6x + 1 = 0$
Solution:

<u>Step 1</u> : Divide the equation by the coefficient of the x^2 term. Recall that completing the square required a coefficient of one on this term and this will guarantee that we will get that. We don't need to do that for this equation .

<u>Step 2</u> : Set the equation up so that the x's are on the left side and the constant is on the right side.

$$x^2 - 6x = -1$$

<u>Step 3</u> : Complete the square on the left side by taking half the coefficient of x and adding its square to both sides. **Remember the rule, what we do to one side of an equation we need to do to the other side of the equation.**

First, here is the number we add to both sides.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now, complete the square.

$$x^2 - 6x + 9 = -1 + 9$$

$$(x-3)^2 = 8$$

<u>Step 4</u> : Now, at this point notice that we can use the square root principle on this equation. That was the purpose of the first three steps. Doing this will give us the solution to the equation.

$$x - 3 = \pm\sqrt{8}$$

$$x = 3 \pm \sqrt{8}$$

Remember this means that there are two solutions here, $x = 3 + \sqrt{8}$ and $x = 3 - \sqrt{8}$.

HOW TO Find a Quadratic Equation given roots:

If you have the roots $x = 1$ and $x = -3$
You can recover the quadratic equation by forming $(x - 1)(x + 3) = 0$. Then proceed by foiling to get the equation.

Now try the following problems:

1) Solve the following quadratics by factoring.

a) $x^2 - 3x + 2 = 0$
b) $x^2 + 7x + 12 = 0$
c) $x^2 + 3x - 30 = 0$
d) $x^2 - x - 30 = 0$
e) $2x^2 + 7x + 3 = 0$
f) $3x^2 + x - 2 = 0$
g) $x^2 + 12x + 36 = 0$
h) $x^2 - 2x = -1$
i) $x^2 = x + 20$
j) $3x^2 + x = 10$
k) $5x^2 + 13x - 6 = 0$
l) $6x^2 + 5x - 6 = 0$
m) $15x^2 - 11x - 12 = 0$

2) Solve the following quadratics using the quadratic formula.

a) $x^2 - 3x + 2 = 0$
b) $x^2 + 7x + 12 = 0$
c) $x^2 + 3x - 30 = 0$
d) $x^2 - 5x + 5 = 0$
e) $2x^2 - 8x + 5 = 0$
f) $3x^2 + 5x - 8 = 0$
g) $x^2 + 12x + 36 = 0$
h) $x^2 - 2x = -1$
i) $x^2 = x + 20$
j) $3x^2 + x = 10$

3) Solve the following quadratics by completing the square.

a) $x^2 - 2x - 2 = 0$
b) $x^2 + 4x - 6 = 0$
c) $x^2 + 3x + 1 = 0$
d) $x^2 - 5x - 5 = 0$
e) $2x^2 - 10x - 10 = 0$
f) $2x^2 - 4x - 4 = 0$

4) Use the discriminant to determine the type of solution for each of the following quadratic equations.

a) $-3x^2 + 2x + 1 = 0$
b) $2x^2 - 5x - 3 = 0$
c) $-4x^2 + 6x - 5 = 0$
d) $-5x^2 - 3 = 2x$
e) $4x^2 + 12x = -9$
f) $9x^2 - 12x + 4 = 0$

5) In each of the following cases, find a quadratic equation having the given roots.

a) 2, 3
b) 6 only
c) -5, $\frac{3}{2}$
d) $2+\sqrt{3}$, $2-\sqrt{3}$

Please check HW problems as well.

3.2 Complex Numbers

* The **imaginary number** i is defined as the number such as

$$i = \sqrt{-1} \text{ and } i^2 = -1$$

Using this definition all the square roots of the following examples become,

$$\sqrt{-9} = 3i$$

$$\sqrt{-100} = 10i$$

$$\sqrt{-5} = \sqrt{5}i$$

$$\sqrt{-290} = \sqrt{290}i$$

* These are all examples of **complex numbers**.
    - A **complex number** is a number of the form $a+bi$ (**standard form**), where a and b are real numbers.
    - When in the standard form ***a*** is called the **real part** of the complex number and ***b*** is called the **imaginary part** of the complex number.

*** Operations with complex numbers:**

Now we need to discuss the basic operations for complex numbers.

1) **Addition and Subtraction of complex numbers:** The easiest way to think of adding and/or subtracting complex numbers is to think of each complex number as a polynomial and do the addition and subtraction in the same way that we add or subtract polynomials. (To add or subtract complex numbers, **Just combine like terms)**.

Example 1:

- **Simplify (2 + 3*i*) + (1 – 6*i*).**

  Solution:

To simplify complex-valued expressions, you combine "like" terms and apply the various other methods you learned for working with polynomials.

$(2 + 3i) + (1 - 6i) = (2 + 1) + (3i - 6i) = 3 + (-3i) = \mathbf{3 - 3i}$

- **Simplify (5 – 2*i*) – (–4 – *i*), express answer in standard form.**

$(5 - 2i) - (-4 - i) = 5 - 2i + 4 + i$ **distribute the negative sign**

$= (5 + 4) + (-2i + i)$ **Combine like terms**

$= (9) + (-i) = \mathbf{9 - i}$ **Answer in standard form**

2) Multiplication of complex numbers: again think of the complex numbers as polynomials so multiply them out as you would with polynomials (**FOIL/Distributive property**). The one difference will come in the final step; ***i*^{2} reduces to the number –1**

Example 2:

* **Multiply** $(1+3i)(2-4i)$

Solution:

$(1+3i)(2-4i) = 2-4i+6i-12i^2$ **Multiply (use Foil)**

$= 2+2i-12i^2$ **Add the real parts and imaginary parts**

$= 2+2i-12(-1)$ **Note that** $i^2 = -1$

$= 14+2i$ **Simplify**

- **Simplify (2 – *i*)(3 + 4*i*).**

$(2 - i)(3 + 4i) = (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)$ **Multiply (use Foil)**

$= 6 + 8i - 3i - 4i^2$ **Add the real parts and imaginary parts**

$= 6 + 5i - 4(-1)$ **Note that** $i^2 = -1$

$= 6 + 5i + 4 =$ **$10 + 5i$** **Simplify**

3) Division of complex numbers: Multiply the numerator and denominator by the **conjugate** of the denominator we will be able to eliminate the *i* from the denominator.

