* 1.1 Operations with Polynomials:

Let's start by defining some words.

Term: A term is a number, variable or the product of a number and variable(s).

For example: $3x$, $5y^3$, $2xy^3z$, a

Coefficient: A coefficient is the numeric factor of the term.

Here are the coefficients of the terms listed above: 3,5,2,1,1 respectively.

A **monomial** is a polynomial that consists of exactly one term. For example, $5x$.

A **binomial** is a polynomial that consists of exactly two terms. For example, $5x+2$.

A **trinomial** is a polynomial that consists of exactly three terms. For example, $(3x+5)^2 = 9x^2 + 15x + 25$

By definition, all monomials, binomials, and trinomials are also polynomials.

* Adding Polynomials:

To add polynomials, remove the parentheses (or any grouping symbol), then combine like terms.

Like terms: are terms that contain the same variables raised to exactly the same powers.

Example 1: Add the following polynomials.

a. $(11x^3 - 12x^2 + x - 3) + (x^3 - 10x + 5)$

Solution: $(11x^3 - 12x^2 + x - 3) + (x^3 - 10x + 5) = 11x^3 - 12x^2 + x - 3 + x^3 - 10x + 5$

$= 12x^3 - 12x^2 - 9x + 2$

b. $(14x^4 - 6x^3 + x^2 - 6) + (x^3 - 5x^2 + 1)$

* Subtracting Polynomials:

To subtract polynomials, remove the parentheses (or any grouping symbol) by changing the signs of the terms of the polynomial being subtracted, and then combine like terms.

Example 2: Subtract the following polynomials.

a. $(11x^3 - 12x^2 + x - 3) - (x^3 - 10x + 5)$

Solution: When subtracting polynomials, the first thing we'll do is **distribute the minus sign** through the parenthesis. This means that we will change the sign on every term in the second polynomial. Note that all we are really doing here is multiplying a "-1" through the second polynomial using the distributive law. After distributing the minus through the parenthesis we again combine like terms.

So, $\left(11x^3 - 12x^2 + x - 3\right) - \left(x^3 - 10x + 5\right) = 11x^3 - 12x^2 + x - 3 - x^3 + 10x - 5$ = $10x^3 - 12x^2 + 11x - 8$

b. $\left(14x^4 - 6x^3 + x^2 - 6\right) - \left(x^3 - 5x^2 + 1\right)$

*** Multiplying Polynomials:**

To multiply any two polynomials, use the **distributive property** and multiply each term of one polynomial by each term of the other polynomial. Then combine like terms.

** When multiplying Binomials, use **FOIL (F**irst-**O**uter-**I**nner-**L**ast)

Example 3*:* Multiply each of the following.

(a) $4x^2\left(x^2 - 6x + 2\right)$
(b) $(3x+5)(x-10)$

***Solution:* (a)** $4x^2\left(x^2 - 6x + 2\right)$

This is a quick application of the distributive property.

$4x^2\left(x^2 - 6x + 2\right) = 4x^4 - 24x^3 + 8x^2$

(b) $(3x+5)(x-10)$

Here we will use the FOIL method for multiplying the two binomials.

$$(3x+5)(x-10) = 3x^2 - 30x + 5x - 50 = 3x^2 - 25x - 50$$

***Special products:**

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Warning: Be careful not to make the following mistakes!

$$(a+b)^2 \neq a^2 + b^2$$
$$(a-b)^2 \neq a^2 - b^2$$
$$(a+b)^3 \neq a^3 + b^3$$
$$(a-b)^3 \neq a^3 - b^3$$

These are very common mistakes that students often make when they first start learning how to multiply polynomials.

Example 4*:* Multiply each of the following.

- **(a)** $(6x+2)(3x-2)$
- **(b)** $(3x+5)^2$
- **(c)** $(3x-5)^2$
- **(d)** $(3x+5)^3$
- **(e)** $(3x-5)^3$

Solution:

(a) $(6x+2)(3x-2)$

Use FOIL $(3x+2)(3x-2)=9x^2-6x+6x-4=9x^2-4$
In this case the middle terms drop out.

(b) $(3x+5)^2=9x^2+30x+25$
(c) $(3x-5)^2=9x^2-30x+25$

*** Multiplying Three or More Polynomials:**
To multiply three or more polynomials, more than one method may be needed.
Example 5*:* Multiply $(x+3)(x-3)(x^2-9)$
Solution: First Multiply $(x+3)(x-3)=(x^2-9)$ **then,** multiply
$(x^2-9)(x^2-9)=x^4-18x^2+81$

*** Dividing Polynomials Using Long Division (No Missing Terms):**

Example 6: $x-2\overline{)\,3x^3-2x^2+x-1}$

Solution:

1. Arrange the terms of both the dividend and the divisor in descending order, and check that there are no missing terms.

   This is already done for you here.

2. **Divide the first term** in the dividend by the first term in the divisor.

   $$\frac{3x^3}{x}=3x^2$$

   The result is the first term of the quotient.

   $$\begin{array}{r} 3x^2 \\ x-2\overline{)\,3x^3-2x^2+x-1} \end{array}$$

3. **Multiply** every term in the divisor by the first term in the quotient.

   $$3x^2(x-2)=3x^3-6x^2$$

Write the resulting product beneath the dividend
with like terms lined up.

$$\begin{array}{r} 3x^2 \\ x-2\overline{)3x^3-2x^2+x-1} \\ 3x^3-6x^2 \end{array}$$

4. **Subtract** the product from the dividend.

$$\begin{array}{r} 3x^2 \\ x-2\overline{)3x^3-2x^2+x-1} \\ -3x^3+6x^2 \end{array}$$

5. Bring down the next term in the original dividend and write it next to the remainder to form a new dividend.

6. Use this new expression as the dividend and repeat this process until the remainder can no longer be divided. **This will occur when the degree of the remainder is less than the degree of the divisor.**

Please see class notes for complete solution.

