REPRESENTATIONS AND ACTIONS OF HOPF ALGEBRAS

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ABSTRACT

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The larger area of my thesis is Algebra; more specifically, my work belongs to the following two major branches of Algebra: ***representation theory*** and ***invariant theory***. In brief, the objective of representation theory is to investigate algebraic objects through their actions on vector spaces; this allows the well-developed toolkit of linear algebra to be brought to bear on complex algebraic problems. The theory has played a crucial role in nearly every subdiscipline of pure mathematics. Outside of pure mathematics, representation theory has been successfully used, for instance, in the study of symmetries of physical systems and in describing molecular structures in physical chemistry. Invariant theory is another classical algebraic theme permeating virtually all areas of pure mathematics and some areas of applied mathematics as well, notably coding theory. The theory studies actions of algebraic objects, traditionally groups and Lie algebras, on algebras, that is, vector spaces that are equipped with a multiplication.

The representation theory of (associative) algebras provides a useful setting in which to study many aspects of the two most classical flavors of representation theory under a common umbrella: representations of groups and of Lie algebras. However, it turns out that general algebras fail to capture certain features of group representations and the same can be said for representations of Lie algebras as well. The additional structure that is needed in order to access these features is naturally provided by the important class of ***Hopf algebras***. Besides unifying the representation theories of groups and of Lie algebras, Hopf algebras serve a similar purpose

in invariant theory, allowing for a simultaneous treatment of group actions (by automorphisms) and Lie algebras (by derivations) on algebras. More importantly, actions of Hopf algebras have the potential of capturing additional aspects of the structure of algebras they act on, uncovering features that cannot be accessed by ordinary groups or Lie algebras.

Presently, the theory of Hopf algebras is still nowhere near the level that has been achieved for groups and for Lie algebras over the course of the past century and earlier. This thesis aims to make a contribution to the representation and invariant theories of Hopf algebras, focusing for the most part on Hopf algebras that are not necessarily finite dimensional. Specifically, the contributions presented here can be grouped under two headings:

(i) **Invariant Theory:** Hopf algebra actions and prime spectra, and

(ii) **Representation Theory:** the adjoint representation of a Hopf algebra.

In the work done under the heading (i), we were able to use the action of cocommutative Hopf algebras on other algebras to "stratify" the prime spectrum of the algebra being acted upon, and then express each stratum in terms of the spectrum of a commutative domain. Additionally, we studied the transfer of properties between an ideal in the algebra being acted upon, and the largest sub-ideal of that ideal, stable under the action. We were able to achieve results for various families of acting Hopf algebras, namely ***cocommutative*** and ***connected*** Hopf algebras.
The main results concerning heading (ii) concerned the subalgebra of locally finite elements of a Hopf algebra, often called the finite part of the Hopf algebra. This is a subalgebra containing the center that was used successfully to study the ring theoretic properties of group algebras, Lie algebras, and other classical structures. We prove that the finite is not only a subalgebra, but a coideal subalgebra in general, and in the case of (almost) cocommuative Hopf algebra, it is indeed a Hopf subalgebra. The results in this thesis generalize earlier theorems that were proved for the prototypical special classes of Hopf algebras: group algebras and enveloping algebras of Lie algebras.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Martin Lorenz, for his constant aid and guidance during my time at Temple. I am very happy to have had the opportunity to learn from someone with such an elegant mind and a generous character. Additionally I would like to thank the Temple math department. I was lucky to be part of such an accepting and supportive community, I will miss you all.

To Maria and Maher

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

1.1.1 Representation theory and invariant theory

The objective of Representation Theory is to investigate algebraic objects through their linear actions on vector spaces; this allows the well-developed toolkit of linear algebra to be brought to bear on more complex algebraic problems. The theory has played a crucial role in nearly every subdiscipline of pure mathematics. Outside of pure mathematics, Representation Theory has been successfully used, for instance, in the study of symmetries of physical systems [51] and in describing molecular structures in physical chemistry [12].

Invariant Theory is another classical algebraic theme permeating virtually all areas of pure mathematics and some areas of applied mathematics as well, notably coding theory; see, e.g., [50] and the references therein. The theory studies actions of algebraic objects, traditionally groups or Lie algebras, on algebras, that is, vector spaces that are equipped with a multiplication. Depending on the algebra under consideration, the theory has a strong geometric component. For example, the ordinary polynomial algebra $\Bbbk[x_1, \ldots, x_n]$ is closely tied to affine n-space $\Bbbk^n$ and group actions on $\Bbbk[x_1, \ldots, x_n]$ are thought of as symmetries of affine space.

We now give some brief background on the main "flavors" of Representation

Theory and Invariant Theory insofar as they are relevant to this thesis.

1.1.2 Group representation theory

Historically, the first application of representation theory in its current form was to the study of groups. In detail, a ***representation*** of a group G over a field $\Bbbk$ is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V), \quad g \mapsto g_V,$$

where V is a $\Bbbk$-vector space and $\mathrm{GL}(V)$ denotes the group of invertible linear transformations of V. The dimension of V, which may be infinite, is called the ***degree*** of the representation. The representation ρ is called ***irreducible*** if $V \neq 0$ and no subspace of V other than 0 and V itself is stable under the transformations g_V for all $g \in G$. Such representations play a crucial role in group representation theory. Indeed, if the group G is finite and the characteristic of the base field $\Bbbk$ does not divide the order $|G|$, then every representation of G over $\Bbbk$ can be decomposed, essentially uniquely, into a direct sum of irreducible representations. This reduces the problem of describing all representations of G over $\Bbbk$ to the case of irreducible representations. The following key result from classical group representation theory, due to Frobenius [17], severely narrows the possibilities for the irreducible representations of G.

Frobenius' Theorem. Let G be a finite group and let $\Bbbk$ be an algebraically closed field whose characteristic does not divide the order of G. Then the degree of every irreducible representation of G over $\Bbbk$ is a divisor of the order of G.

While the complete description of all irreducible representations of a given finite group is generally still a formidable task, this has in fact been achieved for many groups of great interest. Foremost amount them are the symmetric groups, where a description of the irreducible representations can be given in combinatorial terms using the so-called Young graph of partitions; see Okounkov and Vershik [42], [53] (who elaborate on earlier work of Young [58]). Additionally, it is worth noting that Representation Theory was the main tool used in the proof of the celebrated Classification Theorem of finite simple groups.

1.1.3 Invariant theory of groups

Invariant Theory, in algebraic terms, is concerned with study of the relationship between an algebra A on which a group G acts by algebra automorphisms. The main goal is to describe the set of all fixed points in A under the action of G; this is in fact a subalgebra of A, called the subalgebra of ***G-invariants*** and denoted by A^G. The most traditional setting of invariant theory arises from a representation $\rho\colon G \to \mathrm{GL}(V)$ of G as above. The representation ρ gives rise to an action of G by algebra automorphisms on the symmetric algebra $A = S(V)$. If $\dim_{\Bbbk} V = n$, then a choice of basis for V yields an isomorphism $S(V) \cong \Bbbk[x_1, \ldots, x_n]$. This type of action is commonly called a ***linear action*** and the resulting algebra of invariants $S(V)^G$ is often referred to as an algebra of ***polynomial invariants***. The ring theoretic properties of polynomial invariants have been thoroughly explored, especially for finite groups G. Early work of Hilbert [21] and of E. Noether [41] established that $S(V)^G$ is an integrally closed affine domain over $\Bbbk$ and $S(V)$ is a finitely generated $S(V)^G$-module. One of the most celebrated results on polynomial invariants is the following.

Shephard-Todd-Chevalley Theorem ([48], [11]). Suppose that the finite group G acts linearly on the polynomial algebra $S(V) \cong \Bbbk[x_1, \ldots, x_n]$ and that the characteristic of $\Bbbk$ does not divide the order of G. Then the invariant algebra $S(V)^G$ is a polynomial algebra over $\Bbbk$ precisely if G acts as a pseudoreflection group on V.

Here, an element $g \in G$ is called a pseudoreflection on V if the linear transformation $\mathrm{Id}_V - g_V$ of V has rank at most 1; the group G acts as a pseudoreflection group on V if G can be generated by pseudoreflections on V.

1.1.4 From groups and other structures to algebras

As has first been observed by Emmy Noether, group representation theory can be embedded into the more general representation theory of associative algebras. This is accomplished by associating to each group G and each base field $\Bbbk$ an associative algebra, the so-called group algebra $\Bbbk G$. The precise definition of $\Bbbk G$, while not difficult, is omitted here, but we do at least mention that the operative fact concerning the group algebra $\Bbbk G$, in the context of representation theory, is that its representations of $\Bbbk G$ are in natural one-to-one correspondence with the representations of G over $\Bbbk$. In particular, irreducible representations of G over $\Bbbk$ correspond to irreducible $\Bbbk G$-modules in the usual ring-theoretic sense. Here, a ***representation*** of a $\Bbbk$-algebra A is a homomorphism of $\Bbbk$-algebras

$$\rho\colon A \to \operatorname{End}_{\Bbbk}(V), \quad a \mapsto a_V,$$

where V is a $\Bbbk$-vector space. Representations of A can equivalently be described in the language of left A-modules.

Similar reductions to the case of algebras exist for the representations of other algebraic structures as well. For example, in the case of a Lie algebra $\mathfrak{g}$, the algebra in question is the so-called enveloping algebra of $\mathfrak{g}$; for a quiver Γ, the vehicle is the path algebra of Γ.

1.1.5 Hopf algebras

While the representation theory of associative algebras provides a useful setting in which to study many aspects of group representation theory, it turns out that general associative algebras fail to capture certain features of group representations, and the same can be said for representations of Lie algebras as well. Additional structure is needed in order to access these features, and this structure is naturally provided by the important class of ***Hopf algebras***. The formal definition of Hopf algebras is famously unwieldy and will be discussed in more detail below. In brief, a Hopf $\Bbbk$-algebra is a $\Bbbk$-algebra H—so there is a multiplication and unit—but there are three additional structure maps: the comultiplication $\Delta\colon H \to H \otimes H$, the counit $\epsilon\colon H \to \Bbbk$, and the antipode $\mathrm{S}\colon H \to H$. All these maps must satisfy certain axioms, which we will spell out in the next chapter. For example, there is the coassociativity axiom for the comultiplication, which dualizes (in the category-theoretic sense of reversing arrows) the familiar associativity axiom for the multiplication.

A remarkable feature of the Hopf algebra axioms is their self-duality. This makes it possible to equip the linear dual $H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ of any finite-dimensional Hopf algebra H with the structure of a Hopf algebra by employing the transposes of the structure maps of H as the structure maps of H^*.

The comultiplication, counit and antipode of a Hopf algebra H impart additional structure on the category of representations of H; this category will be denoted by $\mathfrak{Rep}\, H$. Specifically, the comultiplication Δ allows to form the tensor product representation $V \otimes W$ of any two given $V, W \in \mathfrak{Rep}\, H$ and the counit ϵ yields a distinguished object of $\mathfrak{Rep}\, H$, the so-called trivial representation $\mathbb{1}$. With these constructions, $\mathfrak{Rep}\, H$ becomes a ***monoidal category***. Furthermore, using the antipode S, the linear dual $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ of any $V \in \mathfrak{Rep}\, H$ becomes a representation of H in its own right.

Besides unifying the representation theories of groups and of Lie algebras, Hopf algebras serve a similar purpose in invariant theory, allowing for a simultaneous treatment of group actions (by automorphisms) and Lie algebras (by derivations) on algebras. In addition, due to the relevance of some aspects of Hopf algebras in theoretical physics, certain Hopf algebras are now commonly referred to as ***quantum groups***. The resulting invariant theory, often called ***quantum invariant theory***, is one of the main research areas in the current mainstream of noncommutative algebra. Furthermore, monoidal categories with a notion of a dual are presently the subject of intense investigation, in part due to the fact that they provide the mathematical underpinnings of the emerging theory of ***quantum computing***; see [46], [16]. Hopf algebras feature prominently in this endeavor. However, the theory of Hopf algebras is still nowhere near the level that has been achieved for groups and for Lie algebras. For example, the generalization of Frobenius' Theorem to the context of Hopf algebra is a longstanding open conjecture due to Kaplansky [24] and the Shephard-Todd-Chevalley Theorem also awaits its proper extension. The ring-theoretic properties of Hopf algebras, their actions and their representations, present a fertile research area where many fundamental results are waiting to be discovered.

1.2 Summary of research

My work during my time as a PhD student aimed to make a contribution to the theory Hopf algebras focusing on their representations and their actions on algebras. My main contributions thus far can be grouped under two headings:

Invariant Theory: Hopf algebra actions and prime spectra, and

Representation Theory: the adjoint representation of a Hopf algebra.

I will describe my findings, and some ideas for future work, in more detail below. In the forthcoming, H will denote a Hopf algebra over a field $\Bbbk$, with antipode S and counit ϵ. I will also use the Sweedler comultiplication notation

$$\Delta h = h_{(1)} \otimes h_{(2)} \text{ for } h \in H.$$

Moreover, A will always be an ***H-module algebra***, that is, A is a $\Bbbk$-algebra that is also a left H-module, via an "action" homomorphism $H \otimes A \to A$, $h \otimes a \mapsto h.a$, satisfying two compatibility conditions: $h.(ab) = (h_{(1)}.a)(h_{(2)}.b)$ and $h.1_A = \epsilon(h)1_A$ for $h \in H$ and $a, b \in A$. When specialized to group algebras, these conditions state that the group in question acts by algebra automorphisms; for enveloping algebras of Lie algebras, they state that the Lie algebra acts by derivations.

1.2.1 Hopf algebra actions and (semi-)primeness.

An ideal I of A such that $H.I \subseteq I$ will be called an ***H-ideal*** of A. For an arbitrary ideal I, we may take the sum of all H-ideals that are contained in I; this is the unique largest H-ideal of A that is contained in I, called the ***H-core*** of I. We will denote the H-core of I by $I{:}H$ below. The relationship between ideals and their H-cores is interesting in general, and it is of particular interest for prime ideals. Indeed, in studying a given $\Bbbk$-algebra A, one of the prime objectives is to understand the set of prime ideals of A, commonly denoted by $\operatorname{Spec} A$.