The **conjugate of a complex number** $a + bi$ is the same number, but with the opposite sign in the middle: $a - bi$. When you multiply conjugates, you are multiplying to create something in the pattern of a difference of squares:

$$(a+bi)(a-bi) = a^2 - abi + abi - (bi)^2$$
$$= a^2 - b^2(i^2)$$
$$= a^2 - b^2(-1)$$
$$= a^2 + b^2$$

Note that the *i*'s disappeared, and the final result was a *sum* of squares.

This is what the conjugate is for, and here's how it is used:

Example 3: Divide $\dfrac{3}{2+i}$

Solution: $\frac{3}{2+i}=\frac{3}{2+i}\cdot\frac{2-i}{2-i}=\frac{3(2-i)}{(2+i)(2-i)}$ **Multiply the numerator and denominator by the conjugate of the denominator $2-i$.**

$$=\frac{6-3i}{4-2i+2i-i^2}=\frac{6-3i}{4-(-1)}=\frac{6-3i}{4+1}=\frac{6-3i}{5}$$

$$=\frac{6}{5}-\frac{3}{5}i$$

Note that in the last step, the fraction was split into two pieces. This is because a complex number is in two parts, the real part and the imaginary (*i*) part. To write the answer in standard form, split the complex fraction into its two separate terms.

** The next topic that we want to discuss is **powers of *i*.** Let us take a look at what happens when we start looking at various powers of *i*.

$i^1=i$	$i^1=i$
$i^2=-1$	$i^2=-1$
$i^3=i.i^2=-i$	$i^3=-i$
$i^4=(i^2)^2=(-1)^2=1$	$i^4=1$
$i^5=i.i^4=i$	$i^5=i$
$i^6=i^2.i^4=(-1)(1)=-1$	$i^6=-1$
$i^7=i.i^6=-i$	$i^7=-i$
$i^8=i^4.i^4=(1)(1)=1$	$i^8=1$

NOTE: All powers of *i* can be reduced down to one of four possible answers (*i, -1, -i , or 1*) and they repeat every four powers. This can be a convenient fact to remember.

- **The Quadratic formula:** When the Formula gives you a negative underneath the square root, you can now simplify that by using complex numbers. You cannot graph a complex number on the *x*,*y*-plane. So this "solution to the equation" is not an *x*-intercept.

Example 5: Use the quadratic formula to solve the following.

$$x^2+3x+3=0$$

Solution: $a=1$ $b=3$ $c=3$

$$x=\frac{-3\pm\sqrt{(3)^2-4(1)(3)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9-12}}{2}=\frac{-3\pm\sqrt{-3}}{2}=\frac{-3\pm\sqrt{3}i}{2}$$

$$=\frac{-3}{2}\pm\frac{\sqrt{3}}{2}i$$ **Answer in standard form**

Now, try the following problems.

1) Perform the indicated operation and write answers in standard form.

(a) $(-4+7i)+(5-10i)$
(b) $(4+12i)-(-3-15i)$
(c) $(\sqrt{2}+i)+(-\sqrt{2}-i)$
(d) $(1+2i)+(3-5i)$
(e) $(1+i)-(2-i)$

2) Multiply each of the following and write answers in standard form.

(a) $7i(-5+2i)$

(b) $(1-5i)(-9+2i)$

(c) $(4+i)(2+3i)$

(d) $(1-8i)(1+8i)$
(e) $(-3+4i)(5-2i)$

3) Divide the following, write answers in standard form.

(a) $\dfrac{3-i}{2+7i}$

(b) $\dfrac{3}{9-i}$

(c) $\dfrac{8i}{1+2i}$

(d) $\dfrac{6-9i}{2i}$

(e) $\dfrac{1}{-3+2i}$

4) **Use the quadratic formula to solve the following.**

(a) $3x^2+x+1=0$

(b) $x^2+4=0$

(c) $2x^2+x-5=0$

3.3 Equations Reducible to Quadratic Equations:

In this section we will look at equations that are called **quadratic in form** or **reducible to quadratic in form**. What this means is that we will be looking at equations and determine whether we can make them look like quadratic equations. A ***u-substitution*** will help to solve these equations by the quadratic formula or factoring.

Why do we use substitution?

1) **It is useful in solving an equation that contains a repeated variable expression.**

   For example: $(x^2+x)^2-8(x^2+x)+12=0$ and $(x+4)-3(x+4)^{\frac{1}{2}}=0$

2) **It is used to transform a non quadratic equation into a quadratic equation.**

   For example: $x^{\frac{2}{3}}-3x^{\frac{1}{3}}-10=0$

The following equations are in quadratic form since the degree of the leading term (the first term) is twice the degree of the middle term: $x^4-7x^2+12=0$, $x^{\frac{2}{3}}-3x^{\frac{1}{3}}-10=0$, $(x^2+x)^2-8(x^2+x)+12=0$, $(x+4)-3(x+4)^{\frac{1}{2}}=0$

We will illustrate this technique in the following example.

Example 1: Solve $x^4-7x^2+12=0$

Solution

Notice that

$$x^4=(x^2)^2$$

Notice here that the variable portion of the first term (*i.e.* ignore the coefficient) is nothing more than the variable portion of the second term squared. Note as well that the exponent on the first term was twice the exponent on the first term.

This, along with the fact that third term is a constant, means that this equation is reducible to quadratic in form. We will solve this by first defining,

$$u=x^2$$

this means that

$$u^2=(x^2)^2=x^4$$

Therefore, we can write the equation in terms of *u*'s instead of *x*'s as follows,

$$x^4-7x^2+12=0 \qquad \Rightarrow \qquad u^2-7u+12=0$$

The new equation (the one with the *u*'s) is a quadratic equation and we can solve that. In fact this equation is factorable, so the solution is:

$$u^2-7u+12=(u-4)(u-3)=0 \quad \Rightarrow \quad u=4, u=3$$

So, we get the two solutions shown above. These are not the solutions that we are looking for. **We want values of *x*, not values of *u*.** Recall that we defined

$$u = x^2$$

To get values of *x*, all we need to do is plug in *u* into this equation and solve that for *x*.