***Dividing Polynomials Using Long Division (Missing Terms):**

Example 7: Divide $5x^3-x^2+6$ by x-4

Solution:

Let us first set up the problem.

$$x-4\overline{)5x^3-x^2+0x+6}$$

Recall that we need to have the terms written down with the exponents in **decreasing order** and to make sure we don't make any mistakes we **add in any missing terms with a zero coefficient.**

Now we ask ourselves what we need to multiply $x-4$ to get the first term in first polynomial. In this case that is $5x^2$. So multiply $x-4$ by $5x^2$ and subtract the results

from the first polynomial

$$\begin{array}{l} \qquad\quad 5x^2 \\ x-4\overline{)5x^3-x^2+0x+6} \\ \quad\underline{-\left(5x^3-20x^2\right)} \\ \qquad\qquad\quad 19x^2+0x+6 \end{array}$$

The new polynomial is called the **remainder**. We continue the process until the degree of the remainder is less than the degree of the **divisor**, which is $x-4$ in this case. So, we need to continue until the degree of the remainder is less than 1.

Recall that the **degree** of a polynomial is the highest exponent in the polynomial. Also, recall that a constant is thought of as a polynomial of degree zero. Therefore, we'll need to continue until we get a constant in this case. Here is the rest of the work of this example.

$$\begin{array}{rl}
 & 5x^2+19x+76 \\
x-4 \big) & \overline{5x^3-x^2+0x+6} \\
 & \underline{-\left(5x^3-20x^2\right)} \\
 & \qquad 19x^2+0x+6 \\
 & \qquad \underline{-\left(19x^2-76x\right)} \\
 & \qquad\qquad 76x+6 \\
 & \qquad\qquad \underline{-\left(76x-304\right)} \\
 & \qquad\qquad\qquad 310
\end{array}$$

Now, try the following problems:

1. Perform the indicated operations and simplify when needed.

(a) $\left(2x^3-3x^2+x+5\right)+\left(2x^2+x-1\right)$ (b) $\left(2x^3-3x^2+x+5\right)-\left(2x^2+x-1\right)$

(c) $\left(2x^3-3x^2+x+5\right)\left(2x^2+x-1\right)$ (d) $2x^2+x-1\overline{\big)2x^3-3x^2+x+5}$

(e) $x+1\overline{\big)3x^3+x+1}$ (f) $\left(2x+3y\right)^2$

(g) $\left(2x-3y\right)^2$ (h) $\left(2x+3y\right)\left(2x-3y\right)$

(i) $\left(3x+2\right)\left(4x-3\right)$ (j) $\left(x+1\right)^3$

(k) $x+3\overline{\big)x^3+4x^2+2x-3}$

Please check HW problems as well.

1.2 Factoring Polynomials:

Factoring is the reverse process of multiplying. It is the process of writing a polynomial as a product of factors.
In other words, to factor a polynomial means to write it as a product of other polynomials.

* Methods for factoring Polynomials:

1) Greatest Common Factor: The GCF for a polynomial is the largest monomial that divides (is a factor of) each term of the polynomial.

The first method for factoring polynomials will be factoring out the **greatest common factor**. When factoring in general this will also be the first method that we should try as it will often simplify the problem.

Example 1: Factor the following:
(a) $8x+4$
(b) $6x^2+3x^3$
(c) $8x^4-4x^3+10x^2$
(d) $x(3x-1)+5(3x-1)$ **See class notes**

Solution:
(a) $8x+4$
The greatest common factor of the terms *8x* and *4* is *4*.

$$8x+4=4(2x+1)$$

(b) $6x^2+3x^3$
The greatest common factor of the terms is $3x^2$.

$$6x^2+3x^3=3x^2(2+x)$$

(c) $8x^4-4x^3+10x^2$
First we notice that we can factor a 2 out of every term. Also note that we can factor an x^2 out of every term. Here then is the factoring for this problem.

$$8x^4-4x^3+10x^2=2x^2(4x^2-2x+5)$$

Note: We can always check our factoring by multiplying the terms out to make sure we get the original polynomial.

2) Factoring by Grouping: is used with polynomials that have four or more terms.

Hint: you will almost always use factoring by grouping on polynomials with degree greater than 3.

Example 2: Factor the following:

(a) $3x^2 - 2x + 12x - 8$

(b) $x^4 + x - 2x^3 - 2$

Solution

(a) $3x^2 - 2x + 12x - 8$

In this case we *group* the first two terms and the final two terms as shown here,

$$(3x^2 - 2x) + (12x - 8)$$

Now, factor an *x* out of the first grouping and a *4* out of the second grouping, which gives:

$$3x^2 - 2x + 12x - 8 = x(3x - 2) + 4(3x - 2)$$

Notice that we can factor out a common factor of *3x-2*, so the final factored form.

$$3x^2 - 2x + 12x - 8 = (3x - 2)(x + 4)$$

And we are done factoring by grouping.

Note that if we multiply our answer out, we do get the original polynomial.

(b) $x^4 + x - 2x^3 - 2$

In this case we will do the same initial step, but this time notice that both of the final two terms are negative so we will factor out a "-" as well when we group them. Doing this gives,

$$(x^4 + x) - (2x^3 + 2)$$

Again, we can always distribute the "-" back through the parenthesis to make sure we get the original polynomial.

At this point we can see that we can factor an *x* out of the first term and a 2 out of the second term. This gives,

$$x^4 + x - 2x^3 - 2 = x(x^3 + 1) - 2(x^3 + 1)$$

We now have a common factor that we can factor out to complete the problem.

$$x^4 + x - 2x^3 - 2 = (x^3 + 1)(x - 2)$$

Be careful. When the first term of the second group has a minus sign in front of it, you want to put the minus in front of the second ( ). When you do this you need to change the sign of BOTH terms of the second ( ) as shown above.

* Special Factorization Patterns:

1) Difference of Squares:

Using the technique described above, $x^2 - 16$ factors as $(x+4)(x-4)$.

This type of expression is called a "**difference of squares."**

Notice that, in this case, the "middle term" disappears when you multiply these two factors.

Special Note: In fact, a difference of squares will always factor according to the rule:

$$a^2 - b^2 = (a+b)(a-b)$$

Another example is: $9p^2 - 1 = (3p+1)(3p-1)$

Warning: When working with real numbers, a "sum of squares" will not factor according to this rule (or any other rule).

Try it on $x^2 + 9$.

Example: $25x^2 - 16y^2$

2) Difference and Sum of Cubes:

We have just seen that a difference of squares always factors. The same is true for a "difference of cubes."