More specifically denoting the set of all ideals of A by $\operatorname{Ideals} A$ and the subset of all H-ideals by $H\text{-}\operatorname{Ideals} A$, we have the core operator,

$$\,.\,{:}H\colon \operatorname{Ideals} A \longrightarrow H\text{-}\operatorname{Ideals} A.$$

This operator evidently preserves intersections and it is the identity on the subset $H\text{-}\operatorname{Ideals} A \subseteq \operatorname{Ideals} A$. We are interested in the question as to when this operator preserves primeness or semiprimeness. Recall that I is said to be semiprime if A/I has no nonzero nilpotent ideals or, equivalently, I is an intersection of prime ideals. The semiprimeness question may seem a mere technicality, but it can be reformulated in various alternative ways, and semiprimeness is a valuable ring-theoretic commodity without which the investigation of H-module algebras can be rather daunting. In joint work with Lorenz and Nguyen [35], we have shown that the operator $\,.\,{:}H$ does indeed preserve semiprimeness when the Hopf algebra H is cocommutative and $\operatorname{char} \Bbbk = 0$.

In subsequent work [57], I studied actions of a ***connected*** Hopf algebras on A, and showed that the core operator $\,.\,{:}H$ does in fact preserve primeness in this case (and hence semiprimeness as well) when $\operatorname{char} \Bbbk = 0$.

Theorem 1.2.1. *Let H be a connected Hopf algebra over a field of characteristic 0, let A be an H-module algebra, and let I be an ideal of A. If I is prime (semi-prime, completely prime), then so is $I{:}H$.*

Recall that a Hopf algebra is said to be connected if the base field $\Bbbk$ is its only simple subcoalgebra. Examples of connected Hopf algebras include universal enveloping algebras of Lie algebras, and coordinate rings of affine algebraic unipotent groups [54]. In the special case of enveloping algebras of Lie algebras, the above theorem is due to Dixmier; see [15, 3.3].

1.2.2 Hopf algebra actions and prime spectra: stratification.

The existence of an action of a Hopf algebra H on A leads to a useful grouping of the prime ideals of A, called the ***H-stratification*** of $\operatorname{Spec} A$. This stratification employs the H-core operator discussed in the previous paragraph. Specifically, it turns out that if $I = P{:}H$ is the H-core of some prime $P \in \operatorname{Spec} A$, then I always satisfies a condition resembling the familiar defining condition of "primeness," but restricted to H-ideals: if $J_1 J_2 \subseteq I$ for H-ideals J_i, then at least one of the J_i must be contained in I. Such ideals I are called ***H-prime***. Thus, denoting the collection of all H-primes of A by $H\text{-}\operatorname{Spec} A$, the H-core operator restricts to a map

$$\boldsymbol{.}{:}H\colon \operatorname{Spec} A \to H\text{-}\operatorname{Spec} A.$$

Under mild additional conditions on A or the H-action on A, it is known that this map is surjective. For example, this holds whenever $\dim_{\Bbbk} H.a < \infty$ for all $a \in A$, which is automatic for actions of finite dimensional Hopf algebras, or when A is Noetherian. The fibers of this map, that is, the sets

$$\operatorname{Spec}_I A \stackrel{\text{def}}{=} \{P \in \operatorname{Spec} A \mid P{:}H = I\} \qquad (I \in H\text{-}\operatorname{Spec} A),$$

are called the ***H-strata*** of $\operatorname{Spec} A$ and the resulting partition

$$\operatorname{Spec} A = \bigsqcup_{I \in H\text{-}\operatorname{Spec} A} \operatorname{Spec}_I A$$

is the aforementioned H-stratification. Stratifications of this type were pioneered by Goodearl and Letzter [20] in the case of group actions or, equivalently, actions of group algebras $H = \Bbbk G$; they have proven to be a useful tool in the investigation of many questions in non-commutative ring theory, such as the Dixmier-Moeglin equivalence (to be discussed in detail later), the catenarity problem [56], [55], and the computation of various ring-theoretic invariants, such as heights of prime ideals or Gelfand–Kirillov dimensions [18].

For rational actions of a connected affine algebraic $\Bbbk$-group G (with $\Bbbk$ algebraically closed), one has a description of each stratum $\operatorname{Spec}_I A$ in terms of the prime spectrum of a suitable ***commutative*** algebra [31, Theorem 9]. In joint work with Lorenz and Nguyen [35], we were able to generalize this result to the context of cocommutative Hopf algebras: Assuming H to be cocommutative and the H-action on A to be "integral," a technical condition that replaces "rationality" and "connectedness" for algebraic groups and their actions, we describe each H-stratum $\operatorname{Spec}_I A$ in terms of the spectrum of a commutative algebra that can be calculated from I. The detailed formulation of this result is too technical to be included here. For the precise statement, see Theorem 4.1.1 below.

1.2.3 The adjoint representation of a Hopf algebra

The (left) ***adjoint action*** is the action of H on itself defined by:

$$h.k = h_{(1)} k \, \mathrm{S}(h_{(2)}) \qquad (h, k \in H).$$

The Hopf algebra H, equipped with this particular H-action, is a left H-module algebra that will be denoted by H_{ad}. It is easy to see that the invariant subalgebra of H_{ad} is given by $(H_{\text{ad}})^H = \mathcal{Z}H$, the center of H. Our main interest is in the ***locally finite part***,

$$H_{\text{ad fin}} = \{h \in H \mid \dim_{\Bbbk} H.h < \infty\};$$

this is always a subalgebra of H containing $\mathcal{Z}H$, but $H_{\text{ad fin}}$ has additional properties that generally fail for the center. Of course, if H is finite dimensional, then $H_{\text{ad fin}} = H$; so we are mostly concerned with infinite-dimensional Hopf algebras below.

In the special case of a group algebra, $H = \Bbbk G$, the adjoint action is the familiar G-action by conjugation. It turns out that $(\Bbbk G)_{\text{ad fin}} = \Bbbk G_{\text{fin}}$, the subgroup algebra of

$$G_{\text{fin}} = \{g \in G \mid g \text{ has finitely many } G\text{-conjugates}\},$$

the so-called FC-center of G. In particular, $H_{\text{ad fin}}$ always is a Hopf subalgebra of $\Bbbk G$, whereas this holds for $(\Bbbk G)^H_{\text{ad}} = \mathcal{Z}(\Bbbk G)$ only under very severe restrictions on the group G. In the literature, G_{fin} is traditionally denoted by $\Delta(G)$, a notation introduced by Passman before Hopf algebras entered the mainstream of research in algebra and not to be confused with the notation for the Hopf comultiplication; this notation has led to the nomenclature "Δ-methods" in the study of group algebras. It turns out that a significant number of ring-theoretic properties of group algebras are controlled by the FC-center; see [43]. The locally finite part of the adjoint representation has since also been explored for enveloping algebras [5], [6], [7] and for more general Hopf algebras H [4], [23], [29]. In the case where H is the enveloping algebra of a Lie algebra over a field $\Bbbk$ of characteristic 0, it has been shown in the above references that $H_{\text{ad fin}}$ is an enveloping algebra itself, and hence it is again a Hopf subalgebra of H. While this no longer remains true for a quantized enveloping algebra $H = U_q(\mathfrak{g})$—see [4, Example 2.8] or [29] for counterexamples—Joseph and Letzter [23], [29] have shown that $U_q(\mathfrak{g})_{\text{ad fin}}$ is at least a left coideal subalgebra of $U_q(\mathfrak{g})$, that is, a subalgebra that is also a left coideal. In recent joint work with Kolb, Lorenz, and Nguyen [25], we were able to prove that this holds generally: $H_{\text{ad fin}}$ is always a left coideal subalgebra of H, for any Hopf algebra H. Furthermore, if H is virtually cocommutative (i.e., H is finitely generated as right module over some cocommutative Hopf subalgebra), then $H_{\text{ad fin}}$ is in fact a Hopf subalgebra of H.

Theorem 1.2.2. (a) *$H_{\text{ad fin}}$ is always a left coideal subalgebra of H, for any Hopf algebra H.*

(b) *If H is virtually cocommutative, then $H_{\text{ad fin}}$ is a Hopf subalgebra of H.*

1.3 Layout of this thesis and future projects

Chapter 2 serves to assemble the preliminaries needed for the exposition of the work descrived in the previous paragraph. The core of this thesis is Chapter 3, where I present my work descrived in 1.2.1 above in detail, elaborating on the published version [57]. In the concluding Chapter 4, I describe my joint work with my adviser and other co-authors in a brief survey format without proofs.

I am currently working on analyzing the structure of the finite part $H_{\text{ad fin}}$ for a Hopf algebra H that is ***connected***. Computing $H_{\text{ad fin}}$ for many examples of connected Hopf algebras indicates that $H_{\text{ad fin}}$ is possibly actually a Hopf subalgebra in this case, and not only a coideal subalgebra. I hope to settle this question in future work. Other possible avenues for future research will be indicated throughout this thesis, especially in Chapter 4.

Notations and conventions

Throughout this thesis, $\Bbbk$ denotes an arbitrary field; all further assumptions on $\Bbbk$ will be stated as they are needed. Further, H will denote a Hopf algebra over a field $\Bbbk$, with antipode S and counit ϵ. I will use the Sweedler comultiplication notation

$$\Delta h = h_{(1)} \otimes h_{(2)} \qquad (h \in H).$$

Moreover, A will always be an ***H-module algebra***, that is, A is a $\Bbbk$-algebra that is also a left H-module, via an "action" homomorphism $H \otimes A \to A$, $h \otimes a \mapsto h.a$, satisfying two compatibility conditions: $h.(ab) = (h_{(1)}.a)(h_{(2)}.b)$ and $h.1_A = \epsilon(h)1_A$ for $h \in H$ and $a, b \in A$. Tensors will all be over the field $\Bbbk$ by default, i.e. $\otimes = \otimes_{\Bbbk}$. The notation $< \ .\,, \ . \ >: V^* \otimes V \longrightarrow \Bbbk$ will be used to represent the evaluation map. Additional explanations of the above notation will be given below and further, more specialized, notations will be introduced as well.

CHAPTER 2

PRELIMINARIES

This chapter introduced the background, additional notation, definitions, and results used throughout this thesis. The reader is assumed to be familiar with the basics of the representation and invariant theories of associative algebras and groups. The chapter begins with a detailed definition of a Hopf algebra and a brief introduction to their representation theory including several important theorems. We then go on to give the definition of a rational prime ideals and highlight their role in connecting our work in invariant theory to representation theory.

Throughout this chapter, $\Bbbk$ denotes an arbitrary field and all tensor products are assumed to be over $\Bbbk$ unless otherwise denoted.

2.1 Hopf algebras

2.1.1 Coalgebras

The familiar axioms of a $\Bbbk$-algebra with multiplication $m\colon A \otimes A \to A$ and unit $\mu\colon \Bbbk \to A$ can be expressed by the commutativity of the following diagrams in the category of $\Bbbk$-vector spaces:

Associativity

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{Id} \otimes m"] \arrow[d, "m \otimes \mathrm{Id}"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

Unit

$$\begin{tikzcd} A \otimes A \arrow[dr, "m"] & \Bbbk \otimes A \arrow[l, "\mu \otimes \mathrm{Id}"'] \arrow[d, leftrightarrow, "\wr"] \\ A \otimes \Bbbk \arrow[u, "\mathrm{Id} \otimes \mu"] \arrow[r, leftrightarrow, "\sim"] & A \end{tikzcd}$$

Reversing the direction of all arrows in the above diagrams, we obtain commutative diagrams describing the defining axioms of coalgebras. In detail, a $\Bbbk$-***coalgebra*** is a $\Bbbk$-vector space, C, that is equipped with two linear maps, the comultiplication $\Delta\colon C \to C \otimes C$ and the counit $\epsilon\colon C \to \Bbbk$, which satisfy the ***coassociativity*** and ***counit*** axioms:

Coassociativity

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{Id} \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes \mathrm{Id}"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Counit

$$\begin{tikzcd} C \otimes C \arrow[r, "\epsilon \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \epsilon"'] & \Bbbk \otimes C \arrow[d, leftrightarrow, "\wr"] \\ C \otimes \Bbbk \arrow[r, leftrightarrow, "\sim"] & C \arrow[ul, "\Delta"'] \end{tikzcd}$$

For example, if A is a finite-dimensional $\Bbbk$-algebra, then the $\Bbbk$-linear dual $C = A^*$ becomes a $\Bbbk$-coalgebra by taking the dual maps $\Delta = m^*$ and $\epsilon = \mu^*$.

Without special notation, computations using the comultiplication Δ quickly become unwieldy. This dissertation will make use of an abbreviated notation known as Sweedler notation (after Moss Sweedler, one of the first to research Hopf algebras). In this notation the element $\Delta(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)}$ will be abbreviated by

$$\Delta(c) = c_{(1)} \otimes c_{(2)} ,$$

where summation is assumed. Using this notation, the counit axiom can be written simply as

$$\langle \epsilon, c \rangle = \langle \epsilon, c_{(1)} \rangle c_{(2)} = c_{(1)} \langle \epsilon, c_{(2)} \rangle$$

and the coassociativity axiom can be expressed as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = (\mathrm{Id} \otimes \Delta)\Delta(c) = (\Delta \otimes \mathrm{Id})\Delta(c) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} .$$

We will write the map $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$ more simply as Δ^2 and it is also customary to write $\Delta^2(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. Inductively, for any number n of iterations of the comultiplication, one obtains a linear map $\Delta^n : C \to C^{\otimes(n+1)}$ that will be written as

$$\Delta^n(c) = c_{(1)} \otimes c_{(2)} \otimes \ldots \otimes c_{(n+1)} .$$

A subcoalgebra of a coalgebra C is defined exactly as expected: it is a subspace D of C such that $\Delta(D) \subseteq D \otimes D$. Given two coalgebras C and D, a $\Bbbk$-linear map $\varphi : C \to D$ is a ***morphism of coalgebras*** if $\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C$ and $\epsilon_C = \epsilon_D \circ \varphi$. A ***coideal*** of a coalgebra C is a subspace $I \subseteq C$ such that $\Delta(I) \subseteq I \otimes H + H \otimes I$ and $I \subseteq \mathrm{Ker}(\epsilon)$. These are exactly the conditions necessary to make the coalgebra structure maps descend to the vector space C/I, thus giving it the structure of a coalgebra. As in the case of associative algebras, it remains true that coideals are exactly the kernels of coalgebra morphisms.