$u = 3$: $\quad x^2 = 3 \quad \Rightarrow \quad x = \pm\sqrt{3}$

$u = 4$: $\quad x^2 = 4 \quad \Rightarrow \quad x = \pm\sqrt{4} = \pm 2$

So, we have four solutions to the original equation, $x = \pm\sqrt{3}$ and $x = \pm 2$.

*** Solving Equations with Rational Expressions:**

If an equation involves rational expressions, you need to perform the following steps:

1) Find the LCD.

2) Clear the denominators by multiplying **BOTH** sides of the equation by the least common denominator (LCD).

3) Solve the resulting quadratic equation (**factor or quadratic formula**).

4) Check your solutions.

Example 2: **Solve** $x + \dfrac{12}{x-3} = 1 + \dfrac{4x}{x-3}$

Solution:

Step 1: The LCD is $x-3$, where $x \neq 3$

Step 2: Multiply **BOTH** sides of the equation by the LCD, Clear the denominators

$$(x-3)\left(x + \frac{12}{x-3}\right) = (x-3)\left(1 + \frac{4x}{x-3}\right)$$

$x^2 - 3x + 12 = x - 3 + 4x$ **denominators are eliminated**

Step 3: Solve the resulting quadratic equation; **factor or quadratic formula**

$$x^2 - 8x + 15 = 0$$

$$(x-3)(x-5) = 0$$

$x = 3$ or $x = 5$

Step 4: Check your solutions.

Checking shows $x = 3$ is an extraneous root, and $x = 5$ is the only valid solution.

*** Solving Equations with Radicals:** The main idea is to **isolate** the term containing the radical and then **square** both sides of the equation to eliminate the radical. If, after squaring, there is still another radical, repeat the process. Solve the resulting quadratic equation and check your solutions.

Example 3: Solve $x + \sqrt{x} = 12$

Solution:

Step 1: Isolate the radical term.

$$\sqrt{x} = 12 - x$$

Step 2: Square both sides to remove the radical sign.

$$\left(\sqrt{x}\right)^2 = (12 - x)^2$$

$$x = 144 - 24x + x^2$$

Step 3: Solve the resulting equation.

$$0 = 144 - 25x + x^2$$

This is a quadratic equation which factors as follows:

$$0 = (x - 9)(x - 16)$$

So we have $x = 9$ **or** $x = 16$

Step 4: Check your answers in the original equation.

Checking $x = 9$,

$$9 + \sqrt{9} = 12$$

$$9 + 3 = 12$$ ✓

Checking $x = 16$,

$$16 + \sqrt{16} = 12$$

$$16 + 4 = 12$$ ✗

Therefore, the original equation only has one solution, namely $x = 9$.

Example 4: Solve $\sqrt{3x+4}-\sqrt{7x}=-2$

Solution:

Step 1: Isolate one radical term.

$$\sqrt{3x+4}=\sqrt{7x}-2$$

Step 2: Square both sides to remove the radical sign.

$\left(\sqrt{3x+4}\right)^2=\left(\sqrt{7x}-2\right)^2$ **use** $(a-b)^2$ **to simplify the right hand side**

$3x+4=(7x)-4\sqrt{7x}+4$ **simplify**

$-4x=-4\sqrt{7x}$ **isolate the radical; divide both sides by** -4

$x=\sqrt{7x}$ **Square both sides,**

$x^2=7x$

Step 3: Solve the resulting equation.

$$x^2-7x=0$$

This is a quadratic equation which factors as follows:

$$0=x(x-7)$$

So we have $x=0$ **or** $x=7$

Step 4: Check your answers in the original equation.

Checking $x=0$,

$$\sqrt{3(0)+4}=\sqrt{7(0)}-2$$

$$\sqrt{4}=-2$$

$2=-2$ ✗ **extraneous root**

Checking $x=7$,

$$\sqrt{3(7)+4} = \sqrt{7(7)} - 2$$

$$\sqrt{25} = \sqrt{49} - 2$$

$$5 = 7 - 2$$

$$5 = 5$$ ✓

Therefore, the original equation only has one solution, namely $x = 7$.

Now, try the following problems.

1) Solve the following.

a) $3x^4 + 5x^2 - 2 = 0$

b) $(x-1)^6 - 7(x-1)^3 - 8 = 0$

c) $2x + 3\sqrt{x} - 2 = 0$

d) $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10 = 0$

e) $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$

2) Solve the following.

a) $2x - 1 = \sqrt{x+1}$

b) $5 + \sqrt{x+7} = x$

c) $\sqrt{2x-1} - \sqrt{x+3} = -1$

d) $\sqrt{2x-1} - \sqrt{x-4} = 2$

3) Solve the following.

a) $\dfrac{x}{x-1} + \dfrac{6}{x} = \dfrac{7}{x-1}$

b) $\dfrac{x^2}{x-5} + 3 = \dfrac{5x}{x-5}$

c) $\dfrac{x}{2x+1} - \dfrac{2x^2+5}{2x^2-5x-3} = \dfrac{3}{x-3}$

d) $\dfrac{-18}{6x^2-x-1} + \dfrac{3x}{2x-1} = \dfrac{4x}{3x+1}$

3.4 Quadratic Inequalities

An inequality with a quadratic expression in one variable on one side and zero on the other, is called a **quadratic inequality** in one variable. **It can be written in one of the following standard forms:**

$$ax^2 + bx + c < 0$$
or
$$ax^2 + bx + c \le 0$$
or
$$ax^2 + bx + c > 0$$
or
$$ax^2 + bx + c \ge 0$$

In other words, a quadratic inequality is in standard form when the inequality is set to 0.

Just like in a quadratic equation, the degree of the polynomial expression is two.

Also remember that, multiplication or division of both sides of an inequality by a negative number reverses the direction of the inequality.