For example: $y^3 - 64 = (y-4)(y^2 + 4y + 16)$

The general rule for a **"difference of cubes"** is:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Remember, you can verify this rule by multiplying out the factored form.

Example: $8x^3 - 27y^3$

Note: Although a sum of squares will not factor, a "sum of cubes" will factor, for example, $x^3 + 27 = (x+3)(x^2 - 3x + 9)$.

The general rule for a **"sum of cubes"** is:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Example: $27x^3 + 125y^3$

***Factoring Quadratic Polynomials:**

First, note that quadratic is another term for second degree polynomial. So we know that the largest exponent in a quadratic polynomial will be a 2. In these problems, we will be attempting to factor quadratic polynomials into two first degree (hence forth linear) polynomials. Until you become good at these, we usually end up doing these by trial and error although there are a couple of processes that can make them somewhat easier.

Example 3: Factor each of the following polynomials.

(a) $x^2 + 2x - 15$

(b) $x^2 - 10x + 24$

(c) $x^2 + 6x + 9$

(d) $x^2 + 5x + 1$

(e) $3x^2 + 2x - 8$

(f) $5x^2 - 17x + 6$

(g) $4x^2 + 10x - 6$

Solution:

(a) $x^2 + 2x - 15$

Since the first term is x^2 we know that the factoring must take the form.

$$x^2 + 2x - 15 = (x + \underline{\quad\quad})(x + \underline{\quad\quad})$$

We know that it will take this form because when we multiply the two linear terms the first term must be x^2, and the only way to get that to show up is to multiply *x* by *x*. Therefore, the first term in each factor must be an *x*. To finish this we just need to determine the two numbers that need to go in the blank spots.

We can narrow down the possibilities considerably. Upon multiplying the two factors out, these two numbers will need to multiply out to get -15. In other words these two numbers must be factors of -15. Here are all the possible ways to factor -15 using only integers.

$$(-1)(15) \qquad (1)(-15) \qquad (-3)(5) \qquad (3)(-5)$$

Now, we can just plug these in one after another and multiply out until we get the correct pair. However, there is another trick that we can use here to help us out. The correct pair of numbers must add to get the coefficient of the *x* term. So, in this case the third pair of factors $(-3) and (5)$ will add to "+2" and so that is the pair we are after.

Here is the factored form of the polynomial.

$$x^2 + 2x - 15 = (x-3)(x+5)$$

Again, we can always check that we got the correct answer by doing a quick multiplication.

Note that the method we used here will only work if the coefficient of the x^2 term is one. If it is anything else, this will not work and we will be back to trial and error to get the correct factoring form.

(b) $x^2 - 10x + 24$

Let's write down the initial form again,

$$x^2 - 10x + 24 = (x + \_\_\_)(x + \_\_\_)$$

Now, we need two numbers that multiply to get 24 and add to get -10. It looks like -6 and -4 are the right factors. So the factored form of this polynomial is,

$$x^2 - 10x + 24 = (x-6)(x-4)$$

(c) $x^2 + 6x + 9$

Let's start with the initial form,

$$x^2 + 6x + 9 = (x + \_\_\_)(x + \_\_\_)$$

This time we need two numbers that multiply to get 9 and add to get 6. In this case 3 and 3 will be the correct pair of numbers. **Do not forget that the two numbers can be the same number on occasion as they are here.**

Here is the factored form for this polynomial.

$$x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$$

Note as well that we further simplified the factoring to acknowledge that it is a perfect square. You should always do this when it happens.

(d) $x^2 + 5x + 1$

Once again, here is the initial form,

$$x^2 + 5x + 1 = (x + \_\_\_)(x + \_\_\_)$$

Note that, we need two numbers that multiply to get 1 and add to get 5. There aren't such two integers, so this quadratic is **not factorable**. This will happen on occasion.

(e) $3x^2 + 2x - 8$

Here, we no longer have a coefficient of 1 on the x^2 term, however, we can still make a guess as to the initial form of the factoring. Since the coefficient of the x^2 term is a 3 and there are only two positive factors of 3 there is really only one possibility for the initial form of the factoring.

$$3x^2 + 2x - 8 = (3x + \_\_)(x + \_\_)$$

Since the only way to get a 3 x^2 is to multiply a 3*x* and an *x* these must be the first two terms. However, finding the numbers for the two blanks will not be as easy as the previous examples. We will need to start off with all the factors of -8.

$$(-1)(8) \qquad (1)(-8) \qquad (-2)(4) \qquad (2)(-4)$$

At this point the only option is to pick a pair plug them in and see what happens when we multiply the terms out. Let's start with the fourth pair. Let's plug the numbers in and see what we get.

$$(3x + 2)(x - 4) = 3x^2 - 10x - 8$$

Well the first and last terms are correct, but the middle term is not. So this is not the correct factoring of the polynomial.

Now, flip the order and see what we get.

$$(3x - 4)(x + 2) = 3x^2 + 2x - 8$$

So, we got it. We did guess correctly the first time we just put them into the wrong spot.

Note that, in these problems don't forget to check both places for each pair to see if either will work.

(f) $5x^2 - 17x + 6$

Again the coefficient of the x^2 term has only two positive factors so we've only got one possible initial form.

$$5x^2 - 17x + 6 = (5x + \_\_)(x + \_\_)$$

Next we need all the factors of 6. Here they are.

$(1)(6) \quad (-1)(-6) \quad (2)(3) \quad (-2)(-3)$

Do not forget the negative factors. They are often the ones that we want. In fact, upon noticing that the coefficient of the *x* is negative we can be assured that we will need one of the two pairs of negative factors since that will be the only way we will get negative coefficient there. With some trial and error we can get that the factoring of this polynomial is, $5x^2 - 17x + 6 = (5x-2)(x-3)$

(g) $4x^2 + 10x - 6$

Here, we have a harder problem. The coefficient of the x^2 term now has more than one pair of positive factors. This means that the initial form must be one of the following possibilities:

$$4x^2 + 10x - 6 = (4x + \_\_)(x + \_\_)$$

$$4x^2 + 10x - 6 = (2x + \_\_)(2x + \_\_)$$

To fill in the blanks we will need all the factors of -6. Here they are,

$(1)(-6) \quad (-1)(6) \quad (2)(-3) \quad (-2)(3)$

With some trial and error we can find that the correct factoring of this polynomial is,

$$4x^2 + 10x - 6 = (2x-1)(2x+6)$$

Note as well that in the trial and error phase we need to make sure and plug each pair into both possible forms and in both possible orderings to correctly determine if it is the correct pair of factors or not.