We now discuss two important constructions. Given an algebra A, we can construct it opposite algebra A^{op} in the familiar way. A similar construction is available for coalgebras. Namely, given a coalgebra C, its ***coopposite coalgebra*** C^{cop} is the vector space C with comultiplication given by $\Delta^{cop} = \tau \circ \Delta$, where $\tau : C \otimes C \to C \otimes C$ simply switches the order of the tensor factors, and with $\epsilon^{cop} = \epsilon$. Next, given two coalgebras C and D, we can give their tensor product the structure of a coalgebra by defining $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$ and $\Delta_{C\otimes D} = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_C \otimes \Delta_D)$ or, in Sweedler notation,

$$\Delta(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \,.$$

Example 2.1.1. One of the simplest and most useful examples of a $\Bbbk$-algebra is the matrix ring $M_n(\Bbbk)$. Its dual $M_n(\Bbbk)^*$ gives us one of the simplest and most useful examples of a coalgebra. Let $E_{i,j} \in M_n(\Bbbk)$ be the matrix with a 1 in position (i, j) and a 0 everywhere else and take $\{d_{i,j}\}$ to be the basis of $M_n(\Bbbk)^*$ that is dual to $\{E_{i,j}\}$. In terms of this basis the coalgebra structure of $M_n(\Bbbk)^*$ is given by the equations below:

$$\begin{aligned} \Delta(d_{i,j}) &= \sum_{k=1}^{n} d_{i,k} \otimes d_{k,j} \\ \epsilon(d_{i,j}) &= \delta_{i,j} \qquad \text{(Kronecker delta)} \end{aligned}$$

Example 2.1.2. The vector space $\Bbbk[x]$ admits a coalgebra structure with structure maps as given below:

$$\begin{aligned} \Delta(x^n) &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k \\ \epsilon(x^n) &= \delta_{n,0} \end{aligned}$$

This is an example of an infinite-dimensional $\Bbbk$-coalgebra.

2.1.2 Convolution product

Given a $\Bbbk$-algebra A and a $\Bbbk$-coalgebra C, we can give $\mathrm{Hom}_{\Bbbk}(C, A)$ the structure of an algebra. The multiplication is called the ***convolution product***, it is denoted by $*$ and defined by $* = m \circ (\,.\otimes .\,) \circ \Delta \colon\ \mathrm{Hom}_{\Bbbk}(C, A) \otimes \mathrm{Hom}_{\Bbbk}(C, A) \to \mathrm{Hom}_{\Bbbk}(C, A)$. In explicit elementwise form, for $c \in C$ and $f, g \in \mathrm{Hom}_{\Bbbk}(C, A)$,

$$\langle f * g, c\rangle = \langle f, c_{(1)}\rangle\langle g, c_{(2)}\rangle \,.$$

The unit map is given by it is straightforward to verify that $\mu \circ \epsilon \colon C \to \Bbbk \to A$ serves as a unit element for $*$.

2.1.3 Bialgebras and Hopf algebras

A ***bialgebra*** is a coalgebra in the category of algebras or, equivalently, an algebra in the category of coalgebras. More explicitly a $\Bbbk$-bialgebra B is both a $\Bbbk$-algebra and a $\Bbbk$-coalgebra with the two structures being compatible in the sense that the following two equivalent conditions are satisfied:

(a) The comultiplication Δ and the counit ϵ of B are algebra maps;

(b) the multiplication m and the unit u of B are coalgebra maps.

Checking the conditions (a) and (b) is equivalent to checking that the following formulas hold, for $x, y \in B$:

$$\Delta(xy) = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} = \Delta(x)\Delta(y) \qquad < \epsilon, xy >=< \epsilon, x >< \epsilon, y >,$$

as well as

$$\Delta(1) = 1 \otimes 1 \qquad\qquad < \epsilon, 1 >= 1_{\Bbbk}.$$

For more details, [33] can serve as a reference.

Example 2.1.3. The vector space $\Bbbk[x]$ is a bialgebra, where $\Bbbk[x]$ has the usual polynomial algebra structure and the coalgebra structure described in Example 2.1.2.

Let H be a $\Bbbk$-bialgebra. Then $\mathrm{End}_{\Bbbk}(H) = \mathrm{Hom}_{\Bbbk}(H, H)$ is an algebra via the convolution product, where the first H is viewed as a coalgebra structure and the second H as an algebra. If there exists a two-sided inverse, $S \in \mathrm{End}_{\Bbbk}(H)$, to the identity morphism, $\mathrm{Id}_H \in \mathrm{End}_{\Bbbk}(H)$, then H is called a ***Hopf Algebra***. The element S is then called the ***antipode*** of H. In Sweedler notation, the defining property of the antipode can be written as

$$S(h_{(1)})h_{(2)} = \epsilon(h)\mu(1) = h_{(1)}S(h_{(2)})\,.$$

It is worth noting that the antipode is always an antialgebra and anticoalgebra map [40]. Explicitly, for any $a, b, h \in H$,

$$S(ab) = S(b)S(a) \quad \text{and} \quad S(h_{(2)}) \otimes S(h_{(1)}) = S(h)_{(1)} \otimes S(h)_{(2)}\,.$$

A remarkable feature of Hopf algebras is that their axioms are self dual. Thus given a finite-dimensional Hopf algebra H, the $\Bbbk$-linear dual H^* also has the structure of a Hopf algebra, where the structure maps of H^* come from applying the dualizing functor $\,.^*$ to the structure maps of H.

Example 2.1.4. Given a group G and a field $\Bbbk$, we can construct the group algebra $\Bbbk G$: it is the $\Bbbk$ vector space with basis the elements of G; multiplication is given by linear extension of the group multiplication; and unit element of $\Bbbk G$ is given by the unit element of G. The group algebra is in fact a Hopf algebra, with coalgebra structure maps and the antipode as given below, for $g \in G$:

$$\begin{aligned}\Delta(g) &= g \otimes g \\ \epsilon(g) &= 1 \\ S(g) &= g^{-1}\end{aligned}$$

Example 2.1.5. When $|G| < \infty$ we can construct the Hopf algebra $(\Bbbk G)^*$, the $\Bbbk$-linear dual of $\Bbbk G$. Let $\{\rho_x \mid x \in G\}$ denote the basis of $(\Bbbk G)^*$ that is dual to the basis G of $\Bbbk G$. Then for $g, h \in G$ the Hopf algebra structure of $(\Bbbk G)^*$ is given by:

$$\begin{aligned}\mu(1) &= \sum_{g \in G} \rho_g = \epsilon & \rho_g \rho_h &= \delta_{g,h} \rho_g \\ \epsilon(\rho_g) &= \rho_g(1) = \delta_{g,1} & \Delta(\rho_g) &= \sum_{h \in G} \rho_h \otimes \rho_{h^{-1}g} \\ S(\rho_g) &= \rho_{g^{-1}} & &\end{aligned}$$

Group algebras are particularly nice examples of Hopf algebras in that they are ***cocommutative***, meaning that, $\tau \circ \Delta = \Delta$ where τ is the twist map as in Section 2.1.1. In other words, $(\Bbbk G)^{cop} = \Bbbk G$. An element h of a Hopf algebra is called ***group-like*** if $\Delta(h) = h \otimes h$ and $\epsilon(h) = 1$. In this case $S(h)$ will also be a group-like element, and $S(h)$ is the multiplicative inverse of h. As the name suggests, the collection of all group-like elements of any Hopf algebra H forms a subgroup of the group of units of H.

Example 2.1.6. Another classical example of a cocommutative Hopf algebra is the enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. The images of $\Bbbk$-basis of $\mathfrak{g}$ in $U\mathfrak{g}$

generate $U\mathfrak{g}$ as an algebra. The space generated by these images is the space of ***primitive*** elements of $U\mathfrak{g}$. Here, an element x of a Hopf algebra is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $<\epsilon, x> = 0$. Note that Example 2.1.3 is an example of an enveloping algebra, namely that of the one-dimensional Lie algebra.

The smallest non-cocommutative example of a Hopf algebra is a four-dimensional example named after Sweedler, who first constructed it.

Example 2.1.7. Assume that $\Bbbk$ has characteristic $\neq 2$. The ***Sweedler algebra***, denoted H_4, is the unique non-commutative and non-cocommutative Hopf algebra of dimension 4. The algebra structure of H_4 is defined by $H_4 = \Bbbk\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx\rangle$. The coalgebra structure and the antipode of H_4 are defined by:

$$\begin{aligned} \Delta(g) &= g \otimes g\,, & \Delta(x) &= x \otimes 1 + g \otimes x\,, \\ \epsilon(g) &= 1\,, & \epsilon(x) &= 0\,, \\ S(g) &= g\,, & S(x) &= -gx\,. \end{aligned}$$

2.1.4 Hopf subalgebras and quotient Hopf algebras

A ***Hopf subalgebra*** of a Hopf algebra H, by definition, is a subalgebra of H that is also a subcoalgebra and is stable under the antipode. Likewise, a ***Hopf ideal*** of H is an ideal of H that is also a coideal and is stable under the antipode.

Given a subalgebra, A, of H we can define $\mathscr{H}(A)$ to be the subalgebra of A that is generated by all Hopf subalgebras of H that are contained in A. It is a simple exercise to see that this subalgebra is in fact a Hopf subalgebra, and thus is the unique largest Hopf subalgebra of H contained in A. Of particular interest is sometimes the largest Hopf subalgebra contained in the center of H. For brevity, we will simply write

$$\zeta(H) = \mathscr{H}\mathcal{Z}(H)\,.$$

A similar process can be done starting with an ideal I of H to construct the largest Hopf ideal, $\mathscr{H}(I)$, contained in I: define $\mathscr{H}(I)$ to be the sum of all Hopf ideals of H that are contained in I. Given an H-module, M, we can use this construction to construct the largest Hopf ideal of H contained in the annihilator of M;

this ideal will be called the ***Hopf kernel*** of M and denoted by

$$\mathscr{H}\,\mathrm{Ker}(M)\,.$$

A representation of H that is given by an H-module M with Hopf kernel $\mathscr{H}\,\mathrm{Ker}(M) = 0$ will be called ***inner faithful***.

Example 2.1.8. Given a Hopf algebra H, we always have the augmentation ideal, $H^+ := \mathrm{Ker}(\epsilon)$. This is in fact a Hopf ideal. The quotient Hopf algebra $\overline{H} = H/H^+$ is isomorphic t o $\Bbbk$ as an algebra and the coalgebra structure is given by $\Delta_{\overline{H}}(c) = c \otimes 1$. Given a Hopf subalgebra K in H, we can form the left ideal HK^+ of H. It is easy to see that HK^+ is also a coideal of H. In Section 2.2.4, we will discuss under which circumstances HK^+ is in fact a Hopf ideal of H.

Example 2.1.9. For any group G, the Hopf subalgebras of the group algebra $\Bbbk G$ are exactly the various $\Bbbk H$, where H is a subgroup of G. The Hopf ideals of $\Bbbk G$ are exactly the ideals of the form $\Bbbk G \Bbbk H^+$, where H is a normal subgroup of G. Furthermore, $\Bbbk G/\Bbbk G \Bbbk H^+ \cong \Bbbk[G/H]$. An inner faithful representation of $\Bbbk G$ is just a representation of G such that no group element $g \in G$ other than $g = 1$ acts as the identity transformation.

It is important to note that, while all Hopf ideals of group algebras arise from Hopf subalgebras, this is not always the case in general. This is illustrated in the next example.

Example 2.1.10. Observe the set $G := \{1, g\} \subseteq H_4$ in the Sweedler algebra is a group. The Hopf algebra $\Bbbk G$ is the only Hopf subalgebra of the Sweedler algebra other then $\Bbbk$ and H_4. The space $H_4 \Bbbk G^+$ has basis $\{x + gx, 1 - g\}$ which is not an ideal of H_4 since it does not contain $x - gx = (1 - g)x$. Now look at the Hopf ideal of H_4 given by the $\Bbbk$-span of $\{x, gx\}$. This is in fact the Jacobson radical of H_4. Since this Hopf ideal did not arise from the unique nontrivial Hopf subalgebra $\Bbbk G$ it could not have arisen from any Hopf subalgebra. The corresponding quotient Hopf algebra is isomorphic to the group algebra $\Bbbk[C_2]$, where C_2 is the cyclic group of order two.

2.2 Representation theory

As is the case with groups, Hopf algebras have a representation theory that has additional features not present in the representation theory of general associative algebras. Throughout this dissertation we will focus on left modules. This is only for consistency as the theory could be formulated equally well for right modules.

In the following, H denotes a Hopf $\Bbbk$-algebra and $\mathfrak{Rep} H$ denotes the category of left H-modules that are finite-dimensional over $\Bbbk$. All further assumptions will be explicitly stated when they are needed.

2.2.1 The representation ring and character algebra

The coalgebra structure of H allows us to endow the category of left H-modules with the structure of a tensor category. The precise definition of a tensor category is not needed and hence will be omitted. The key fact that is needed is that, given two left H-modules V and W, the tensor product $V \otimes W$ has an H-module structure with $h \in H$ acting by

$$h.(v \otimes w) = h_{(1)}.v \otimes h_{(2)}.w \,. \tag{2.1}$$

The counit of a Hopf algebra gives rise to a representation, which will be called the ***trivial representation*** and denoted by $\mathbb{1}$. Explicitly, $\mathbb{1} = \Bbbk$ with $h \in H$ acting by $h.1 = \langle \epsilon, h \rangle$. It is easy to see that, for any $V \in \mathfrak{Rep} H$, the isomorphisms $V \otimes \mathbb{1} \cong V \cong \mathbb{1} \otimes V$ hold in $\mathfrak{Rep} H$. Using the trivial representation we get a notion of the ***invariants*** of any $V \in \mathfrak{Rep} H$,

$$V^H = \{ v \in V \mid h.v = \langle \epsilon, h \rangle v \} \,.$$

The antipode of the Hopf algebra allows us to give the $\Bbbk$-linear dual of a module the structure of an H-module. Given $V \in \mathfrak{Rep} H$, the vector space V^* becomes an H-module with action defined by

$$\langle h.f, v \rangle = \langle f, S(h).v \rangle \,. \tag{2.2}$$

We can work with the ***representation ring*** $\mathscr{R}(H)$ of H, which is defined as the abelian group with generators the isomorphism classes $[V]$ of finite-dimensional representations $V \in \mathfrak{Rep} H$ and with relations $[U]+[W] = [V]$ for each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathfrak{Rep} H$. The multiplication of $\mathscr{R}(H)$ comes from the tensor product of representations: $[V][W] = [V \otimes W]$. By extension of scalars from $\mathbb{Z}$ to $\Bbbk$, we obtain the $\Bbbk$-algebra $\mathscr{R}_{\Bbbk}(H) := \Bbbk \otimes \mathscr{R}(H)$; this algebra will be called the ***representation algebra*** of H.