*** Solving Quadratic Inequalities:**

To solve quadratic inequalities, perform the following steps:

1) The inequality should be written so that one side consists only of zero.
2) If possible, factor the expression on the nonzero side of the inequality. Otherwise, use the quadratic formula to solve for x.
3) Set each of the factors equal to zero, and solve for x. **Remember that, the solutions to the equation become boundary points for the solution to the inequality.**
4) Plot the values on the number line, and break up the number line into intervals. **Remember to make the boundary points solid circles if the original inequality includes equality; otherwise, make the boundary points open circles**.
5) Use a test point in each interval to calculate the sign of the expression.
6) Select the interval(s) on which the inequality is satisfied; Shade the intervals.
7) Write the solution in interval notation.

***Example* 1:** Solve $x^2 - 3x - 10 < 0$.

Solution

There is a fairly simple process to solving inequalities. If you can remember it, you will always be able to solve these kinds of inequalities.

<u>Step 1</u> : Get a zero on one side of the inequality. It does not matter which side has the zero, however, we are going to be factoring in the next step so keep that in mind as you do this step. Make sure that you've got something that's going to be easy to factor.

$$x^2 - 3x - 10 < 0$$

<u>Step 2</u> : If possible, factor the polynomial. Note that it is not always be possible to factor the given inequality, but remember that quadratic formula is another option.

$$(x-5)(x+2) < 0$$

<u>Step 3</u> : Set each of the factors equal to zero, and solve for *x*.

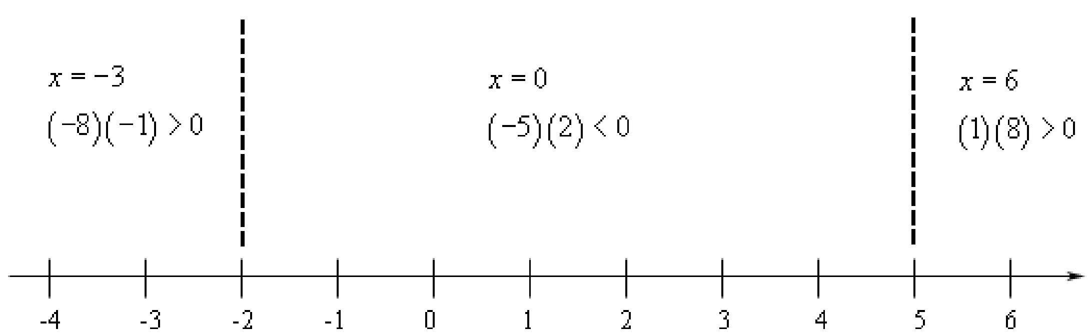
$$x - 5 = 0 \Rightarrow x = 5$$
$$x + 2 = 0 \Rightarrow x = -2$$

The number line is then divided into three regions. In each region if the inequality is satisfied by one point from that region then it is satisfied for ALL points in that region. Likewise, if the inequality is not satisfied for some point in that region, it is not satisfied for ANY point in that region.

This leads us into the next step.

<u>Step 4,5</u> : Graph the points where the polynomial is zero (*i.e.* the points from the previous step) **on a number line** and **pick a test point** from each of the regions. Plug each of these test points into the polynomial and **determine the sign** of the polynomial at that point.

This is the step in the process that has all the work. Here is the number line for this problem.



When you pick test points make sure that you pick easy numbers to work with. So, do not choose large numbers or fractions unless you are forced to by the problem.

Also, note that we plugged the test points into the factored from of the polynomial and all **we are really after here is whether or not the polynomial is positive or negative** (You may also plug them in the original problem). Therefore, we did not actually bother with values of the polynomial just the **sign** and we can get that from the product shown. The product of two negatives is a positive, and the product of two positives is a positive.

Step 6 : Write down the answer. Recall that we discussed earlier that if any point from a region satisfied the inequality then **ALL points** in that region satisfied the inequality and likewise if any point from a region did not satisfy the inequality then **NONE** of the points in that region would satisfy the inequality.

This means that all we need to do is look up at the number line above. If the test point from a region satisfies the inequality then that region is part of the solution **(Shade the region)**. If the test point does not satisfy the inequality then that region is not part of the solution **(do not shade the region)**.

Now, also notice that any value of *x* that will satisfy the original inequality will also satisfy the inequality from Step 2 and likewise, if an *x* satisfies the inequality from Step 2 then it will satisfy the original inequality.

So, that means that all we need to do is determine the regions in which the polynomial from Step 2 is negative. For this problem that is only the middle region. The interval notation for the solution to this inequality is,

$$(-2,5)$$

***Example* 2:** Solve $x^2 + x \geq 6$

Step 1 : Get a zero on one side of the inequality.

Step 2 : If possible, factor the polynomial.

Step 3 : Set each of the factors equal to zero, and solve for *x*.

Step 4 : Plot the values on the number line, and break up the number line into intervals. **Remember to make the boundary points solid circles if the original inequality includes equality; otherwise, make the boundary points open circles**.

Step 5 : Use a test point in each interval to calculate the sign of the expression.

Step 6 : Select the interval(s) on which the inequality is satisfied; Shade the intervals.

Step 7 : Write the solution in interval notation.

*** Quadratic Rational Inequalities:**

Example 3 Solve $\dfrac{x^2+4x+3}{x-1} > 0$

Solution

1) We already have zero on one side.
2) Factor the numerator and denominator.

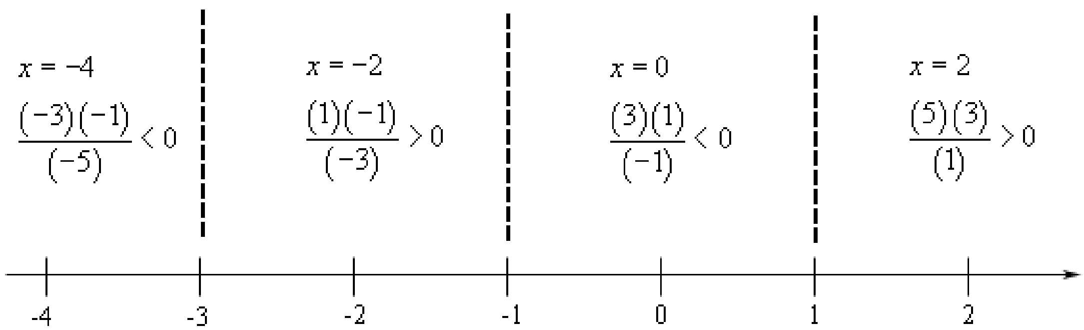
$$\frac{(x+1)(x+3)}{x-1} > 0$$

3) set each factor equal to zero and solve for *x*, remember that the values of *x* are the boundary values.