We can actually go one more step here and factor a 2 out of the second term. This gives,

$$4x^2 + 10x - 6 = 2(2x-1)(x+3)$$

This is important because we could also have factored this as,

$$4x^2 + 10x - 6 = (4x-2)(x+3)$$

However, in this case we can, from the beginning, factor a 2 out of the first term to get:

$$4x^2 + 10x - 6 = 2(2x-1)(x+3)$$

CAUTION: Not every polynomial is factorable. Just like not every number has a factor other than 1 or itself. A prime number is a number that has exactly two factors, 1 and itself. 2, 3, and 5 are examples of prime numbers.

The same thing can occur with polynomials. **If a polynomial is not factorable we say that it is a prime polynomial.**

Example: $x^2 + 5x + 1$

Now try the following problems,

1) Factor completely the following polynomials:

(a) $2x^2 + 11x + 12$

(b) $2x^3 + 8x^2 + 3x + 12$

(c) $5x^3 + x^2 - 20x - 4$

(d) $10x^2 - 29x + 10$

(e) $x^2 + 4x + 4$

(f) $4x^2 - 25$

(g) $3x^2 - 12x + 12$

(h) $x^2 + 81$

(i) $8x^3 + 64y^3$

(j) $8x^3 - 64y^3$

(k) $4x^2 - 8x$

(l) $x^2 + 10xy + 25y^2$

More factoring problems:

(a) $x^2 - 10x + 25$

(b) $4x^2 + 12x + 9$

(c) $9x^2 - 4$

(d) $6x^2 + 10x - 3x - 5$

(e) $x^2 + 4x - 12$

(f) $2y^3 - 22y^2 + 48y$

(g) $2x^2 + x - 3$

(h) $12x^2 + 7x - 10$

(i) $2x^4 - 16x$

(j) $6x^4 - 8x^3 - 2x^2$

(k) $x^4 - y^4$

For more practice, please check the homework sheet.

R.4 Fractional Expressions:
1.3 Operations with Rational Expressions:

*** Equivalent fractions:**

*** Rational Expressions:**

A **rational expression** is the quotient of two polynomials.

Ex: $\frac{2}{x-3}$, $\frac{x^2-4}{x^2-4x+5}$

*** Simplifying Rational Expressions:**

NOTE: To simplify a rational expression factor completely the numerator and denominator the cancel common factors.

A rational expression is **reduced to lowest terms** if all common factors from the numerator and denominator are canceled.

Example 1a: Reduce $\frac{4}{12}$ to lowest terms

Not reduced to lowest terms $\Rightarrow \frac{4}{12} = \frac{(4)(1)}{(4)(3)} = \frac{1}{3} \Leftarrow$ reduced to lowest terms

With rational expressions it works exactly the same way.

Not reduced to lowest terms $\Rightarrow \frac{(x+3)(x-1)}{x(x+3)} = \frac{x-1}{x} \Leftarrow$ reduced to lowest terms

** We have to be careful with canceling. There are some common mistakes that students often make with these problems. Remember that in order to cancel a factor it must multiply the whole numerator and the whole denominator. So, the x+3 above could cancel since it multiplied the whole numerator and the whole denominator. However, the x's in the reduced form can not be cancelled, since the x in the numerator is not times the whole numerator.

To see why the x's don't cancel in the reduced form above put a number in and see what happens. Let's plug in x=4.

$\frac{4-1}{4} = \frac{3}{4}$ $\qquad$ $\frac{4-1}{4} = -1$ (If 4 gets canceled)

Clearly the two answers are not the same number!

Note: Only COMMON FACTORS of the numerator and denominator can be canceled.

Example 1b: Simplify the following:

$$\frac{-6axy}{8y},\ \frac{4-x}{x-4},\ \frac{2ax-2x}{2ax+2x},\ \frac{2x-3y+4z}{4ax-6ay+8az}\ ,\ \frac{ax-ay+bx-by}{x-y}$$

*** Multiplying Rational Expressions:**

1) Completely factor each numerator and denominator.
2) Multiply the numerators and multiply the denominators.
3) Simplify the result as far as possible by **cancelling common factors**.

Example 2:

$$\frac{3x+1}{2x}\cdot\frac{2x-4}{3x^2-2x-1}$$

$$=\frac{3x+1}{2x}\cdot\frac{2(x-2)}{(3x+1)(x-1)}$$

$$=\frac{(3x+1)2(x-2)}{2x(3x+1)(x-1)}$$

$$=\frac{x-2}{x(x-1)}$$

Example 3: $\frac{x^2-2xy}{3x+3y}\cdot\frac{x^2-y^2}{xy-2y^2}$

- **Dividing Rational Expressions:**

1) Completely factor each numerator and denominator.
2) Change to multiplication.
3) Invert (flip) the second fraction and proceed as in multiplication.

Example 4:

$$\frac{x^2-16}{x^2+5x+4}\div\frac{x-4}{x^2-3x-4}$$

$$=\frac{(x-4)(x+4)}{(x+4)(x+1)}\div\frac{x-4}{(x-4)(x+1)}$$

$$=\frac{(x-4)(x+4)}{(x+4)(x+1)}\cdot\frac{(x+1)(x-4)}{x-4}=\frac{(x-4)(x+4)(x+1)(x-4)}{(x+4)(x+1)(x-4)}=x-4$$

*** Finding the Least Common Denominator (LCD):**

1) Completely factor each **denominator.**
2) The LCD is the product of all unique factors each raised to the greatest power that appears in any factored denominator.

Example 5:

1) $\frac{2}{3x^5y^2}, \frac{3z}{5xy^3}$

2) $\frac{z}{z-1}, \frac{7}{z+1}$

3) $\frac{7}{m^2-10m+25}, \frac{2m}{2m^2-9m-5}, \frac{m-1}{m^2-25}$

4) $\frac{x}{x^2-4}, \frac{x}{6-3x}$

Hint: If **opposite factors** occur, do not use both in the LCD. Instead, factor -1 from one of the opposite factors so that the factors are then identical.