It is a standard fact that the representation algebra $\mathscr{R}_{\Bbbk}(H)$ embeds into the linear dual H^* via the ***character map*** and this embedding is a homomorphism of $\Bbbk$-algebras for the convolution algebra structure of H^*; see [33, Proposition 12.10]. Explicitly, for any finite-dimensional $V \in \mathfrak{Rep} H$, the ***character*** χ_V is the linear form on H that is defined by $\langle \chi_V, h \rangle = \operatorname{trace}(h_V)$, where $h_V \in \operatorname{End}_{\Bbbk}(V)$ denotes the operator given by the action of $h \in H$. The character map is given by

$$\begin{array}{rcl} \chi\colon \mathscr{R}_{\Bbbk}(H) & \hookrightarrow & H^* \\ \cup & & \cup \\ [V] & \longmapsto & \chi_V \end{array}$$

The image of the character map in H^* is called ***character algebra*** of H and is denoted $R(H)$. A $\Bbbk$-basis of $R(H)$ is given by the irreducible characters of H, that is, the characters of a full set of non-isomorphic irreducible finite-dimensional representations of H. If H is semisimple and $\Bbbk$ is a splitting field for H (e.g., if $\Bbbk$ is algebraically closed), then the character algebra $R(H)$ coincides with the subspace of all ***trace forms*** on H, that is, the linear forms on H that vanish on the subspace $[H, H]$ spanned by the Lie commutators $[h, k] = hk - kh$ for $h, k \in H$. Thus, the space of trace forms is isomorphic to $(H/[H, H])^*$; it can equivalently be thought of as the set of cocommutative elements of H^*.

Example 2.2.1. Let G be a finite group. Then the set of irreducible characters of $\Bbbk G^*$ are the elements of G and hence $\Bbbk G = R(\Bbbk G^*)$.

2.2.2 Comodules

Given an algebra $A = (A, m, \mu)$, the familiar axioms of a left A-module M can be expressed by the existence of a $\Bbbk$-linear "action" map $a\colon A \otimes M \to M$ such that the following diagrams are commutative:

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{Id} \otimes a"] \arrow[d, "m \otimes \mathrm{Id}"'] & A \otimes M \arrow[d, "a"] \\ A \otimes M \arrow[r, "a"] & M \end{tikzcd} \qquad \begin{tikzcd} A \otimes M \arrow[dr, "a"] & \Bbbk \otimes M \arrow[l, "\mu \otimes \mathrm{Id}"'] \arrow[d, leftrightarrow, "\wr"] \\ & M \end{tikzcd}$$

Dually, if $C = (C, \Delta, \epsilon)$ is a $\Bbbk$-coalgebra, then a $\Bbbk$-vector space N is called a left C-***comodule*** if there is a $\Bbbk$-linear "coaction" map $\rho\colon N \to C \otimes N$ such that the following diagrams commute:

$$\begin{tikzcd} C \otimes C \otimes N & C \otimes N \arrow[l, "\mathrm{Id} \otimes \rho"'] \\ C \otimes N \arrow[u, "\Delta \otimes \mathrm{Id}"] & N \arrow[l, "\rho"'] \arrow[u, "\rho"'] \end{tikzcd} \qquad \begin{tikzcd} C \otimes N \arrow[r, "\epsilon \otimes \mathrm{Id}"] & \Bbbk \otimes N \arrow[d, leftrightarrow, "\wr"] \\ & N \arrow[ul, "\rho"'] \end{tikzcd}$$

As with coalgebras it is customary to use a version of the Sweedler notation when dealing with comodules: for $n \in N$,

$$\rho(n) = n_{(-1)} \otimes n_{(0)} \,.$$

For right comodules, defined by the obvious modification of the above, we will instead use the notation

$$\rho(n) = n_{(0)} \otimes n_{(1)} \in N \otimes C \,.$$

Let M be a right C-comodule. Then M can be viewed as a left a left module over the convolution algebra C^* via

$$c^*.m = m_{(0)} \langle c^*, m_{(1)} \rangle \qquad (c^* \in C^*, m \in M) \,.$$

If C is finite-dimensional, then all left C^*-modules arise in this fashion; so there is equivalence between the categories of right C-comodules and left C^*-modules. Thus, as one would expect, there are analogs of all constructions and properties of modules for comodules. For example, a subcomodule of a left comodule M is a subspace $V \subseteq M$ such that $\rho(V) \subseteq C \otimes V$. When C is a bialgebra, one subcomodule of particular interest is given by the ***coinvariants***,

$$^{\mathrm{coInv}}M := \{m \in M \mid \rho(m) = 1 \otimes m\}.$$

A comodule M is called ***simple*** if M contains no subcomodules other than 0 and M itself, and M is called ***indecomposable*** if it can not be expressed as the direct sum of two nonzero subcomodules. A coalgebra C is called ***cosemisimple*** if all C-comodules are direct sums of simple comodules. We will not go into detail on cosemisimplicity as, in the case where C is finite-dimensional, it is equivalent to the dual algebra C^* being semisimple by our remark above. A Hopf algebra with cosemisimple coalgebra structure is called cosemisimple.

Example 2.2.2. The group algebra $\Bbbk G$ is cosemisimple. All simple comodules are one dimensional and have the form $\Bbbk g$ for $g \in G$ with structure map given by $\rho(g) = g \otimes g$.

Example 2.2.3. The coalgebra $M_n(\Bbbk)^*$ of Example 2.1.1 is also cosemisimple. Up to isomorphism it has exactly one simple left comodule. This module can be viewed as $\Bbbk\{e_{i,1}\}$ with $\rho(e_{i,1}) = \sum_k e_{i,k} \otimes e_{k,1}$. If $\Bbbk$ is algebraically closed, then dualizing the Artin-Wedderburn structure theorem of semisimple algebras, one sees that all cosemisimple coalgebras over $\Bbbk$ are isomorphic to a direct sum of duals of matrix algebras.

Example 2.2.4. A classical ***non***-example is the enveloping algebra of a Lie algebra. Indeed, if $\mathfrak{g}$ is a $\Bbbk$-Lie algebra and $\mathrm{U}\,\mathfrak{g}$ is its enveloping algebra, then $\Bbbk \oplus \Bbbk\mathfrak{g}$ is a sub-colagebra of $\mathrm{U}\,\mathfrak{g}$ that is not decomposable into simples.

2.2.3 Module algebras

Quantum invariant theory is concerned with actions of Hopf algebras on associative algebras. In detail, let A be an associative $\Bbbk$-algebra and assume that A is a left module over the Hopf $\Bbbk$-algebra H with action map $H \otimes A \to A$, $h \otimes a \mapsto h.a$. If the multiplication $m\colon A \otimes A \to A$ and the unit map $\mu\colon A \to \Bbbk = \mathbb{1}$ are maps of H-modules, then we say A is a H-***module algebra***. We will sometimes use the categorical notation $A \in {}_H\mathsf{Alg}$. These axioms can be stated in Sweedler notation as follows: for $b, c \in A$ and $h \in H$, we have

$$h.(bc) = (h_{(1)}.b)(h_{(2)}.c) \qquad \text{and} \qquad h.1_A = \langle \epsilon, h\rangle 1_A\,.$$

Example 2.2.5. Let A be an associative algebra and let G be a subgroup of the autmorphism group $\mathrm{Aut}_{\mathrm{alg}}(A)$. Then the G-action on A extends to an action of the group algebra $\Bbbk G$, making A a $\Bbbk G$-module algebra.

Example 2.2.6. Let A be an associative $\Bbbk$-algebra and $\mathfrak{g}$ be a $\Bbbk$-Lie algebra that embeds into the derivations Lie algebra $\mathrm{Der}_{\Bbbk}(A)$. We then have that the $\mathfrak{g}$-action on A extends to an action of $\mathrm{U}\,\mathfrak{g}$, the enveloping algebra of $\mathfrak{g}$, on A. This makes A a $\mathrm{U}\,\mathfrak{g}$-module algebra.

2.2.4 Adjoint representation

Let V be an H-bimodule, that is, a left module over the algebra $H \otimes H^{\mathrm{op}}$. Then we can form a left adjoint module of V, denoted by V_{ad}, by defining

$${}^{h}v := h_{(1)}vS(h_{(2)}) \qquad (h \in H, v \in V)\,.$$

Naturally, there is also an analogous right adjoint action, given by

$$v^h = S(h_{(1)})vh_{(2)}\,.$$

However, by passing to the opposite-cooposite Hopf algebra, this action can be reduced to the above action; so we will mainly focus on the left-handed version here. In fact, we will mostly be interested in the case where $V = H$ is the regular H-bimodule, with left and right actions given by multiplication. As in the Introduction, the corresponding representation of H will be called the adjoint representation. The algebra H, with the (left) adjoint H-action, is an example of an H-module algebra as introduced in the previous section.

For any H-bimodule V, the ordinary H-actions on V and the adjoint action are related by:

$$hv = ({}^{h_{(1)}}v)h_{(2)} \,. \tag{2.3}$$

The H-invariants of V_{ad} are given by:

$$(V_{\text{ad}})^H = \mathcal{Z}V := \{v \in V \mid hv = vh \text{ for all } h \in H\} \,. \tag{2.4}$$

Indeed, for $v \in \mathcal{Z}V$ and $h \in H$, we have ${}^h v = h_{(1)}S(h_{(2)})v = \langle \epsilon, h\rangle v$; so $v \in (V_{\text{ad}})^H$. Conversely, if $v \in (V_{\text{ad}})^H$ and $h \in H$, then (2.3) gives $hv = ({}^{h_{(1)}}v)h_{(2)} = \langle \epsilon, h_{(1)}\rangle v h_{(2)} = vh$.

Example 2.2.7. For G a group the adjoint action of $\Bbbk G$ is given by the $\Bbbk$-linear extension of G acting on itself by conjugation.

Example 2.2.8. Another classical example of the adjoint action of a Hopf algebra on itself is when H is an enveloping algebra of a Lie algebra. When $\mathfrak{g}$ is a Lie algebra and $\mathrm{U}\,\mathfrak{g}$ is its enveloping algebra, the adjoint action of an element $x \in \mathfrak{g}$ on an element $u \in \mathrm{U}\,\mathfrak{g}$ is given by $x.u = x_{(1)}uS(x_{(2)}) = xu - ux = [x, u]$. This action can be extended to an action of $\mathrm{U}\,\mathfrak{g}$ on itself.

Example 2.2.9. To he adjoint action of H_4 is given on algebra generators by:

$$\begin{array}{llll} {}^g g = g & {}^g x = -x & {}^g 1 = 1 & {}^g gx = -gx \\ {}^x g = -2gx & {}^x x = 0 & {}^x 1 = 0 & {}^x gx = 0 \end{array}$$

If a Hopf subalgebra K of H is stable under the left and right adjoint actions of H, then we say K is ***normal***. In this case, the coideal HK^+ of H is also a two-sided ideal of H, and hence it is a Hopf ideal. If H is finite dimensional, then the converse holds as well: for any Hopf subalgebra K of H, the Hopf ideal HK^+ is an ideal of H if and only if K is normal; see [40, Corollary 3.4.4]. Famous and useful examples of normal Hopf subalgeras of a Hopf algebra are the center, $\mathcal{Z}(H)$, and the finite part $H_{\text{ad fin}}$.

2.2.5 Integrals, semisimplicity and cosemisimplicity

A $\Bbbk$-algebra, A, is said to be ***augmented*** if A is equipped with a given algebra map $\epsilon\colon A \to \Bbbk$, called the ***augmentation map*** of A. Thus, a Hopf algebra is always an augmented algebra, the augmentation map ϵ being the counit. A ***left integral*** in an augmented algebra (A, ϵ) is an element $\Lambda \in A$ such that $a\Lambda = \epsilon(a)\Lambda$ for all $a \in A$. Right integrals are defined similarly by the condition $\Lambda a = \Lambda\epsilon(a)$ for all $a \in A$. Throughout, Λ will always be used to denote an integral, superscripts of L or R will be used to distinguish left and right integrals. The space of all left integral of an augmented algebra A will be denoted $\int_A^L$ and similarly the space of right integrals will be denoted $\int_A^R$. The following theorem of Larson and Sweedler shows that finite-dimensional Hopf algebras always have integrals, and they are unique up to scalar multiples [28].

Theorem 2.2.10. *Let H be a finite-dimensional Hopf algebra. Then the spaces $\int_H^L$ and $\int_H^R$ are one dimensional.*

As H^* is also a Hopf algebra it is an augmented algebra with augmentation map μ^*. Thus Theorem 2.2.10 also implies the existence of integrals of H^*. These integrals will commonly be denoted with the lowercase Greek letter λ and superscripts of L and R will be used to distinguish left and right integrals.

Observe for all $h \in H$ we have $h\Lambda^R$ is also a right integral and since the space of right integrals is one dimensional this gives $h\Lambda^R = \langle \alpha, h\rangle \Lambda^R$ for some $\alpha \in H^*$. The element α is easily seen to be an algebra map; thus it is a group-like element of H^*. The element α is called the ***distinguished group-like element of*** H^*. A Hopf algebra is called ***unimodular*** if the distinguished group like element of H^* is the counit of H or equivalently if H contains a central integral. Identifying H with H^{**} we also get a distinguished group-like element of H.

The integral of a Hopf algebra can be used to easily determine when the Hopf algebra is semisimple via the following theorem.

Maschke's Theorem for Hopf Algebras [28]**.** *A Hopf algebra H is semisimple iff H is finite dimensional and $\epsilon(\int_H^L) \neq 0$ or, equivalently, $\epsilon(\int_H^R) \neq 0$.*

The theorem, though due to Larson and Sweedler, is generally named after Heinrich Maschke, who proved the special case of group algebras: $\Bbbk G$ is semisimple if and only if G is finite and $\operatorname{char}\Bbbk$ does not divide the order $|G|$ [47].

It is an immediate consequence of Maschke's Theorem that a semisimple Hopf algebra is unimodular. Another immediate consequence is that semisimple Hopf algebras are separable: $F\otimes H$ remains semisimple for all field extensions $F/\Bbbk$. This follows, because any integral of H is also an integral of $F \otimes H$. If $\operatorname{char}\Bbbk = 0$, then a theorem of Larson and Radford [27] gives that H is also cosemisimple. Moreover, in this case, the antipode of H and H^* must satisfy $S \circ S = \mathrm{Id}$. A Hopf algebra with the latter property is called ***involutory*** [27]. It was shown in [27] that, for an involutory unimodular Hopf algebra, the character of the regular representation is an integral of H^*. We will make frequent use of this fact in this thesis.

Example 2.2.11. For a finite group G, the standard integral of $\Bbbk G$ is $\sum_{g\in G} g$. It is easily seen to be central and hence group algebras are unimodular. Applying Maschke's Theorem for Hopf algebras, we get back Maschke's original result that $\Bbbk G$ is semisimple iff G is finite and $\operatorname{char}\Bbbk$ does not divide $|G|$. Even in the case where $\operatorname{char}\Bbbk$ divides $|G|$, the group algebra $\Bbbk G$ is still involutory as, clearly, the antipode $g \mapsto g^{-1}$ composed with itself is the identity.