Numerator: $x=-1$, $x=-3$

Denominator: $x=1$

Here is the number line for this problem.



In the problem we are after values of *x* that make the inequality strictly positive and so that looks like the second and fourth region and we **will not** include any of the endpoint here. The solution in **interval notation** is then,

$$(-3,-1)\cup(1,\infty)$$

Now try the following problems.

1) Solve the following inequalities by testing intervals. State your answers in interval notation. Graph your solutions.

a) $x^2+21<10x$

b) $-x^2+5x-4\le 0$

c) $2x^2-x-1>2x+1$

d) $x^2-12\ge x$

e) $\dfrac{x-2}{x+4}\le 0$

f) $\dfrac{x^2-x-12}{x^2-4}\ge 0$

For more practice, check Homework problems as well.

3.5 Applications of Quadratic equations: (Projectile Problems)

General Strategy for Solving Word Problems:

1. Read the problem carefully.

2. Use diagrams/charts if you think it will make the information clearer.

3. Find the relationship or formula relevant to the problem.

4. Identify the unknown quantity (or quantities), and label them, using one variable.

5. Write an equation involving the unknown quantity, using the relationship or formula

from step 3.

6. Solve the equation.

7. Answer the question.

8. Check the answer in the original words of the problem.

* **Projectile Problems:**

Def: A "projectile" is any object that is thrown, shot, or dropped.
Usually the object is moving straight up or straight down.

Remember, our main goal in this section is to construct an equation which describes the relationships among the relevant quantities in a given problem.

The equations related to the motion of gravity are $y = -16t^2 + v_0 t + y_0$, in this formula, -16 is a constant is based on the gravitational force of the earth and represents ½ ***g*** = ½(-32 ft/sec^2) = -16 ft/sec^2, and $v = -32t + v_0$
where v_0 = the initial velocity
y_0 = the height with respect to the ground at time t = 0. Here, we assume that $y = 0$ at the ground level.

Once you have constructed and solved an equation that models the application problem, you need to remember to do the following:

1) Use your solution(s) to answer the question(s) posed by the original problem statement.

2) Check that your answer(s) "make sense" in the context of the problem.

Example 1: An object is launched directly upward at 64 feet per second from a platform 80 feet high.
Solution:

If an object is launched straight up into the air, its height in feet after t seconds is given by the equation:

$$y = -16t^2 + 64t + 80$$

a) **When** will it strike the ground?

Here, we want to determine the time at which the object hits the ground.

That is, we want to find the value(s) of the variable t for which y = 0.

So we should consider the equation height = 0

$$0 = -16t^2 + 64t + 80$$

Solve the equation for t. Divide through by *-16.*Then use factoring or the quadratic formula to solve the equation.

$$0 = t^2 - 4t - 5$$

$$0 = (t-5)(t+1)$$

$$(t-5) = 0 \Rightarrow t = 5\sec$$
$$(t+1) = 0 \Rightarrow t = -1\sec$$

We will disregard negative values.

Answer: **The object hits the ground after *t=5* seconds.**

Note: Be sure to check that your final answers "make sense" in the context of the original problem.

In this problem, for example, our equation needs to yield a non-negative number for the value of "t". That is, it wouldn't "make sense" for the time to be a negative number.

b) **When** will the object attain its maximum height?

Here, we want to determine the time at which the object reaches its maximum height. Since $y = -16t^2 + 64t + 80$ is a quadratic equation, we know that the maximum value of the equation occurs at the **vertex** of the graph of the equation. The first coordinate of the vertex gives the time t at which the object reaches its maximum height. It is

$$t = -\frac{b}{2a} = -\frac{64}{2(-16)} = 2\sec$$

Or, when the object reaches its maximum height, its vertical velocity is equal to zero. Use $v = -32t + v_0$ and plug 0 for $v, v_0 = 64$ and solve for t.

$$0 = -32t + 64$$

$$32t = 64$$

$$t = \frac{64}{32} = 2\sec$$

c) **What** will be the object's maximum height?

It will take the object 2 seconds to reach its maximum height. At this point we need to find the height of the object at that point. This is done by evaluating the equation $y = -16t^2 + 64t + 80$ when $t=2$.

$$y = -16(2)^2 + 64(2) + 80 = -16(4) + 64(2) + 80 = -64 + 128 + 80 = 144\, feet.$$

Answer: The object reaches a maximum height of 176 feet 2 seconds after it has been launched.

Example 2: A ball is thrown directly upward from an **initial** **height** of 200 feet with an **initial** **velocity** of 96 feet per second. After how many seconds will the ball reach its maximum height? And, what is the maximum height?

Solution: To analyze our problems, we will be using a formula $y = -16t^2 + v_0 t + y_0$.

Begin by substituting known values for variables in the formula:

$$y = -16t^2 + v_0 t + y_0$$

$$y = -16t^2 + 96t + 200$$

when the object reaches its maximum height, its vertical velocity is equal to zero. Use $v = -32t + v_0$ and plug 0 for $v, v_0 = 96$ and solve for t.

$$0 = -32t + 96$$

$$32t = 96$$

$$t = \frac{96}{32} = 3\sec$$

OR,

Since the formula represents a parabola, we must find the **vertex** of the parabola to find the time it takes for the ball to reach its maximum height as well as the maximum height (called the apex) . Using the vertex formula,

$$t = -\frac{b}{2a} = -\frac{96}{2(-16)} = 3\sec$$

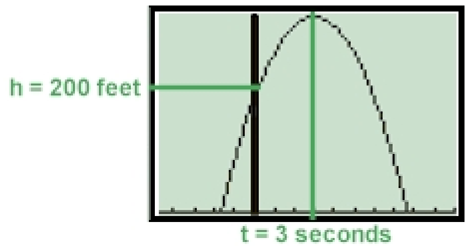
Substituting into the projectile motion formula we have:

$y = -16t^2 + 96t + 200$

$y = -16(3)^2 + 96(3) + 200$

$y = 344$ feet

Therefore, if a ball is thrown directly upward from an initial height of 200 feet with an initial velocity of 96 feet per second, after 3 seconds it will reach a maximum height of 344 feet.