Ex: If you have factors like *x-2* and *2-x*, these are called **opposite factors.** Notice that you can factor a *-1 from 2-x* so that the factors are identical.

- **Adding or Subtracting Rational Expressions:**

1) With the same denominator:

If the denominators are the same, keep the same denominator just add numerators. Then simplify if possible.

Example 6:

$$\frac{x}{x^2+5x+4} + \frac{2x+3}{x^2+5x+4}$$

$$= \frac{x+2x+3}{x^2+5x+4} = \frac{3x+3}{x^2+5x+4} = \frac{3(x+1)}{(x+4)(x+1)} = \frac{3}{x+4}$$

Example 7:

$$\frac{x^2}{x+7} - \frac{49}{x+7}$$

$$= \frac{x^2-49}{x+7} = \frac{(x+7)(x-7)}{x+7} = x-7$$

2) **With different denominators: (LCD is NEEDED)**
   a. Factor completely each denominator.
   b. Find the LCD of the rational expression.
   c. Write each rational expression as an equivalent rational expression whose denominator is the LCD found in step (b)
   d. Add or subtract numerators, and write the result over the common denominator.
   e. Simplify the resulting rational expression if possible.

Example 8:

a) $\dfrac{5x}{x^2-4}-\dfrac{2}{x^2+x-2}$

b) $\dfrac{3}{x+2}+\dfrac{2x}{x-2}$

c) $\dfrac{2x-1}{2x^2-9x-5}+\dfrac{x+3}{6x^2-x-2}$

d) $\dfrac{3}{x^2-9}-\dfrac{x}{x^2-6x+9}+\dfrac{1}{x+3}$

e) $\dfrac{2x-6}{x-1}-\dfrac{4}{1-x}$

* **Complex Fractions:**

A complex fraction is a rational expression whose numerator, denominator, or both contain one or more rational expressions. Examples are

$$\frac{\frac{1}{a}}{\frac{b}{2}} \qquad \frac{\frac{x}{2y^2}}{\frac{6x-2}{9y}} \qquad \frac{x+\frac{1}{y}}{y+1} \qquad \frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$$

They can be simplified by treating the numerator and the denominator as separate problems.

Then we have a "division" problem.

For example, to simplify $\dfrac{x-\frac{9}{x}}{4x-12}$,

we first complete the subtraction problem contained in the numerator of the entire fraction: $x-\dfrac{9}{x}=\dfrac{x^2-9}{x}$.

Now we have the division problem: $\dfrac{\left(\dfrac{x^2-9}{x}\right)}{\left(\dfrac{4x-12}{1}\right)}$

We invert and multiply: $\left(\dfrac{x^2-9}{x}\right)\left(\dfrac{1}{4x-12}\right)$.

As before, we should now factor in order to reduce:

$$\left(\frac{(x+3)(x-3)}{x}\right)\left(\frac{1}{4(x-3)}\right)$$

Now cancel the common factor "$(x-3)$."

So our final answer in factored form, reduced to lowest terms is:

$$\frac{x+3}{4x}.$$

Remember, you do not need a common denominator when multiplying (or dividing) fractions.

Steps for simplifying complex fractions:

1) Simplify the numerator and the denominator of the complex fraction so that each is a single fraction.
2) Perform the indicated division by multiplying the numerator of the complex fraction by the reciprocal of the denominator of the complex fraction.
3) Simplify if possible.

Example 9: Simplify the following:

a) $\dfrac{\dfrac{x}{3y^2}}{\dfrac{6x-2}{9y}}$

b) $\dfrac{\dfrac{x}{y^2}+\dfrac{1}{y}}{\dfrac{y}{x^2}+\dfrac{1}{x}}$

c) $\dfrac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$

d) $\dfrac{x+\dfrac{1}{y}}{y+1}$

Solution:

a) $\dfrac{\dfrac{x}{3y^2}}{\dfrac{6x-2}{9y}} = \dfrac{x}{3y^2} \div \dfrac{6x-2}{9y} = \dfrac{x}{3y^2} \cdot \dfrac{9y}{6x-2} = \dfrac{3x}{y(6x-2)}$

b) First we simplify the numerator and denominator separately so that each is a single fraction.

$\dfrac{\dfrac{x}{y^2}+\dfrac{1}{y}}{\dfrac{y}{x^2}+\dfrac{1}{x}} = \dfrac{\dfrac{x}{y^2}+\dfrac{1.y}{y.y}}{\dfrac{y}{x^2}+\dfrac{1.x}{x..x}}$ Note the LCD of the fractions that are in the numerator is y^2 and the LCD of the fractions that are in the denominator is x^2.

$= \dfrac{\dfrac{x+y}{y^2}}{\dfrac{y+x}{x^2}} = \dfrac{x+y}{y^2} \cdot \dfrac{x^2}{y+x} = \dfrac{x^2}{y^2}$

Now, try the following problems:

1. Perform the indicated operations and simplify your answers.

(a) $\dfrac{x}{x-3}+\dfrac{3}{3-x}$

(b) $\dfrac{1}{4x^2}-\dfrac{2x+1}{3x^3}+\dfrac{3}{12x}$

(c) $\dfrac{y-3}{y^2-4}-\dfrac{y+2}{y^2-4y+4}-\dfrac{2}{2-y}$

(d) $\dfrac{4}{2x-1}\cdot\dfrac{10x-5}{16}$

(e) $\dfrac{x+1}{x-x^2}\cdot\dfrac{x^2-2x+1}{x^2-1}$

(f) $\dfrac{4x^2-4x+1}{2x^2+5x-3}\div\dfrac{2x^2-3x-2}{2x^2+7x+3}$

(g) $\dfrac{\dfrac{x^2}{y^2}-1}{\dfrac{x}{y}+1}$

(h) $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

(i) $\dfrac{a+b^{-1}}{b+a^{-1}}$

1.4 Rational Exponents

We have already discussed integer exponents in Section P.5. We can also define exponents that are rational.

We use "rational (i.e. fractional) exponents" to represent radicals.