Example 2.2.12. The standard integral of $\Bbbk G^*$ is ρ_1 where 1 is the identity element of $\Bbbk G$. Since $\Bbbk G^*$ is commutative, it is clearly unimodular.

Example 2.2.13. The Sweedler algebra H_4 is our first example of a non-unimodular Hopf algebra. The space of left integrals is spanned by $gx+x$, and the space of right integral by $-gx + x$. The distinguished group like element of the dual is defined on algebra generators by $\alpha(x) = 0$ and $\alpha(g) = -1$. Since $\epsilon(gx + x) = 0$, Maschke's Theorem for Hopf algebras tells us that H_4 is not semisimple. In fact, as we have pointed out in Example 2.1.10, the Jacobson radical of H_4 is the 2-dimensional ideal of H_4 that is generated by the element x .

2.2.6 Chevalley property

For a non-semisimple Hopf algebra the tensor product of two completely reducible modules fails to be completely reducible in general. It is of interest to know when the tensor product inherits complete reduciblity from its factors. Following [1], a Hopf algebra H (not necessarily finite-dimensional) is said to have the ***Chevalley property*** if the tensor product of any two finite-dimensional completely reducible H-modules is again completely reducible. A classical result of Chevalley [11] states that group algebras of arbitrary groups over fields of characteristic zero do in fact have this property; see also [22, Theorem VII.2.2]. We will say that a left H-module M has the ***Chevalley property*** if all tensor powers $M^{\otimes n}$ are completely reducible or, equivalently, the H-module $\mathsf{T}(M) := \bigoplus_{n \in \mathbb{Z}_+} M^{\otimes n}$ is completely reducible. The Chevalley property for H, in the above sense, is evidently equivalent to the Chevalley property for the direct sum of all finite-dimensional irreducible H-modules.

2.3 Motivation and background: rationality

In this section we give some background to serve as a motivation for Section 4.1. The study of prime spectra of algebras is interesting and useful in its own right, but has additional importance through its applications and connections to other fundamental questions as well.

2.3.1 Symmetric rings of quotients

In this section, R denotes any ring (with 1). We briefly recall some background material on the symmetric ring of quotients $\mathrm{Q}R$ and its center. The symmetric ring of quotients will not be defined here. For details, see [33].

Extended center

The center of $\mathrm{Q}R$, $\mathcal{C}R := \mathcal{Z}(\mathrm{Q}R)$, is called the ***extended center*** of R. See [33, Appendix E] for details. The ring R is a subring of $\mathrm{Q}R$ and $\mathcal{C}R$ coincides with the centralizer of R in $\mathrm{Q}R$. In particular, $\mathcal{Z}R \subseteq \mathcal{C}R$. If $\mathcal{Z}R = \mathcal{C}R$, then R is called ***centrally closed***. In general, we may consider the following subring of $\mathrm{Q}R$, possibly strictly larger than R:

$$\widetilde{R} := R(\mathcal{C}R) \subseteq \mathrm{Q}R.$$

If R is semiprime, then $\widetilde{R}$ is a centrally closed ring [3, Theorem 3.2], called the ***central closure*** of R. If R is a $\Bbbk$-algebra, then so is $\widetilde{R}$, because $\mathcal{Z}R \subseteq \mathcal{C}R = \mathcal{Z}\widetilde{R}$.

2.3.2 Rational primes

Rational and H-rational ideals are certain prime and H-prime ideals, respectively, that are of particular interest. Ultimately, the notion of a rational prime ideal is motivated by the classical (weak) Nullstellensatz of Hilbert; we will discuss the connection in greater detail below. The definition of rational and H-rational ideals relies on a suitable quotient ring of A: For any H-prime ideal I, let $\mathcal{C}_H(I)$ denote the subalgebra of H-invariants of the center of the ring of the so-called H-symmetric quotient $\mathrm{Q}_H(A/I)$. It is known that $\mathcal{C}_H(I)$ is a field containing the base field $\Bbbk$ [37]. If the extension $\mathcal{C}_H(I)/\Bbbk$ is algebraic, then we say that I is H-***rational***. For a trivial H-action on A, we obtain the notion of rationality for a prime ideal P: Letting $\mathcal{C}(P)$ denote the center of the ordinary symmetric ring of quotients $\mathrm{Q}(A/P)$, the prime P is said to be ***rational*** if the field extension $\mathcal{C}(P)/\Bbbk$ is algebraic. Denoting the collection of all rational and H-rational ideals of A by $\operatorname{Rat} A$ and H-$\operatorname{Rat} A$, respectively, Lorenz [34] has shown that the H-core operator gives well-defined map

$$\operatorname{Rat} A \longrightarrow H\text{-}\operatorname{Rat} A\,, \qquad P \mapsto P{:}H\,.$$

This map has been explored in detail for rational actions of an affine algebraic group G on A. In this case, the map is surjective and its fibers are exactly the G-orbits in

$\mathrm{Rat}\, A$ [31]. In the current more general context of Hopf actions, however, none of this is known.

An interesting project would be to study the map $\mathrm{Rat}\, A \to H\text{-}\mathrm{Rat}\, A$. In particular, describe its image and its fibers under suitable hypotheses on the H-action on A. We do, however, expect this project to be a long-term one. Indeed, while there is a notion of H-orbits in $\mathrm{Spec}\, A$, proposed by Skryabin [49], its usefulness has only been tested under very special circumstances thus far (e.g., for H-module algebras A that are finite over their centers or else noetherian with H finite dimensional). Despite the fact that much ground work remains to be done, the existence of the above framework is valuable inasmuch as it also captures, besides the aforementioned case of group actions, the much-studied notion of Poisson-rational primes [10], [19].

2.3.3 Connections to representation theory

When trying to understand the category of representations of A, the first goal is often the description of the set of irreducible representations, $\mathrm{Irr}\, A$. This task, however, is usually a daunting one, and often unachievable in any useful way. However, a helpful connection exists between the representation and ideal theories of A; this link is provided by the kernel $\mathrm{Ker}\, V$ of a representation V of A, that is, the kernel of the corresponding homomorphism $A \to \mathrm{End}_{\Bbbk}(V)$. When $V \in \mathrm{Irr}\, A$, we say that the ideal $P = \mathrm{Ker}\, V$ is ***primitive***. The set of all such ideals is denoted by $\mathrm{Prim}\, A$; so we have a surjection of sets,

$$\mathrm{Irr}\, A \twoheadrightarrow \mathrm{Prim}\, A\,, \qquad V \mapsto \mathrm{Ker}\, V\,.$$

Even though this map is generally far from being bijective, ring-theoretic tools allow us to study $\mathrm{Prim}\, A$, while the fibers of that surjection give us a rough idea about the set $\mathrm{Irr}\, A$. There is one caveat with this approach, however: In principle, showing that a given prime ideal P of A is actually primitive still requires us to come up with an irreducible representation of A for which P is the kernel. To have a useful strategy, one would like an intrinsic characterization of "primitivity," one that allows us to detect primitivity directly from P, much in the way the classical (weak) Nullstellensatz of Hilbert, for an affine commutative algebra A, equates maximality of P with algebraicity of A/P over $\Bbbk$.

It is in this connection that the above notion of rationality first arose. To complete the picture, we recall that $\operatorname{Spec} A$ carries the so-called Jacobson-Zariski topology: closed subsets of $\operatorname{Spec} A$ in this topology are precisely the sets of the form

$$\{P \in \operatorname{Spec} A \mid P \supseteq I\}$$

for some ideal I of A. A prime ideal P of A is said to be ***locally closed*** if the one-point subset $\{P\} \subseteq \operatorname{Spec} A$ is locally closed. It is a fortuitous fact that, for many algebras A that are of interest, the following equivalence has been established, for every $P \in \operatorname{Spec} A$,

$$P \text{ is locally closed} \iff P \text{ is primitive} \iff P \text{ is rational}\,.$$

This equivalence is called the ***Dixmier-Moeglin equivalence***; when it holds, one obtains a topological (locally closed) and a ring-theoretic (rational) characterization of the representation-theoretic notion of primitivity, thereby widening the set of tools that can be used to investigate $\operatorname{Prim} A$. The implications $\Longrightarrow$ above hold for wide classes of algebras, being a consequence of the non-commutative Nullstellensatz, but the reverse implications are rather more fickle. Nonetheless, the Dixmier-Moeglin equivalence has been shown to hold for affine PI algebras [44], enveloping algebras of finite-dimensional Lie algebras [14], [39], and for various quantum groups [9].

Even though the Dixmier-Moeglin equivalence makes no reference to any H-action on A, such actions and the aforementioned H-stratification of $\operatorname{Spec} A$ have proven to be helpful in establishing the equivalence for certain algebras. So far this approach to the Dixmier-Moeglin equivalence has been predominantly pursued for certain group actions; see [31], [9]). It would be interesting to put the larger context of Hopf algebra actions on A and the resulting the stratifications of $\operatorname{Spec} A$ to use in analyzing the representation theory of A and the Dixmier-Moeglin equivalence in particular.

CHAPTER 3

ACTIONS OF HOPF ALGEBRAS: PRIMENESS AND SEMIPRIMENESS

In this chapter, I present my main results on Hopf algebra actions and invariant theory. We keep the earlier notation: H is a Hopf algebra and A is an H-module algebra. An ideal I of A such that $H.I \subseteq I$ will be called an ***H-ideal*** of A. For an arbitrary ideal I, we may take the sum of all H-ideals that are contained in I; this is the unique largest H-ideal of A that is contained in I, called the ***H-core*** of I. We will denote the H-core of I by $I{:}H$ below. The relationship between ideals and their H-cores is full of mysteries, and it is of particular interest for prime ideals.

More specifically denoting the set of all ideals of A by $\operatorname{Ideals} A$ and the subset of all H-ideals by $H\text{-}\operatorname{Ideals} A$, we have the core operator,

$$\,\bullet\,{:}H\colon\ \operatorname{Ideals} A \longrightarrow H\text{-}\operatorname{Ideals} A.$$

This operator evidently preserves intersections and it is the identity on the subset $H\text{-}\operatorname{Ideals} A \subseteq \operatorname{Ideals} A$. We are interested in the question as to when this operator preserves primeness or semiprimeness. Recall that I is said to be semiprime if A/I has no nonzero nilpotent ideals or, equivalently, I is an intersection of prime ideals. The above question may seem a mere technicality, but it can be reformulated in various alternative ways, and semiprimeness is a valuable ring-theoretic commodity without which the investigation of H-module algebras can be rather daunting. In joint work with Lorenz and Nguyen [35], we have shown that the operator $\boldsymbol{.}:H$ does indeed preserve semiprimeness when the Hopf algebra H is cocommutative and $\operatorname{char} \Bbbk = 0$.

In subsequent work [57], I studied actions of a ***connected*** Hopf algebras on A, and showed that the core operator $\boldsymbol{.}:H$ does in fact preserve primeness in this case (and hence semiprimeness as well) when $\operatorname{char} \Bbbk = 0$. Recall that a Hopf algebra is said to be connected if the base field $\Bbbk$ is its only simple subcoalgebra. Examples of connected Hopf algebras include universal enveloping algebras of Lie algebras, and coordinate rings of affine algebraic unipotent groups [54].

3.1 Semiprimeness

3.1.1 Reformulations

We start this section by giving several reformulations, in terms of the semiprime radical operator $\sqrt{\boldsymbol{.}}$. The semiprime radical of a subset $X \subseteq A$, by definition, is the unique smallest semiprime ideal of A containing X:

$$\sqrt{X} = \bigcap_{\substack{P \in \operatorname{Spec} A \\ P \supseteq X}} P\,.$$

We continue to assume that $A \in {}_H\mathsf{Alg}$; the Hopf algebra H can be arbitrary for now.

Lemma 3.1.1. *The following are equivalent:*

(i) If J is an H-ideal of A, then so is $\sqrt{J}$;

(ii) for all ideals I of A, the H-core $\sqrt{I}{:}H$ is semiprime;

(iii) $H.\sqrt{I} \subseteq \sqrt{H.I}$ for any ideal I of A.

Proof. (i) $\Rightarrow$ (ii). We may assume that I is semiprime. Then $\sqrt{I{:}H} \subseteq \sqrt{I} = I$, since $\sqrt{\centerdot}$ preserves inclusions. In fact, $\sqrt{I{:}H} \subseteq I{:}H$, because $\sqrt{I{:}H}$ is an H-ideal by (i). The reverse inclusion being trivial, it follows that $I{:}H = \sqrt{I{:}H}$ is semiprime.

(ii) $\Rightarrow$ (iii). Let J denote the ideal of A that is generated by the subset $H.I \subseteq A$. Then $\sqrt{I} \subseteq \sqrt{J} = \sqrt{H.I}$ and J is easily seen to be an H-ideal. (If the antipode S is bijective, then $J = H.I$ [33, Exercise 10.4.3].) Thus, $J = J{:}H \subseteq \sqrt{J}{:}H$ and the latter ideal is semiprime by (ii). It follows that $\sqrt{J} = \sqrt{J{:}H} \subseteq \sqrt{J}{:}H$. Again, the reverse inclusion is clear; so $\sqrt{J} = \sqrt{J}{:}H$. Therefore, $H.\sqrt{I} \subseteq H.\sqrt{J} = \sqrt{J} = \sqrt{H.I}$.

(iii) $\Rightarrow$ (i). Specialize (iii) to the case where $I = J$ is an H-ideal. $\square$

3.1.2 Extending the base field

For the proof of Theorem 3.1.4, we may work over an algebraically closed base field. This follows by taking K to be an algebraic closure of $\Bbbk$ in the argument below.

Let $K/\Bbbk$ be any field extension and put $H' = H \otimes K$ and $A' = A \otimes K$. Then $A' \in {}_{H'}\mathsf{Alg}$ and H' is cocommutative if H is so. Assuming Theorem 3.1.4 to hold for A', our goal is to show that it also holds for A. So let I be a semiprime ideal of A. Viewing A as being contained in A' in the usual way, IA' is an ideal of A' satisfying $IA' \cap A = I$. By Zorn's Lemma, we may choose an ideal I' of A' that is maximal subject to the condition $I' \cap A = I$. Then I' is semiprime. For, if J is any ideal of A' such that $J \supsetneqq I'$, then $J \cap A \supsetneqq I$ by maximality of I', and so $(J \cap A)^2 \nsubseteq I$ by semiprimeness of I. Since $(J \cap A)^2 \subseteq J^2 \cap A$, it follows that $J^2 \nsubseteq I'$, proving that I' is semiprime. Therefore, by our assumption, the core $I'{:}H'$ is semiprime. Since the extension $A \hookrightarrow A'$ is centralizing, it follows that $(I'{:}H') \cap A$ is a semiprime ideal of A. Finally, $(I'{:}H') \cap A = \{a \in A \mid H'.a \subseteq I'\} = \{a \in A \mid H.a \subseteq I' \cap A = I\} = I{:}H$, giving the desired conclusion that $I{:}H$ is semiprime.