Example 3: An object is shot straight up from the ground with an initial velocity of 112 ft/sec.

a) Find the **interval of time** *t* during which the object is 160 feet above the ground or higher.
b) At **what time** does the object reach its highest point and what is the highest point reached by the object?

Please Check HW problems as well.

3.6 Circles and Parabolas:

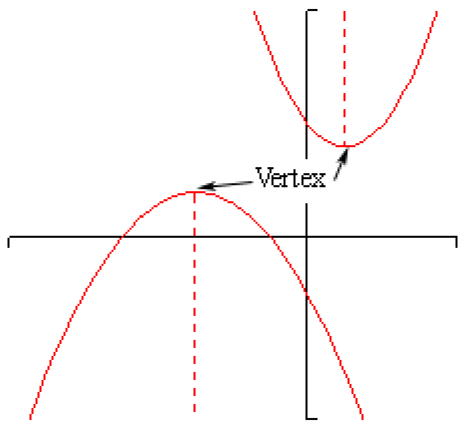
* Parabolas:

1) Parabolas that open up or down:

In this section we want to look at the graph of a quadratic equation. The general form of a quadratic equation is,

$$y = ax^2 + bx + c$$

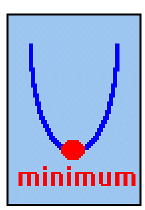
The graphs of quadratic equations are called **parabolas**. Here are some examples of parabolas.



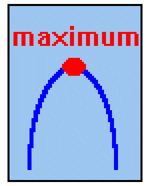
Facts about parabolas:

1. All parabolas are "U" shaped and the will have a highest or lowest point that is called the **vertex.** The vertex of the parabola is given by:

   ($x = \frac{-b}{2a}, y = \frac{4ac - b^2}{4a}$ **).**
2. Parabolas may open up or down, this depends on the sign of ***a.***
3. If $a > 0$, the parabola opens upward. **The vertex will represent a minimum** (since the parabola opens up).



- If $a < 0$, the parabola opens downward. **The vertex will represent a maximum** (since the parabola opens down).



** Parabolas may or may not have *x*-intercepts and they will always have a single *y*-intercept.

4. The dashed line with each of these parabolas is called the **axis of symmetry.** The axis of symmetry of a parabola is a vertical line and is given by: $x = \frac{-b}{2a}$. Every parabola has an axis of symmetry and, as the graph shows, the graph to either side of the axis of symmetry is a mirror image of the other side. This means that if we know a point on one side of the parabola we will also know a point on the other side based on the axis of symmetry.

5. **Intercepts**: Intercepts are the points where the graph will cross the *x* or *y*-axis.

Finding intercepts is a fairly simple process. To find the ***y*-intercept** of an equation, all we need to do is **set *x=0* and evaluate** to find the *y* coordinate. In other words, the *y*-intercept is the point $(0, c)$.
We find ***x*-intercepts** in pretty much the same way. We **set *y=0* and solve** the resulting equation for the *x* by factoring or using the quadratic formula.

Example 1: Find the axis of symmetry, vertex, the *x*- and *y*-intercepts then graph

$$y = x^2 - 6x + 5$$

Solution:

1. Opens upward since the coefficient of the x^2 is positive

2. Axis of symmetry is a vertical line passing through the vertex is determined by

Axis of symmetry $= x = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$

3. Vertex: $x = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$ and $y = (3)^2 - 6(3) + 5 = 9 - 18 + 5 = -4$

So the **vertex is (3, -4).**

3. *y*-intercept: $y = 0^2 - 6(0) + 5 = 5$

The ***y*-intercept is (0, 5).**

4. *x*-intercepts:

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$

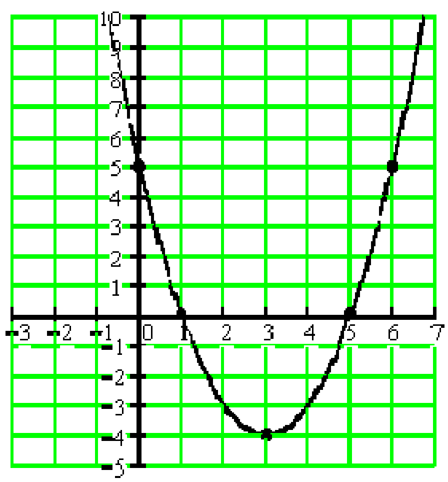
$x - 5 = 0$ or $x - 1 = 0$

$x = 5$ $x = 1$

The ***x*-intercepts are (5,0) and (1,0).**

Plot the points and the axis of symmetry on the coordinate plane.

Then connect the points with a smooth curve.



Note that the **vertex** is a point on the graph. It is either going to be the lowest or highest point on the graph of a quadratic equation.

*The **axis of symmetry** is not actually part of the graph itself, but is important in that the parabola creates a mirrored image about it. Note how it is symmetric about the axis of symmetry. Also, note how it goes through the vertex.

Third, note how there is **one *y*-intercept and two *x*-intercepts**. The quadratic function can **have no, one or two *x*-intercepts.**

2) Parabolas that open to the left or right:

The general form of a parabola that opens to the left or right is determined by:

$$x = ay^2 + by + c$$

1. if a is positive, the parabola opens to the right.
   if a is negative, the parabola opens to the left.
2. **Axis of symmetry:** The axis of symmetry is a horizontal line given by:
   $$y = \frac{-b}{2a}$$
3. **Vertex:** The y-coordinate of the vertex is given by $y = \frac{-b}{2a}$ and we find the x-coordinate by plugging this into the equation.
4. **Intercepts:** To find the ***x*-intercept** of an equation, all we need to do is **set *y=0* and evaluate** to find the x coordinate. In other words, the x-intercept is the point $(c,0)$; the x-intercept is the value of the constant term of the given equation.