*** Definition of $a^{1/n}$:**

If *a* is a real number and *n* is a positive integer greater than 1, then

$$a^{1/n} = \sqrt[n]{a}, \quad \text{where } a \geq 0 \text{ when } n \text{ is even.}$$

The quantity $a^{1/n}$ is called the **nth root of *a*.**

Consider the following examples:

$\sqrt{x}$ is rewritten as $x^{1/2}$

$\sqrt[3]{x}$ is rewritten as $x^{1/3}$

Example 1: Evaluate the following.

(a) $27^{1/3}$

(b) $-36^{1/2}$

(c) $(-8)^{1/3}$

Solution:

(a) Applying the definition of $a^{1/n}$ with *n=3,*

$$27^{1/3} = \sqrt[3]{27} = 3$$

(b) Applying the definition of $a^{1/n}$ with *n=2*

$-36^{1/2} = -\left(\sqrt{36}\right) = 6$. Note that only the number 36 is raised to the $1/2$ power, the result is multiplied by *-1*.

(c) $(-8)^{1/3}$, Try it on your own.

*** Definition of $a^{m/n}$:**

$\sqrt[n]{a^m}$ is rewritten as $a^{m/n}$.

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Notice that, **in a fractional exponent, the denominator identifies the root while the numerator identifies the power.**

Example 2: Evaluate the following.

(a) $(4)^{3/2}$

(b) $(25)^{-3/2}$

(c) $(16)^{5/4}$

Solution:

(a) Using the definition of rational exponents,

$$(4)^{3/2} = \left(\sqrt{4}\right)^3 = (2)^3 = 8$$

(b) To compute $(25)^{-3/2}$,

First remember that a negative exponent means "flip the base:"

$$(25)^{-3/2} = \frac{1}{(25)^{3/2}}$$

Now convert to radical notation:

$$= \frac{1}{\left(\sqrt{25}\right)^3}$$

Remember that it's usually easier to compute the root before you compute the power.

So, as our final answer we have:

$$= \frac{1}{(5)^3} = \frac{1}{125}$$

(c) $(16)^{5/4}$, Try it on your own.

The next thing that we should acknowledge is that all of the properties for exponents that we gave in Section P.5 are still valid for all rational exponents.

* Properties of Rational Exponents:

The basic rules of rational exponents are similar to those of integer exponents. If m and n are rational numbers then the following rules hold:

Rule 1: $x^m x^n = x^{m+n}$

Example: (a) $x^{1/2} x^{1/3} = x^{1/2 + 1/3} = x^{5/6}$

(b) $(x+3)^{5/2}(x+3)^{-1/2} = (x+3)^{5/2 - 1/2} = (x+3)^{{5-1}/2} = (x+3)^2$

That is, **when you multiply like bases, you add the exponents.**

Rule 2: $\dfrac{x^m}{x^n} = x^{m-n}$

Example: $\dfrac{x^{4/3}}{x^{1/3}} = x^{4/3 - 1/3} = x^1$

That is, **when you divide like bases, you subtract the exponents.**

Rule 3: $(x^m)^n = x^{mn}$

Example: $\left(x^{3/8}\right)^4 = x^{3/8 \cdot 4} = x^{12/8} = x^{3/2}$

That is, **when you raise a power to a power, you multiply the exponents.**

****

Rule 4: $x^{-m} = \dfrac{1}{x^m}$

Example: $x^{-4/7} = \dfrac{1}{x^{4/7}}$

That is, **negative exponents result in "flipping fractions over."**

Rule 5: $(xy)^n = x^n y^n$

That is, the *n*th power of a product is equal to the product of the *n*th powers of the factors.

Example: $(x^2 y^4)^{1/6} = x^{2/6} y^{4/6} = x^{1/3} y^{2/3}$

Note that many of these properties were given with only two terms/factors but they can be extended out to as many terms/factors as we need. For example, rule 5 can be extended as follows.

$$(xyz)^n = x^n y^n z^n$$

Warning: DO NOT distribute exponents across addition or subtraction!

For example, $(x+y)^{1/4}$ **DOES NOT EQUAL** $x^{1/4} + y^{1/4}$!

Rule 6: $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

That is, the *n*th power of a quotient is equal to the quotient of the *n*th powers.

Example: $\left(\frac{x}{8}\right)^{1/3} = \frac{x^{1/3}}{8^{1/3}} = \frac{x^{1/3}}{2}$

Warning: DO NOT distribute exponents across addition or subtraction!

For example, $\left(\frac{2x+1}{3x+4}\right)^{3/4}$ **DOES NOT EQUAL** $\frac{2x^{3/4}+1^{3/4}}{3x^{3/4}+4^{3/4}}$!

---

*Zero exponents:

Rule 7: $x^0 = 1$

Example: $(2y)^0 = 1$ **and** $\left(\frac{2x+1}{3x+4}\right)^0 = 1$

Try these on your own.

1. Simplify the following and write the answer in terms of positive exponents:

(a) $\left(27x^3\right)^{2/3}$ **(b)** $\left(16x^8y^{-4}\right)^{1/4}$

(c) $\left(x^4y^4\right)^{1/2}$ **(d)** $3x^{2/3}y^{3/4}(2x^{5/3}y^{1/2})^3$

(e) $\dfrac{\left(8x^2y^{2/3}\right)^{2/3}}{2\left(x^{3/4}y\right)^3}$ **(f)** $\left(16a^{4/3}b^{-3}\right)^{3/2}$

(g) $\dfrac{\left(x^{-7/3}y^{5/2}\right)^3}{y^{-1/2}x^2}$ **(h)** $\left(\dfrac{x^{-1/3}y^{1/2}}{x^{-1/4}y^{1/3}}\right)^6$

(i) $\sqrt[3]{\dfrac{\left(7^{-2}y^{-6}\right)^6}{5^{-2}x^4}}$

2. Express the following in terms of rational exponents.

(a) $\left(\sqrt{(x+1)^3}\right)^5$

(b) $\sqrt{\dfrac{x-1}{x-2}}\left(\sqrt{(x-1)(x-2)}\right)^3$

For more practice, please check the homework problems.

1.5 Operations with Radicals

Radicals and Rational exponents:
Radicals can be rewritten by using exponents, and exponents can be rewritten by using radicals according to a special rule of notation.