3.1.3 Enveloping algebras

For any ring R, let $R[\![X_\lambda]\!]_{\lambda\in\Lambda}$ denote the ring of formal power series in the commuting variables X_λ ($\lambda \in \Lambda$) over R; see [8, Chap. III, §2, n° 11].

Lemma 3.1.2. *Let R be a ring, let Λ be any set, and let S be a subring of $R[\![X_\lambda]\!]_{\lambda\in\Lambda}$ such that S maps onto R under the homomorphism $R[\![X_\lambda]\!]_{\lambda\in\Lambda} \to R$, $X_\lambda \mapsto 0$. If R is prime (resp., semiprime, a domain) then so is S.*

Proof. We write monomials in the variables X_λ as $X^{\boldsymbol{n}} = \prod_\lambda X_\lambda^{\boldsymbol{n}(\lambda)}$ ($\boldsymbol{n} \in M$), where $M = \mathbb{Z}_+^{(\Lambda)}$ denotes the additive monoid of all functions $\boldsymbol{n}\colon \Lambda \to \mathbb{Z}_+$ such that $\boldsymbol{n}(\lambda) = 0$ for almost all $\lambda \in \Lambda$. Fix a total order $<$ on M having the following properties (e.g., [2, Example 2.5]): every nonempty subset of M has a smallest element; the zero function $\mathbf{0}$ is the smallest element of M; and $\boldsymbol{n} < \boldsymbol{m}$ implies $\boldsymbol{n} + \boldsymbol{r} < \boldsymbol{m} + \boldsymbol{r}$ for all $\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{r} \in M$.

For any $0 \neq s = \sum_{\boldsymbol{n}\in M} s_{\boldsymbol{n}} X^{\boldsymbol{n}} \in R[\![X_\lambda]\!]_{\lambda\in\Lambda}$, we may consider its lowest coefficient, $s_{\min} := s_{\boldsymbol{m}}$ with $\boldsymbol{m} = \min\{\boldsymbol{n} \in M \mid s_{\boldsymbol{n}} \neq 0\}$. If R is prime and $0 \neq s, t \in S$ are given, then $0 \neq s_{\min} r t_{\min}$ for some $r \in R$. By assumption, there exists an element $u \in S$ having the form $u = r + \sum_{\boldsymbol{n}\neq\mathbf{0}} u_{\boldsymbol{n}} X^{\boldsymbol{n}}$. It follows that $sut \neq 0$, with $(sut)_{\min} = s_{\min} r t_{\min}$. This proves that S is prime. For the assertions where R is semiprime or a domain, take $s = t$ or $r = 1$, respectively. □

Now let $H = U\mathfrak{g}$ be the enveloping algebra of an arbitrary Lie $\Bbbk$-algebra $\mathfrak{g}$ and assume that $\operatorname{char} \Bbbk = 0$. The primeness assertion of the proposition below is [15, 3.3.2] and the semiprimeness assertion is an easy consequence. We prove all three assertions together below. Recall that an ideal of a ring is said to be ***completely prime*** if the quotient is a domain.

For the proof, we recall the structure of the convolution algebra $\operatorname{Hom}_\Bbbk(H, R)$ for an arbitrary $\Bbbk$-algebra R. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a $\Bbbk$-basis of $\mathfrak{g}$ and fix a total order of the index set Λ. Put $M = \mathbb{Z}_+^{(\Lambda)}$ as in the proof of Lemma 3.1.2 and, for each $\boldsymbol{n} \in M$, put $e_{\boldsymbol{n}} = \prod_\lambda^< \frac{1}{\boldsymbol{n}(\lambda)!} e_\lambda^{\boldsymbol{n}(\lambda)} \in H$, where the superscript $<$ indicates that the factors occur in the order of increasing λ. The elements $e_{\boldsymbol{n}}$ form a $\Bbbk$-basis of H by the Poincaré-Birkhoff-Witt Theorem, and the comultiplication of H is given by

$\Delta e_{\boldsymbol{n}} = \sum_{\boldsymbol{r}+\boldsymbol{s}=\boldsymbol{n}} e_{\boldsymbol{r}} \otimes e_{\boldsymbol{s}}$; see [33, Example 9.5]. Writing $X^{\boldsymbol{n}} = \prod_\lambda X_\lambda^{\boldsymbol{n}(\lambda)}$ as in the proof of Lemma 3.1.2, we obtain an isomorphism of $\Bbbk$-algebras,

$$\varphi\colon \operatorname{Hom}_{\Bbbk}(H,R) \xrightarrow{\sim} R[\![X_\lambda]\!]_{\lambda\in\Lambda}\,, \quad f \mapsto \sum_{\boldsymbol{n}\in M} f(e_{\boldsymbol{n}})X^{\boldsymbol{n}}.$$

Under this isomorphism, the algebra map $u^*\colon \operatorname{Hom}_{\Bbbk}(H,R) \to R$, $f \mapsto f(1)$, coming from the unit map $u = u_H\colon \Bbbk \to H$ translates into the map $R[\![X_\lambda]\!]_{\lambda\in\Lambda} \twoheadrightarrow R$, $X_\lambda \mapsto 0$, as considered in Lemma 3.1.2.

Proposition 3.1.3. *Let $H = U\mathfrak{g}$ be the enveloping algebra of a Lie $\Bbbk$-algebra $\mathfrak{g}$, let $A \in {}_H\mathsf{Alg}$, and let I be an ideal of A. Assume that $\operatorname{char}\Bbbk = 0$. If I is prime, semiprime or completely prime, then $I{:}H$ is likewise.*

Proof. First note that the core $I{:}H$ is identical to the kernel of the map $\delta_I\colon A \to \operatorname{Hom}_{\Bbbk}(H, A/I)$ that is given by $\delta_I(a) = (h \mapsto h.a + I)$. We need to show that the properties of being prime, semiprime, or a domain all transfer from A/I to the subring $\delta_I A \subseteq \operatorname{Hom}(H, A/I)$ or, equivalently, to the subring $S \subseteq (A/I)[\![X_\lambda]\!]_{\lambda\in\Lambda}$ that corresponds to $\delta_I A$ under the above isomorphism φ. Consider the map

$$u^*\colon \operatorname{Hom}_{\Bbbk}(H, A/I) \to A/I, \ \ f \mapsto f(1),$$

and note that $(u^* \circ \delta_I)(a) = a + I$ for $a \in A$. Therefore, S maps onto A/I under the map $(A/I)[\![X_\lambda]\!]_{\lambda\in\Lambda} \to A/I$, $X_\lambda \mapsto 0$. Now all assertions follow from Lemma 3.1.2. $\square$

For an arbitrary cocommutative Hopf algebra H, we cannot expect a result as strong as Proposition 3.1.3: group algebras provide easy counterexamples to the primeness and complete primeness assertions. Indeed, let $H = \Bbbk G$ be the group $\Bbbk$-algebra of the group G and let $A \in {}_H\mathsf{Alg}$. Then $I{:}H = \bigcap_{g\in G} g.I$ for any ideal I of A. If I is semiprime, then so are all $g.I$, because each $g \in G$ acts on A by algebra automorphisms, and hence $\bigcap_{g\in G} g.I$ will be semiprime also. However, primeness and complete primeness, while inherited by each $g.I$, are generally lost upon taking the intersection.

3.1.4 Cocommutative Hopf algebras

Let H be cocommutative Hopf and assume that $\operatorname{char} \Bbbk = 0$ and that $\Bbbk$ is algebraically closed, as we may by §3.1.2. Then H has the structure of a smash product, $H \cong U \# V$, where U is the enveloping algebra of the Lie algebra of primitive elements of H and V is the group algebra of the group of grouplike elements of H; see [52, §13.1] or [45, §15.3].

This will allow us to use what we know about actions of both enveloping algebras of Lie algebras and of group algebras, to prove the following theorem:

Theorem 3.1.4. *Let $A \in {}_H\mathsf{Alg}$ and assume that H is cocommutative and* $\operatorname{char} \Bbbk = 0$. *Then $I{:}H$ is semiprime for every semiprime ideal I of A.*

Proof. Indeed, by the Kostant-Cartier Theorem [], we have that $H \cong U \# V$ as above. Thus, both U and V are Hopf subalgebras of H and $H = UV$, the $\Bbbk$-space spanned by all products uv with $u \in U$ and $v \in V$. Viewing $A \in {}_U\mathsf{Alg}$ and $A \in {}_V\mathsf{Alg}$ by restriction, repeated application of the core operator gives the following equality for any ideal I of A:

$$I{:}H = \{a \in A \mid UV.a \subseteq I\} = \{a \in A \mid V.a \subseteq I{:}U\} = (I{:}U){:}V.$$

If I is semiprime, then so is $I{:}U$ (Proposition 3.1.3). Our remarks on group algebras in the first paragraph of this proof further give semiprimeness of $(I{:}U){:}V$. Thus, $I{:}H$ is semiprime and Theorem 3.1.4 is proved. $\square$

3.2 Connected Hopf algebras

In this section, we will generalize the result of Proposition 3.1.3 to connected Hopf algebras. This class of Hopf algebras includes enveloping algebras of Lie algebras and, in fact, when the characteristic of the field is 0, every ***cocommutative*** connected Hopf algebra is an enveloping algebra of a Lie algebra [38].

Other examples of connected Hopf algebras include the coordinate rings of affine unipotent algebraic groups. Again, if the characteristic of the field is 0, these will

be the only connected ***commutative*** Hopf algebras.
These classifications were achieved by the 1960s, and until recently there were no known examples of non-commutative non-cocommutative connected Hopf algebras over characteristic 0. It wasn't until Zhang [60] that such examples were discovered, which renewed interest in the area.
In addition to generalizing some classical algebras, connected Hopf algebras can provide a starting point into studying pointed Hopf algebras. Every connected Hopf algebra is pointed, and the newly discovered connected Hopf algebras can provide non-cocommutative examples of pointed Hopf algebras.

3.2.1 Result

It turns out that one can achieve a transfer of primeness and semiprimess from an ideal I in an H-module algebra A, to its H-core $I{:}H$. As was previously mentioned, one cannot expect such a result to generalize much further, to, say, pointed Hopf algebras, since it does not hold in the classical case when H is a group algebra.

Theorem 3.2.1. *Let H be a connected Hopf algebra over a field of characteristic 0, let A be an H-module algebra, and let I be an ideal of A. If I is prime (semi-prime, completely prime), then so is $I{:}H$.*

Our main tool in the proof of Theorem 3.2.1 is an analysis of the coradical filtration of H. This will allow us to construct a certain PBW basis of H, which is then used to prove that certain subrings of convolution algebras of H are prime (semiprime, a domain); see Theorem 3.2.7 below. The strategy for the proof is an adaptation of that used in [36, Lemma 16 and Proposition 17]. However, while the convolution algebra in question was a power series ring in [36], this is no longer the case in the present more general context. Theorem 3.2.1 will then be derived from Theorem 3.2.7.

Before we embark on the proof of Theorem 3.2.1, we mention a few things about the history of connected Hopf algebras, and of recent developments that sparked renewed interest.

Notations and conventions. The notation introduced in the foregoing will remain in effect below. We work over a base field $\Bbbk$ and will write $\otimes = \otimes_{\Bbbk}$. Throughout, H will denote a Hopf $\Bbbk$-algebra; the comultiplication $\Delta : H \to H \otimes H$ will be written in the Sweedler notation, $\Delta(h) = h_{(1)} \otimes h_{(2)}$. All further assumptions on $\Bbbk$ and H will be specified as they are needed. We denote by $\mathbb{Z}_+$ the set of non-negative integers and by $\mathbb{Z}_{>0}$ the set of strictly positive integers.

3.2.2 Set-theoretic preliminaries

We introduce here a certain monoid that will be essential for the proof of Theorem 3.2.1.

The monoid M

Let Λ be a set. We consider functions $\mathbf{m}\colon \Lambda \longrightarrow \mathbb{Z}_+$ whose support $\operatorname{supp}\mathbf{m} := \{\lambda \in \Lambda \mid \mathbf{m}(\lambda) \neq 0\}$ is finite. Using the familiar "pointwise" addition of functions, $(\mathbf{m}+\mathbf{m}')(\lambda) = \mathbf{m}(\lambda) + \mathbf{m}'(\lambda)$ for $\lambda \in \Lambda$, we obtain a commutative monoid,

$$M := \mathbb{Z}_+^{(\Lambda)} = \{\mathbf{m}\colon \Lambda \longrightarrow \mathbb{Z}_+ \mid \operatorname{supp}\mathbf{m} \text{ is finite}\},$$

The identity element is the unique function $\mathbf{0} \in M$ such that $\operatorname{supp}\mathbf{0} = \varnothing$; so $\mathbf{0}(\lambda) = 0$ for all $\lambda \in \Lambda$. We will think of Λ as a subset of $M \setminus \{\mathbf{0}\}$, identifying $\lambda \in \Lambda$ with the function $\delta_\lambda \in M$ that is defined by

$$\delta_\lambda(\lambda') = \delta_{\lambda,\lambda'} \qquad (\lambda, \lambda' \in \Lambda),$$

where $\delta_{\lambda,\lambda'}$ is the Kronecker delta. Thus, $\operatorname{supp}\delta_\lambda = \{\lambda\}$ and each $\mathbf{m} \in M$ has the form $\mathbf{m} = \sum_{\lambda\in\operatorname{supp}\mathbf{m}} \mathbf{m}(\lambda)\delta_\lambda$.

Now assume that Λ is equipped with a map $|\,.\,| : \Lambda \longrightarrow \mathbb{Z}_{>0}$, which we will think of as a "degree." We extend $|\,.\,|$ from Λ to M, defining the degree of an element $\mathbf{m} \in M$ by

$$|\mathbf{m}| := \sum_{\lambda\in\operatorname{supp}\mathbf{m}} \mathbf{m}(\lambda)|\lambda| \in \mathbb{Z}_+ .$$

Note that $|\delta_\lambda| = |\lambda|$ and that degrees are additive: $|\mathbf{m} + \mathbf{m}'| = |\mathbf{m}| + |\mathbf{m}'|$ for $\mathbf{m}, \mathbf{m}' \in M$. For a given $d \in \mathbb{Z}_+$, we put

$$M_d := \{\mathbf{m} \in M \mid |\mathbf{m}| = d\} \qquad \text{and} \qquad \Lambda_d := \Lambda \cap M_d\,.$$

Thus, $M = \bigsqcup_{d \in \mathbb{Z}_+} M_d$ and $M_0 = \{\mathbf{0}\}$, $\Lambda_0 = \varnothing$.