We find the ***y*-intercepts** in pretty much the same way. We **set *x=0* and solve** the resulting equation for y by factoring or using the quadratic formula.

***Example 2:* Graph** $x = y^2 - 6y + 5$

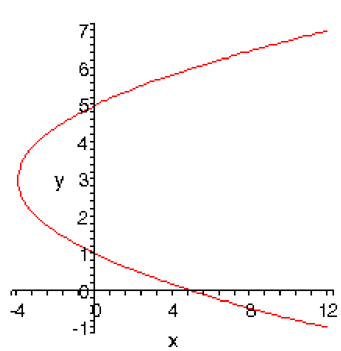
Solution:

This is a parabola that opens to the right (a is positive) and has a vertex at (-4,3).

To graph this we will need y-intercepts. We find these just like we found x-intercepts in the previous couple of problems.

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

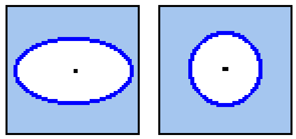
The parabola will have y-intercepts at $y = 1$ and y =5. Here's a sketch of the graph



**Circles:

What does **"circle"** mean?

You are probably thinking "well, it's perfectly round." But what do we mean by "perfectly round"?



Both of these shapes are round, but only the second shape is a circle.

In fact, a circle is "perfectly round" because **each point along the edge of the circle is the same distance from the center point.**

This fact serves as the fundamental property of circles.

A very important example is the equation $\mathbf{x^2 + y^2 = 1}$.

This equation produces **"the unit circle."** That is, the circle centered at (0,0), and with radius 1. You will make much use of this circle when you study trigonometry.

Definition: A circle is a set of points that are equidistant from a fixed point called the center.

Standard form of a circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$

Radius of a circle: is the distance from the center to any point on the circle.

$$d = r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Center of a circle: is determined by (x_0, y_0).

Diameter of a circle: is a line segment that passes through the center of the circle and whose endpoints are on the circle. Diameter is twice the radius.

To graph a circle: First plot the center, then from the center, go ***r - units*** up, down, left, and right, then sketch the circle by hand or using a compass.

Example 3a: Find the standard equation of the circle with the given center and radius, then graph.

1) center *(5,-2)*, *r* =7
2) center *(3,1)*, *r* =2 “ Check class notes”
3) center *(0,0)*, *r* =3 “ Check class notes”

Solution: Since we are given the **center point** and the **radius**, we can use the formula for the standard form of a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$

Therefore, we get the equation

$$(x-5)^2 + (y-(-2))^2 = (7)^2$$
$$(x-5)^2 + (y+2)^2 = 49$$

Example 3b: Find the equation of a circle with center at (4, -2) and passing through

(1, 2).

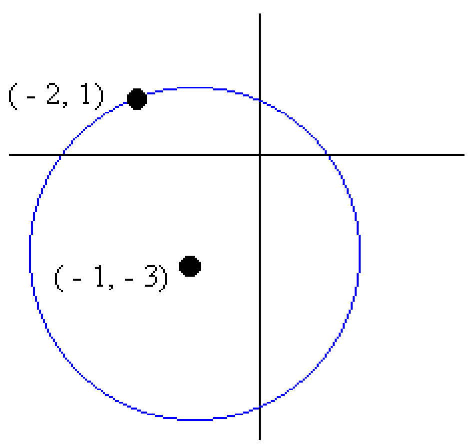
Solution: In order to find the equation of a circle, you must know the **center** and the **radius** of the circle. In this case, you are given the center, but not the radius of the circle. However, if you know the center and a point that is ON the circle, then the distance between these two points is the radius. You must find the distance between the two given points by the **distance formula**.

Now, the distance between the two points given in this problem, and therefore the radius of the circle, is

Radius= $d = r = \sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{(1-4)^2 + (2+2)^2} = \sqrt{(-3)^2 + (4)^2} = \sqrt{9+16}$
$= \sqrt{25} = 5$

Therefore the equation of the circle with **center at (4, -2) and of radius** $\sqrt{25} = 5$ is given by $(x-4)^2 + (y+2)^2 = 25$. As a check, you might want to sketch the graph, and see if it does indeed pass through the point (1, 2).

Example 3c: Find the equation of a circle whose graph is shown below.



Solution:

Step 1: We are given the coordinates of the **center point (- 1, - 3).**

However, we must compute the radius of the circle by applying the distance formula to points (- 1, - 3) and (- 2, 1).

$$d = r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$r = \sqrt{(-2-(-1))^2 + (1-(-3))^2} = \sqrt{(-2+1)^2 + (1+3)^2} = \sqrt{(-1)^2 + (4)^2} = \sqrt{17}$$

Step 2: Use the formula for the standard form of a circle.

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$(x-(-1))^2 + (y-(-3))^2 = \left(\sqrt{17}\right)^2$$

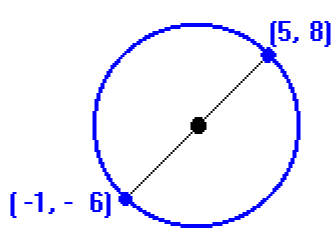
$(x+1)^2 + (y+3)^2 = 17$ **"Equation of the given circle in standard form"**

Example 3c: Write the equation of the circle which has endpoints of a diameter at (5, 8) and (- 1, - 6).

Solution: We always need a **center point and a radius** to find the equation of any circle.

Step 1: Find the coordinates of the center point.

Note: The center point will be located at the midpoint of any diameter of the circle.



We apply the "**midpoint formula**" to the given endpoints of the diameter.

$$\text{Center point} = \left(\frac{5+(-1)}{2}, \frac{8+(-6)}{2}\right) = \left(\frac{4}{2}, \frac{2}{2}\right) = (2,1)$$

Step 2: Compute the radius.