That is, **we use "rational (i.e. fractional) exponents" to represent radicals and visa versa.**

Consider the following examples:

$\sqrt{x}$ is rewritten as $x^{\frac{1}{2}}$

$\sqrt[3]{x}$ is rewritten as $x^{\frac{1}{3}}$

$\sqrt[5]{x^2}$ is rewritten as $x^{\frac{2}{5}}$

In general, $\sqrt[n]{x^m}$ is rewritten as $x^{\frac{m}{n}}$.

Keep in mind that, **in a fractional exponent, the denominator identifies the root while the numerator identifies the power.**

Example: Express each of the following in terms of rational exponents.

(a) $\sqrt{x^2+1}$

(b) $\sqrt[3]{(x-2)^2}$

(c) $\left(\sqrt{(x+1)^3}\right)^5$

(d) $\sqrt[5]{\sqrt{x-1}}$

(e) $\sqrt{\sqrt{\sqrt{x+1}}}$

Solution:

(a) $\sqrt{x^2+1}=(x^2+1)^{\frac{1}{2}}$ **(b)** $\sqrt[3]{(x-2)^2}=(x-2)^{\frac{2}{3}}$ **(c)** $\left(\sqrt{(x+1)^3}\right)^5=(x+1)^{\frac{15}{2}}$

(d) $\sqrt[5]{\sqrt{x-1}}=\left(\ (x-1)^{\frac{1}{2}}\ \right)^{\frac{1}{5}}=(x-1)^{\frac{1}{10}}$ **(e)** $\sqrt{\sqrt{\sqrt{x+1}}}=(x+1)^{\frac{1}{8}}$

In this section, we will be adding, subtracting, multiplying and dividing algebraic expressions containing radicals.

*__Adding or Subtracting Radical expressions:__

To add or subtract radical expressions we have to combine **like radicals.**

Like radicals: are radicals with the same index and the same radicand.

Note: When adding or subtracting radicals, always check first whether any radicals can be simplified.

Example 1: Perform the indicated operation, then simplify the following:

(a) $4\sqrt{11}+8\sqrt{11}=(4+8)\sqrt{11}=12\sqrt{11}$

(b) $5\sqrt[3]{3x}-7\sqrt[3]{3x}=(5-7)\sqrt[3]{3x}=-2\sqrt[3]{3x}$

(c) $2\sqrt{7}+2\sqrt[3]{7}$ Cannot be simplified, since $2\sqrt{7}$ and $2\sqrt[3]{7}$ do not contain like radicals.

(d) $\sqrt{20}+2\sqrt{45}=$

$$= \sqrt{4.5}+2\sqrt{9.5}$$
$$= \sqrt{4}\cdot\sqrt{5}+2\cdot\sqrt{9}\cdot\sqrt{5}$$
$$= 2\cdot\sqrt{5}+2\cdot 3\cdot\sqrt{5}$$
$$= 2\sqrt{5}+6\sqrt{5}$$
$$= 8\sqrt{5}$$

(e) $\sqrt[3]{8y^5}-\sqrt[3]{27y^5}$

(f) $2\sqrt{x-1}+3\sqrt{(x-1)^3}$

*__Multiplying or Dividing Radical expressions:__

To multiply or divide expressions containing radicals:

1) Convert to Rational exponents.
2) Apply the rules of rational exponents.

Example 2: Perform the indicated operation, then simplify the following:

(a) $\sqrt{x-1}\sqrt[4]{(x-1)^3}$ **(b)** $\dfrac{\sqrt[4]{(x-1)^3}}{\sqrt{x-1}}$ **(c)** $\sqrt{\dfrac{a+b}{a-b}}\left(\sqrt{a+b}\right)^3\sqrt{(a-b)^3}$

Solution:

(a) $\sqrt{x-1}\sqrt[4]{(x-1)^3}=(x-1)^{1/2}(x-1)^{3/4}$ **Convert to rational exponents**

$=(x-1)^{1/2+3/4}$ **Same base, Add exponents**

$=(x-1)^{5/4}$ **Answer in exponent form**

$=(x-1)^{1+1/4}$

$=(x-1)(x-1)^{1/4}$

$=(x-1)\sqrt[4]{x-1}$ **Answer in simplified radical form**

(b) $\frac{\sqrt[4]{(x-1)^3}}{\sqrt{x-1}} = \frac{(x-1)^{3/4}}{(x-1)^{1/2}}$ **Convert to rational exponents**

$= (x-1)^{3/4-1/2}$ **Same base, Subtract exponents**

$= (x-1)^{1/4}$ **Answer in exponent form**

$= \sqrt[4]{x-1}$ **Answer in simplified radical form**

(c) $\sqrt{\frac{a+b}{a-b}}\left(\sqrt{a+b}\right)^3\sqrt{(a-b)^3} = \frac{(a+b)^{1/2}}{(a-b)^{1/2}}(a+b)^{3/2}(a-b)^{3/2}$ **Convert to rational exponents**

Then look at the same base and determine whether you have to add or subtract exponents.

$= (a+b)^{1/2+3/2}(a-b)^{3/2-1/2}$

$= (a+b)^2(a-b)$ **Answer in simplified form**

- **To multiply** $(\sqrt{x}+2)(\sqrt{x}+2)$**, we use FOIL.**
- **To multiply** $x(\sqrt{x}+2)$**, we use the distributive property.**
  Check class notes.

Example: Multiply ($\sqrt{x}+\sqrt{y}$ **)and (** $\sqrt{x}-\sqrt{y}$ **)**

($\sqrt{a}-b$) and ($\sqrt{a}+b$)

*** Rationalizing Denominators:**

The process of writing an expression without a radical in the denominator is called **rationalizing the denominator.**

A) Rationalizing Denominators Having One Term:

$\frac{1}{\sqrt[3]{x}}$, $\frac{7}{\sqrt{3y}}$, $\frac{1}{\sqrt{x-1}}$ are expressions with one radical term in the denominator.

To rationalize the denominator of expressions with one radical term in the denominator, you need to multiply by whatever makes the denominator a **perfect square** or **perfect cube or any other power** that can be simplified.

NOTE: Make sure you multiply by whatever makes the *radicand* the smallest possible value to be simplified. This will avoid having to further simplify later on.

Example 3: Rationalize the denominator.