A well-order on M

Now assume that Λ is equipped with a total order $\leq$ such that $\Lambda_1 < \Lambda_2 < \dots$, that is, elements of Λ_i precede elements of Λ_j in this order if $i < j$. We extend $\leq$ to M as follows. First order elements of M by degree, i.e.,

$$\{\mathbf{0}\} = M_0 < M_1 < M_2 < \dots\,.$$

For the tie-breaker, let $\mathbf{n} \neq \mathbf{m} \in M_d$ and put

$$\mu = \mu_{\mathbf{n},\mathbf{m}} := \max\{\lambda \in \Lambda \mid \mathbf{n}(\lambda) \neq \mathbf{m}(\lambda)\}.$$

Note that $\{\lambda \in \Lambda \mid \mathbf{n}(\lambda) \neq \mathbf{m}(\lambda)\}$ is finite, being contained in $\mathrm{supp}(\mathbf{n}) \cup \mathrm{supp}(\mathbf{m})$. If $\mathbf{n}(\mu) > \mathbf{m}(\mu)$, then we define $\mathbf{n} > \mathbf{m}$. For $\lambda \neq \lambda' \in \Lambda$, this becomes $\delta_\lambda > \delta_{\lambda'}$ if and only if $\lambda > \lambda'$.

Lemma 3.2.2. *The above order on M has the following properties:*

(a) *$\leq$ is a total order on M and $\mathbf{0}$ is the unique minimal element of M.*

(b) *If $\mathbf{n}, \mathbf{m} \in M$ are such that $\mathbf{n} < \mathbf{m}$, then $\mathbf{n} + \mathbf{r} < \mathbf{m} + \mathbf{r}$ for every $\mathbf{r} \in M$.*

(c) *If the order of each Λ_d is a well-order, then $\leq$ is a well-order of M.*

Proof. (a) It is clear that $\mathbf{0} < \mathbf{m}$ for any $\mathbf{0} \neq \mathbf{m} \in M$ and that exactly one of $\mathbf{n} = \mathbf{m}$, $\mathbf{n} < \mathbf{m}$ or $\mathbf{n} > \mathbf{m}$ holds for any two $\mathbf{n}, \mathbf{m} \in M$. To check transitivity, consider elements $\mathbf{n} < \mathbf{m} < \mathbf{r}$ of M. Then $\mathbf{n} \neq \mathbf{r}$ and we need to show that $\mathbf{n} < \mathbf{r}$. For this, we may assume that $|\mathbf{n}| = |\mathbf{m}| = |\mathbf{r}|$, because otherwise $|\mathbf{n}| < |\mathbf{r}|$ and we are done. Put $\mu = \mu_{\mathbf{n},\mathbf{m}}$, $\nu = \mu_{\mathbf{m},\mathbf{r}}$ and $\rho = \mu_{\mathbf{n},\mathbf{r}}$. Then $\mathbf{n}(\mu) < \mathbf{m}(\mu)$, $\mathbf{m}(\nu) < \mathbf{r}(\nu)$ and we need to show that $\mathbf{n}(\rho) < \mathbf{r}(\rho)$. Let $\lambda := \max\{\mu, \nu\}$ and note that $\mathbf{n}(\rho) \neq \mathbf{r}(\rho)$ implies $\mathbf{n}(\rho) \neq \mathbf{m}(\rho)$ or $\mathbf{m}(\rho) \neq \mathbf{r}(\rho)$, and hence $\rho \leq \lambda$. If $\lambda = \mu$ then $\mathbf{n}(\lambda) < \mathbf{m}(\lambda) \leq \mathbf{r}(\lambda)$, and if $\lambda = \nu$ then $\mathbf{n}(\lambda) \leq \mathbf{m}(\lambda) < \mathbf{r}(\lambda)$. This proves $\lambda = \rho$ and that $\mathbf{n}(\rho) < \mathbf{r}(\rho)$.

(b) By additivity of degrees, $\mathbf{n}+\mathbf{r} < \mathbf{m}+\mathbf{r}$ certainly holds if $|\mathbf{n}| < |\mathbf{m}|$. Assume that $\mathbf{n} < \mathbf{m}$ but $|\mathbf{n}| = |\mathbf{m}|$ and let $\mu = \mu_{\mathbf{n},\mathbf{m}}$; so $\mathbf{n}(\mu) < \mathbf{m}(\mu)$. Then we also have $\mu = \mu_{\mathbf{n}+\mathbf{r},\mathbf{m}+\mathbf{r}}$ and $(\mathbf{n}+\mathbf{r})(\mu) < (\mathbf{m}+\mathbf{r})(\mu)$.

(c) Our assumption easily implies that the order of Λ is a well-order. Now let $\varnothing \neq S \subseteq M$ be arbitrary. We wish to find a minimal element in S. Put

$$d = d(S) := \min\{d' \in \mathbb{Z}_+ \mid S \cap M_{d'} \neq \varnothing\}.$$

If $d = 0$, then $\mathbf{0} \in S$ and $\mathbf{0}$ is the desired minimal element by (a). So assume $d > 0$ and that all $\varnothing \neq T \subseteq M$ with $d(T) < d$ have a minimal element. The desired minimal element of S must belong to $S \cap M_d$; so we may assume without loss that $S \subseteq M_d$. Write $\mu_{\mathbf{n}} = \mu_{\mathbf{n},\mathbf{0}} = \max(\mathrm{supp}(\mathbf{n}))$ for $\mathbf{0} \neq \mathbf{n} \in M$ and put

$$\lambda_S := \min\{\mu_{\mathbf{n}} \mid \mathbf{n} \in S\} \in \Lambda;$$

this is well defined since we have a well-order on Λ. Consider the subsets

$$S' := \{\mathbf{s} \in S \mid \mu_{\mathbf{s}} = \lambda_S\} \quad \text{and} \quad S'' := \{\mathbf{s} \in S' \mid \mathbf{s}(\lambda_S) \leq \mathbf{s}'(\lambda_S)\ \forall \mathbf{s}' \in S^{'}\}$$

and put $z_S := \mathbf{s}(\lambda_S)$ for any $\mathbf{s} \in S''$; so $z_S \in \mathbb{Z}_{>0}$. Since all elements of S'' are smaller than elements of $S \setminus S''$, it suffices to find a minimal element in S''. Notice that, for $\mathbf{r} \in S'$, the function $\mathbf{r} - z_S\delta_{\lambda_S}$ belongs to M. Indeed, $\mathbf{r}(\lambda) \geq z_S$ for all $\lambda \in \Lambda$ and hence $(\mathbf{r} - z_S\delta_{\lambda_S})(\lambda) = \mathbf{r}(\lambda) - z_S \in \mathbb{Z}_+$. By part (b), comparing two elements $\mathbf{r}, \mathbf{t} \in S^{''}$ is equivalent to comparing $\mathbf{r} - z_S\delta_{\lambda_S}$ and $\mathbf{t} - z_S\delta_{\lambda_S}$. Thus setting $T := \{\mathbf{s} - z_S\delta_{\lambda_S} \mid \mathbf{s} \in S''\}$, our goal is to show that T has a minimal element. But $T \subset M$ and all elements of T are of the form $\mathbf{r} - z_S\delta_{\lambda_S}$ where $\mathbf{r} \in S''$, so the degree of any element in T is $d - |\lambda_S|z_S < d$. Therefore, $d(T) < d$ and, by our inductive hypothesis, T has a minimal element. This finishes the proof. $\square$

3.2.3 The coradical filtration

In this section, we recall some standard definitions and state some known facts for later use.

Coradically graded coalgebras

Let C be a $\Bbbk$-coalgebra. Throughout, we let C_0 denote the ***coradical***, that is, the sum of all its simple sub-coalgebras of C. The coalgebra C is said to be ***connected*** if $C_0 = \Bbbk$. The ***coradical filtration*** of C is recursively defined by

$$C_{j+1} := \{x \in C \mid \Delta(x) \in C_j \otimes C + C \otimes C_0\} \qquad (j \geq 0).$$

We also put $C_{-1} := \{0\}$. This yields a coalgebra filtration [45, Prop. 4.1.5]:

$$\{0\} = C_{-1} \subseteq C_0 \subseteq \cdots \subseteq C_j \subseteq C_{j+1} \subseteq \cdots \subseteq C = \bigcup_{n \geq 0} C_n$$

and

$$\Delta(C_j) \subseteq \sum_{0 \leq i \leq j} C_i \otimes C_{j-i} \qquad (j \geq 0).$$

We consider the graded vector space that is associated to this filtration:

$$\operatorname{gr} C := \bigoplus_{n \geq 0} (\operatorname{gr} C)(n) \quad \text{with} \quad (\operatorname{gr} C)(n) := C_n / C_{n-1}\,.$$

Any element $0 \neq \overline{c} \in (\operatorname{gr} C)(n)$ will be called ***homogeneous of degree*** n. The vector space $\operatorname{gr} C$ inherits a natural coalgebra structure from C; this will be discussed greater detail in 3.2.4 below. The resulting coalgebra $\operatorname{gr} C$ is ***coradically graded*** [45, Prop. 4.4.15]: the coradical filtration of $\operatorname{gr} C$ is given by

$$(\operatorname{gr} C)_n = \bigoplus_{0 \leq k \leq n} (\operatorname{gr} C)(k) \qquad (n \in \mathbb{Z}_+).$$

In particular, the coradical of $\operatorname{gr} C$ coincides with the coradical of C:

$$(\operatorname{gr} C)_0 = (\operatorname{gr} C)(0) = C_0\,.$$

Hence $\operatorname{gr} C$ is connected if and only if C is connected.

Some facts about connected Hopf algebras

Let H be a connected Hopf $\Bbbk$-algebra. By 3.2.3, the graded coalgebra $\operatorname{gr} H$ is connected as well. Below, we list some additional known properties of the coradical filtration (H_n) of H and of $\operatorname{gr} H = \bigoplus_{n \geq 0} H_n/H_{n-1}$:

(a) $\operatorname{gr} H$ is a ***coradically graded Hopf algebra*** [45, Prop. 7.9.4]: The coradical filtration (H_n) is also an algebra filtration, that is, $H_n H_m \subseteq H_{n+m}$ for all n, m. Furthermore, all H_n are stable under the antipode of H. Thus, $\operatorname{gr} H$ inherits a natural Hopf algebra structure from H, graded as a $\Bbbk$-algebra and coradically graded as $\Bbbk$-coalgebra.

(b) $\operatorname{gr} H$ is ***commutative*** as $\Bbbk$-algebra [60, Prop. 6.4].

(c) If $\operatorname{char} \Bbbk = 0$, then $\operatorname{gr} H$ is isomorphic to a ***graded polynomial algebra***: There is an isomorphism of graded algebras,

$$\operatorname{gr} H \cong \Bbbk[z_\lambda]_{\lambda \in \Lambda}$$

for some family $\big(z_\lambda\big)_{\lambda \in \Lambda}$ of homogeneous algebraically independent variables, necessarily of strictly positive degrees [59, Proposition 3.6].

3.2.4 A Poincaré-Birkhoff-Witt basis for H

The classical Poincaré-Birkhoff-Witt (PBW) Theorem (see, e.g., [15], [33]) provides a $\Bbbk$-basis for the enveloping algebra $U\mathfrak{g}$ of any Lie $\Bbbk$-algebra $\mathfrak{g}$ that consists of certain ***ordered monomials***: If $(x_\lambda)_{\lambda \in \Lambda}$ is a $\Bbbk$-basis of $\mathfrak{g}$ and $\leq$ is a total order on Λ, then a $\Bbbk$-basis of $U\mathfrak{g}$ is given by the monomials $x_{\lambda_1} x_{\lambda_2} \dots x_{\lambda_n}$ with $n \in \mathbb{Z}_+$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. In this section, working in characteristic 0, we will construct an analogous basis for an arbitrary connected Hopf algebra H, formed by ordered monomials in specified algebra generators for H. We will call it a PBW basis of H.

We assume from this point onward that H is a connected Hopf $\Bbbk$-algebra and that $\operatorname{char} \Bbbk = 0$.

PBW basis

We fix a family of homogeneous algebra generators $(z_\lambda)_{\lambda\in\Lambda}$ of $\operatorname{gr} H$ as in 3.2.3(c), viewing the isomorphism $\operatorname{gr} H \cong \Bbbk[z_\lambda]_{\lambda\in\Lambda}$ as an identification. Define a map $|\,.\,|\colon \Lambda \to \mathbb{Z}_{>0}$ by

$$|\lambda| := \deg z_\lambda \,.$$

Fix a well-order on each set $\Lambda_d := \{\lambda \in \Lambda \mid |\lambda| = d\}$. As in 3.2.2, 3.2.2 consider the monoid $M = \{f\colon \Lambda \longrightarrow \mathbb{Z}_+ \mid \operatorname{supp}(f) \text{ is finite}\}$, equipped with the well-order $\leq$ coming from the chosen orders on each Λ_d.

We consider the following $\Bbbk$-basis of $\operatorname{gr} H$:

$$z_{\mathbf{n}} := \prod_{\lambda\in\Lambda} \frac{1}{\mathbf{n}(\lambda)!} z_\lambda^{\mathbf{n}(\lambda)} = \prod_{\lambda\in\operatorname{supp}\mathbf{n}} \frac{1}{\mathbf{n}(\lambda)!} z_\lambda^{\mathbf{n}(\lambda)} \qquad (\mathbf{n}\in M). \tag{3.1}$$

Note that $z_{\mathbf{n}}$ is homogeneous of degree $|\mathbf{n}| = \sum_{\lambda\in\Lambda} \mathbf{n}(\lambda)|\lambda|$. Our goal is to lift this basis to obtain a PBW basis of H. In detail, for every positive integer k, let

$$\pi_k\colon H_k \twoheadrightarrow (\operatorname{gr} H)(k) = H_k/H_{k-1}$$

be the canonical epimorphism. For each $\lambda \in \Lambda$, let $e_\lambda \in H_{|\lambda|} \subseteq H$ be a fixed pre-image of z_λ under $\pi_{|\lambda|}$. We put

$$e_{\mathbf{n}} = \prod^{<}_{\lambda\in\operatorname{supp}\mathbf{n}} \frac{1}{\mathbf{n}(\lambda)!} e_\lambda^{\mathbf{n}(\lambda)} \qquad (\mathbf{n}\in M), \tag{3.2}$$

where the superscript $<$ indicates that the factors occur in the order of increasing λ. Note that $e_{\mathbf{n}} \in H_{|\mathbf{n}|}$ and $\pi_{|\mathbf{n}|}(e_{\mathbf{n}}) = z_{\mathbf{n}}$. Furthermore, $e_{\mathbf{0}} = 1$.