We apply the **distance formula** to the center point (2,1) and one of the endpoints, in this case (5,8).

$$r = \sqrt{(5-2)^2 + (8-1)^2} = \sqrt{(3)^2 + (7)^2} = \sqrt{9+49} = \sqrt{58}$$

Step 3: Use the formula for the standard form of a circle.

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$(x-2)^2 + (y-1)^2 = \left(\sqrt{58}\right)^2$$

$$(x-2)^2 + (y-1)^2 = 58$$

* **General form** of a circle is given by:

$$Ax^2 + Ay^2 + Bx + Cy + D = 0$$

Notice that the squared terms have matching coefficients.

If the squared terms have different coefficients, the graph won't be a circle. In fact, it will be an oval shape called an "ellipse."

We can use a technique called **"completing the square"** to rewrite such an equation in **standard form** so that we can identify the circle's **center point and radius**.

4. **Complete the square method for finding the standard equation of a circle:**
    1) Isolate the variables and arrange them in descending order.
    2) Divide by the coefficient of the squared terms if that coefficient is not 1.
    3) Combine the *x*- terms and the *y*- terms.
    4) Complete the square in *x*, by taking half the coefficient of *x*, and adding its square on both sides of the equation.
    5) Complete the square in *y*, by taking half the coefficient of *y*, and adding its square on both sides of the equation.
    6) Express the left side of the equation as squares, and add up the right side.

Example 2: By completing the square in *x* and *y*, find the center and radius of the circle

$$x^2 + y^2 + 8x - 6y = 39$$

Solution: **Before you ever start the problem, you know that this looks like a CIRCLE! The best way to find the center and radius is by completing the square. Since you already have coefficients of** $1x^2$ **and** $1y^2$**, you should begin by re-writing the equation with the *x* terms together and *y* terms together, leaving a space to complete the square:**

$$x^2 + 8x + \_\_\_\_\_\_ + y^2 - 6y + \_\_\_\_\_ = 39 + \_\_\_\_\_ + \_\_\_\_\_$$

When you complete the square, remember that you take half of the coefficient, and square. Half of 8 is 4 and 4^2 **is 16. Half of -6 is -3, and** $(-3)^2$ **is 9.**

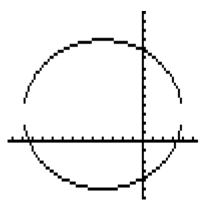
$$x^2 + 8x + \_\_\_\_\_\_ + y^2 - 6y + \_\_\_\_\_ = 39 + \_\_\_\_\_ + \_\_\_\_\_$$

$$x^2 + 8x + 16 + y^2 - 6y + 9 = 39 + 16 + 9$$

$$(x+4)^2 + (y-3)^2 = 64$$

This is a circle with center at (-4, 3) and with radius *r* = $\sqrt{64} = 8$**.**

To graph this circle, start at the origin and count 4 units to the left, then up 3 units. This is the center of the circle. Now, the radius is ***r* = 8** so measure 8 units in each direction from the center (up, down, left, and right). The graph should look like this.



Circle with center at (-4, 3) and with radius *r* = 8.

Example 3: $x^2 + y^2 - 12x + 8y - 48 = 0$

Solution: Use **completing the square.** Begin by re-writing the equation with the *x* terms together and *y* terms together, leaving a space to complete the square. Also, you should add +48 to each side of the equation:

$$x^2 - 12x + \_\_\_\_ + y^2 + 8y + \_\_\_\_ = 48 + \_\_\_\_ + \_\_\_\_$$

When you complete the square, remember that you take **half of the coefficient, and square.** Half of -12 is -6 and $(-6)^2$ is 36. Half of 8 is 4, $(4)^2$ and is16.

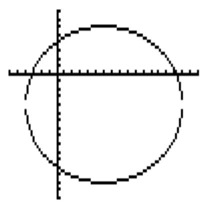
$x^2 - 12x + \_\_\_\_ + y^2 + 8y + \_\_\_\_ = 48 + \_\_\_\_ + \_\_\_\_$

$x^2 - 12x + (-6)^2 + y^2 + 8y + (4)^2 = 48 + (-6)^2 + (4)^2$

$(x-6)^2 + (y+4)^2 = 48 + 36 + 16$

$(x-6)^2 + (y+4)^2 = 100$ **This is a circle with center at (6, -4) and with radius**

***r* =** $\sqrt{100} = 10$. To graph this circle, start at the origin and count 6 units to the right, then down 4 units. This is the center of the circle. Now, the radius is ***r* =** $\sqrt{100} = 10$ so measure 10 units in each direction from the center. The graph should look like this.



Circle with center at (6, -4) and with radius.

Note: When writing an equation of a circle, keep in mind that **you ALWAYS need two pieces of information:**

1. **The center of the circle.**
2. **The radius of the circle.**

Now, try the following problems

1) By completing the square in *x* and *y*, find the center and radius of the circle.

a) $x^2 + y^2 + 6x - 4y = 23$

b) $4x^2 + 4y^2 + 12x - 4y = -1$

2) Find the axis of symmetry and vertex of the parabola. Find the *x*-intercepts and the *y*-intercept of the parabola. Graph the parabola clearly labeling the vertex, the axis of symmetry and the intercepts.

a) $y = -x^2 - 2x + 3$

b) $y = x^2 - 2x - 3$

3) a. Find the radius of a circle, given that the center is at **(2, -3)** and the point **(-1, -2)** lies on the circle.

b. Find the diameter?

4) Find the center of the circle with diameter having endpoints at **(-4, 3)** and **(0, 2).**

Please check the HW problems for more practice.

Formulas needed for test 3:

1) Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2) Square root principle: $x^2 = k,$ $x = \pm\sqrt{k}$

3) Dicriminant: $b^2 - 4ac$

4) standard equation of a circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$

5) radius: $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

6) Standard equation of a parabola: $y = ax^2 + bx + c$

7) Axis of symmetry: $x = \dfrac{-b}{2a}$

8) The equations relating to the motion of gravity are:

$y = -16t^2 + v_0 t + y_0$ and $v = -32t + v_0$

Where: v_0 is the initial velocity (in ft/sec)

y_0 is the height (in feet) with respect to the ground at time $t = 0$.

Here, we assume that $y = 0$ at the ground level.