(a) $\dfrac{1}{\sqrt{x}}$ **(b)** $\dfrac{3}{\sqrt[3]{4x^2}}$ **(c)** $\dfrac{1}{\sqrt{x-1}}$

Solution:

(a) $\dfrac{1}{\sqrt{x}}$

Multiply by a value that will create the smallest perfect square under the radical. This will prevent the need for additional simplifications.

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

Choosing to multiply by $\sqrt{x}$ will create the smallest perfect square under the radical in the denominator.

$$= \frac{\sqrt{x}}{\sqrt{x^2}} = \frac{\sqrt{x}}{x}$$

Replacing $\sqrt{x^2}$ by x, rationalizes the denominator.

(b) $\dfrac{3}{\sqrt[3]{4x^2}}$

Again, Multiply by a value that will create the smallest perfect cube under the radical. This will prevent the need for additional simplifications.

$$\frac{3}{\sqrt[3]{4x^2}} = \frac{3}{\sqrt[3]{4x^2}} \cdot \frac{\sqrt[3]{2x}}{\sqrt[3]{2x}}$$

Choosing to multiply by $\sqrt[3]{2x}$ will create the smallest perfect cube under the radical in the denominator.

$$= \frac{3}{\sqrt[3]{4x^2}} \cdot \frac{\sqrt[3]{2x}}{\sqrt[3]{2x}} = \frac{3\sqrt[3]{2x}}{\sqrt[3]{8x^3}} = \frac{3\sqrt[3]{2x}}{2x}$$

Replacing $\sqrt[3]{8x^3}$ by $2x$, rationalizes the denominator.

(c) $\frac{1}{\sqrt{x-1}}$

Since the Denominator is Just a Single Radical, Multiply the numerator and denominator by the denominator.

So, we need to multiply the numerator and the denominator by $\sqrt{x-1}$

$$\frac{1}{\sqrt{x-1}} = \frac{1}{\sqrt{x-1}} \cdot \frac{\sqrt{x-1}}{\sqrt{x-1}} = \frac{\sqrt{x-1}}{x-1}$$

B) Rationalizing Denominators Having Two Terms: $\frac{x}{\sqrt{x}+\sqrt{y}}$, $\frac{3}{2-\sqrt{x}}$, $\frac{1}{\sqrt{x-1}-\sqrt{x+1}}$ are expressions with two radical terms in the denominator.

When there is more than one term in the denominator, the process is a little tricky. You will need to multiply the denominator by its **conjugate**. The **conjugate** is the same expression as the denominator but with the **opposite sign in the middle.**

Examples of conjugates are:

$\sqrt{x}+\sqrt{y}$ **and** $\sqrt{x}-\sqrt{y}$

$\sqrt{a}-b$ **and** $\sqrt{a}+b$

Example 4: Rationalize the denominator.

(a) $\frac{x}{\sqrt{x}+\sqrt{y}}$ **(b)** $\frac{3}{2-\sqrt{x}}$ **(c)** $\frac{1}{\sqrt{x-1}-\sqrt{x+1}}$

Solution:

(a) $\frac{x}{\sqrt{x}+\sqrt{y}}$

Since the denominator has two terms, we need to multiply the numerator and denominator by the conjugate of the denominator. When multiplying the numerators in this problem, use the **distributive property**. When multiplying the denominators, use **FOIL.**

The conjugate $\sqrt{x}+\sqrt{y}$ **is** $\sqrt{x}-\sqrt{y}$**.**

$$\frac{x}{\sqrt{x}+\sqrt{y}} = \frac{x}{\sqrt{x}+\sqrt{y}} \cdot \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

Multiply top and bottom by the conjugate of the denominator, $\sqrt{x}-\sqrt{y}$. Notice that you are multiplying by 1, which does not change the original expression.

$$= \frac{x\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)}$$

$$=\frac{x\sqrt{x}-x\sqrt{y}}{\sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{y}+\sqrt{x}\sqrt{y}-\sqrt{y}\sqrt{y}}$$

Notice what is happening to the middle terms when you multiply the denominators. **The middle terms will drop out.** Also, the last term has created a perfect square under the square root.

$$=\frac{x\sqrt{x}-x\sqrt{y}}{x-y}$$

Note: Be sure to enclose expressions with multiple terms in ( ). This will help you to remember to FOIL these expressions.

(b) $\frac{3}{2-\sqrt{x}}$

$$\frac{3}{2-\sqrt{x}}=\frac{3}{2-\sqrt{x}}\cdot\frac{2+\sqrt{x}}{2+\sqrt{x}}=\frac{3\left(2+\sqrt{x}\right)}{\left(2-\sqrt{x}\right)\left(2+\sqrt{x}\right)}=\frac{6+3\sqrt{x}}{4+2\sqrt{x}-2\sqrt{x}-\sqrt{x}\sqrt{x}}$$

$$=\frac{6+3\sqrt{x}}{4-x}$$

So, we took the original denominator and changed the sign on the second term and multiplied the numerator and denominator by this new term $2+\sqrt{x}$. By doing this we were able to eliminate the radical in the denominator when we then multiplied out.

(c) $\frac{1}{\sqrt{x-1}-\sqrt{x+1}}$

Again, Multiply top and bottom by the conjugate of the denominator, $\sqrt{x-1}+\sqrt{x+1}$.

$$\frac{1}{\sqrt{x-1}-\sqrt{x+1}}\cdot\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}$$

Notice that you do not change the signs under each radical.

Multiply numerators then multiply denominators. (Check class notes)

In general, the product of an expression and its conjugate will contain no radical terms.

**** Final Note:** Rationalizing the denominator may seem to have no real uses, however, if you are on a track that will take you into a Calculus class you will find that rationalizing is useful on occasion at that level.

1. Perform the indicated operations and simplify. State your answers in radical notation.

(a) $3\sqrt{a^4}+4\sqrt[3]{8a^6}$ **(b)** $\frac{1}{\sqrt{a}}-\frac{2\sqrt{a}}{a}$ **(c)** $\frac{1}{\sqrt{x}-1}-\frac{1}{\sqrt{x}+1}$

2. Rationalize the denominator in each of the following.

(a) $\frac{5}{\sqrt{5x}}$ **(b)** $\frac{x}{\sqrt[3]{x}}$ **(c)** $\frac{1}{\sqrt{x}-1}$ **(d)** $\frac{1}{\sqrt{x+3}-\sqrt{x}}$