Lemma 3.2.3. (a) *$\mathscr{M}_n := \left(e_{\mathbf{n}}\right)_{\mathbf{n}\in M, |\mathbf{n}|\leq n}$ is a basis of H_n ($n \geq 0$). Hence, $\mathscr{M} := \left(e_{\mathbf{n}}\right)_{\mathbf{n}\in M}$ is a basis of H.*

(b) *$e_{\mathbf{n}}e_{\mathbf{m}} - c_{\mathbf{n},\mathbf{m}}e_{\mathbf{n}+\mathbf{m}} \in H_{|\mathbf{n}+\mathbf{m}|-1}$ for some $c_{\mathbf{n},\mathbf{m}} \in \Bbbk^\times$.*

Proof. (a) We first prove linear independence of $\mathscr{M}$. Suppose that there is a linear relation $\sum_{i=1}^r a_i e_{\mathbf{n}_i} = 0$ with $r \geq 1$, $\mathbf{n}_i \in M$, and $0 \neq a_i \in \Bbbk$ for all i. Put $d = \max\{|\mathbf{n}_i|\}_{1\leq i\leq r}$. Then $\sum_{i=1}^r a_i e_{\mathbf{n}_i} \in H_d$ and $\pi_d(\sum_{i=1}^r a_i e_{\mathbf{n}_i}) = \sum_{|\mathbf{n}_i|=d} a_i z_{\mathbf{n}_i} = 0$.

Since $\{z_{\mathbf{n}}\}_{\mathbf{n}\in M}$ are linearly independent in $\operatorname{gr} H$, it follows that $a_i = 0$ for all $i \in \{1, 2, ..., r\}$ such that $|\mathbf{n}_i| = d$, contradicting our assumption that all $a_i \neq 0$.

Next, we show that $\mathscr{M}_n$ spans H_n. For $n = 0$, this is certainly true because $H_0 = \Bbbk$ and $e_{\mathbf{0}} = 1$. Now let $n > 0$. For any $h \in H_n$, the projection $\pi_n(h) \in (\operatorname{gr} H)(n)$ can be written as a $\Bbbk$-linear combination $\pi_n(h) = \sum_{i=1}^r a_i z_{\mathbf{n}_i}$ with $|\mathbf{n}_i| = n$. Thus, $h - \sum_{i=1}^r a_i e_{\mathbf{n}_i} \in H_{n-1}$. Since H_{n-1} is spanned by $\mathscr{M}_{n-1}$ by induction, we deduce that h is in the span of $\mathscr{M}_n$.

(b) Note that $e_{\mathbf{n}+\mathbf{m}} \in H_{|\mathbf{n}+\mathbf{m}|}$ and also $e_{\mathbf{n}}e_{\mathbf{m}} \in H_{|\mathbf{n}|}H_{|\mathbf{m}|} \subseteq H_{|\mathbf{n}+\mathbf{m}|}$, where the last inclusion holds by 3.2.3(a) and additivity of $|\,\boldsymbol{.}\,|$. Furthermore, since $\operatorname{gr} H$ is commutative by 3.2.3(b), we have $z_{\mathbf{n}}z_{\mathbf{m}} = c_{\mathbf{n},\mathbf{m}}z_{\mathbf{n}+\mathbf{m}}$ with $c_{\mathbf{n},\mathbf{m}} = \prod_{\lambda\in\Lambda} \frac{(\mathbf{n}+\mathbf{m})(\lambda)!}{\mathbf{n}(\lambda)!\mathbf{m}(\lambda)!} \in \Bbbk^{\times}$. Therefore, by definition of the multiplication of $\operatorname{gr} H$,

$$\pi_{|\mathbf{n}+\mathbf{m}|}(e_{\mathbf{n}}e_{\mathbf{m}}) = \pi_{|\mathbf{n}|}(e_{\mathbf{n}})\pi_{|\mathbf{m}|}(e_{\mathbf{m}}) = z_{\mathbf{n}}z_{\mathbf{m}} = c_{\mathbf{n},\mathbf{m}}z_{\mathbf{n}+\mathbf{m}} = c_{\mathbf{n},\mathbf{m}}\pi_{|\mathbf{n}+\mathbf{m}|}(e_{\mathbf{n}+\mathbf{m}})$$

and so $e_{\mathbf{n}}e_{\mathbf{m}} - c_{\mathbf{n},\mathbf{m}}e_{\mathbf{n}+\mathbf{m}} \in H_{|\mathbf{n}+\mathbf{m}|-1}$ as claimed. □

The family $\mathscr{M}$ will be our PBW basis for H.

The comultiplication on the PBW basis

We will first prove a result concerning the comultiplication of H.

Lemma 3.2.4. $\Delta(h) \in 1 \otimes h + h \otimes 1 + \sum\limits_{i=1}^{n-1} H_i \otimes H_{n-i}$ *for any* $h \in H_n$.

Proof. We start with some preparations, following the procedure that is described in detail in [45, p. 139ff]. The coradical filtration (H_n) can be used to give H a graded $\Bbbk$-vector space structure such that $H_n = \bigoplus_{k=0}^n H(k)$ for all $n \geq 0$ and $\epsilon(H(n)) = 0$ for all $n \geq 1$. Then there are isomorphisms

$$j_n \colon H(n) \xrightarrow{\sim} H_n/H_{n-1} = (\operatorname{gr} H)(n) \qquad (n \geq 0)$$

and we obtain an isomorphism of graded vector spaces,

$$j := \bigoplus_{n\geq 0} j_n \colon H \xrightarrow{\sim} \operatorname{gr} H\,.$$

Letting $p_n\colon H\otimes H \twoheadrightarrow (H\otimes H)(n) := \bigoplus_{r+s=n} H(r)\otimes H(s)$ denote the natural projection, we define

$$\delta := \bigoplus_{n\geq 0} p_n \circ \Delta\big|_{H(n)} \colon H \to H\otimes H.$$

Then, for any $z \in H(n)$ $(n \geq 0)$,

$$\Delta(z) - \delta(z) \in \bigoplus_{r<n} (H\otimes H)(r) = \bigoplus_{i+j<n} H(i)\otimes H(j). \tag{3.3}$$

Transferring δ to $\operatorname{gr} H$ via j, we obtain a comultiplication $\Delta_{\mathrm{gr}}\colon \operatorname{gr} H \to \operatorname{gr} H \otimes \operatorname{gr} H$ making $\operatorname{gr} H$ a graded coalgebra and a commutative diagram:

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "j"'] & H\otimes H \arrow[d, "j\otimes j"] \\ \operatorname{gr} H \arrow[r, "\Delta_{\mathrm{gr}}"] & \operatorname{gr} H \otimes \operatorname{gr} H \end{tikzcd} \tag{3.4}$$

See [45, p. 139-141] for all this and [45, Props. 4.4.15, 7.9.4] for the fact that the above construction results in a coradically graded Hopf algebra, $\operatorname{gr} H$.

Now let B be a $\Bbbk$-bialgebra whose coradical filtration (B_n) is a bialgebra filtration in the sense of [45, Definition 5.6.2]—we will later take $B = \operatorname{gr} H$, where this condition is satisfied. For any $n > 0$, we put

$$\mathscr{P}_n := \big\{ b \in B_n \mid \Delta(z) \in 1\otimes b + b\otimes 1 + \sum_{i=1}^{n-1} B_i \otimes B_{n-i} \big\}.$$

Clearly, $\mathscr{P}_n \subseteq \mathscr{P}_k$ if $k \geq n$ and $\Bbbk\mathscr{P}_n \subseteq \mathscr{P}_n$ for all n. In addition:

Claim. $\mathscr{P}_n\mathscr{P}_m \subseteq \mathscr{P}_{n+m}$ and $\mathscr{P}_n + \mathscr{P}_m \subseteq \mathscr{P}_{\max(n,m)}$ for all $n, m > 0$.

To check this, let $b \in \mathscr{P}_n$, $c \in \mathscr{P}_m$ and put $r := \max(n,m)$. Then

$$\begin{aligned} \Delta(bc) &= \Delta(b)\Delta(c) \\ &\in \big(1\otimes b + b\otimes 1 + \sum_{i=1}^{n-1} B_i\otimes B_{n-i}\big)\big(1\otimes c + c\otimes 1 + \sum_{i=1}^{m-1} B_i \otimes B_{m-i}\big) \\ &\subseteq 1\otimes bc + bc\otimes 1 + b\otimes c + c\otimes b + \sum_{i=1}^{m+n-1} B_i\otimes B_{m+n-i} \\ &= 1\otimes bc + bc\otimes 1 + \sum_{i=1}^{m+n-1} B_i \otimes B_{m+n-i} \end{aligned}$$

$$\begin{aligned}\Delta(b+c) &= \Delta(b)+\Delta(c)\\ &\in \big(1\otimes b+b\otimes 1+\sum_{i=1}^{n-1} B_i\otimes B_{n-i}\big)+\big(1\otimes c+c\otimes 1+\sum_{i=1}^{m-1} B_i\otimes B_{n-i}\big)\\ &\subseteq 1\otimes b+b\otimes 1+1\otimes c+c\otimes 1+\sum_{i=1}^{r-1} B_i\otimes B_{r-i}\\ &= 1\otimes(b+c)+(b+c)\otimes 1+\sum_{i=1}^{r-1} B_i\otimes B_{r-i}\,,\end{aligned}$$

which proves the Claim.

We now apply the foregoing with $B = \operatorname{gr} H$ using the notation of 3.2.3(c). By [59, Theorem 3.2], we know that $z_\lambda \in \mathscr{P}_{|\lambda|}$ for all $\lambda \in \Lambda$. Therefore, the first inclusion in the claim gives that $z_{\mathbf{n}} \in \mathscr{P}_{|\mathbf{n}|}$; see (3.1). Next, observe that any $z \in (\operatorname{gr} H)_n = (\operatorname{gr} H)(0)\oplus\cdots\oplus(\operatorname{gr} H)(n)$ can be written as $z = \sum_{\mathbf{n}\in M, |\mathbf{n}|\le n} c_{\mathbf{n}} z_{\mathbf{n}}$. The second inclusion of the claim therefore gives that $z \in \mathscr{P}_n$.

To finish, let $h \in H_n$ be given and put $z := j(h) \in (\operatorname{gr} H)_n$. By the foregoing, $z \in \mathscr{P}_n$ and so $\Delta_{\mathrm{gr}}(z) \in 1\otimes z + z\otimes 1 + \sum_{i=1}^{n-1} (\operatorname{gr} H)_i \otimes (\operatorname{gr} H)_{n-i}$. Diagram (3.4) now gives $\delta(h) \in 1\otimes h + h\otimes 1 + \sum_{i=1}^{n-1} H_i\otimes H_{n-i}$ and (3.3) further gives $\Delta(h) \in 1\otimes h + h\otimes 1 + \sum_{i=1}^{n-1} H_i\otimes H_{n-i} + \sum_{i=0}^{n-1} H_i\otimes H_{n-i-1}$. Finally, note that $\sum_{i=0}^{n-1} H_i\otimes H_{n-i-1} \subseteq \sum_{i=1}^{n-1} H_i\otimes H_{n-i}$, because $H_0\otimes H_{n-1} \subseteq H_1\otimes H_{n-1}$ and $H_i\otimes H_{n-i-1} \subseteq H_i\otimes H_{n-i}$. Thus, $\Delta(h) \in 1\otimes h + h\otimes 1 + \sum_{i=1}^{n-1} H_i\otimes H_{n-i}$, which completes the proof. $\square$

Recall that $\mathscr{M}_{\mathbf{n}} = \big(e_{\mathbf{n}}\big)_{\mathbf{n}\in M, |\mathbf{n}|\le n}$ is a basis of H_n (Lemma 3.2.3). For a given $\mathbf{n} \in M$, let $H^{<\mathbf{n}}$ denote the subspace of $H_{|\mathbf{n}|}$ that is spanned by the $e_{\mathbf{i}}$ with $\mathbf{i} < \mathbf{n}$ and put $H^{\le\mathbf{n}} = \Bbbk e_{\mathbf{n}} + H^{<\mathbf{n}}$. Furthermore, we put $(H\otimes H)^{<\mathbf{n}} = \sum_{\mathbf{i}+\mathbf{j}<\mathbf{n}} H^{\le\mathbf{i}}\otimes H^{\le\mathbf{j}}$ and $(H\otimes H)^{\le\mathbf{n}} = \sum_{\mathbf{i}+\mathbf{j}\le\mathbf{n}} H^{\le\mathbf{i}}\otimes H^{\le\mathbf{j}}$; these are the subspaces of $H\otimes H$ that are spanned by the tensors $e_{\mathbf{i}}\otimes e_{\mathbf{j}}$ with $\mathbf{i}+\mathbf{j} < \mathbf{n}$ and $\mathbf{i}+\mathbf{j} \le \mathbf{n}$, respectively

Lemma 3.2.5. *For* $\mathbf{n} \in M$ *we have* $\Delta(e_{\mathbf{n}}) \in \sum_{\mathbf{m}+\mathbf{m}'=\mathbf{n}} e_{\mathbf{m}}\otimes e_{\mathbf{m}'} + (H\otimes H)^{<\mathbf{n}}$.

Proof. We start with the following auxiliary observation. Recall from Lemma 3.2.3(b) that $e_{\mathbf{i}}e_{\mathbf{j}} \in \Bbbk e_{\mathbf{i}+\mathbf{j}} + H_{|\mathbf{i}+\mathbf{j}|-1}$ and observe that the right-hand side is contained in

$H^{\leq \mathbf{i}+\mathbf{j}}$. It follows that $H^{\leq \mathbf{n}} H^{\leq \mathbf{m}} \subseteq H^{\leq \mathbf{n}+\mathbf{m}}$ for any $\mathbf{n}, \mathbf{m} \in M$. Therefore,

$$\begin{aligned}(H \otimes H)^{\leq \mathbf{n}}(H \otimes H)^{<\mathbf{m}} &= \sum_{\substack{\mathbf{i}+\mathbf{j} \leq \mathbf{n} \\ \mathbf{k}+\mathbf{l}<\mathbf{m}}} H^{\leq \mathbf{i}} H^{\leq \mathbf{k}} \otimes H^{\leq \mathbf{j}} H^{\leq \mathbf{l}} \\ &\subseteq \sum_{\substack{\mathbf{i}+\mathbf{j} \leq \mathbf{n} \\ \mathbf{k}+\mathbf{l}<\mathbf{m}}} H^{\leq \mathbf{i}+\mathbf{k}} \otimes H^{\leq \mathbf{j}+\mathbf{l}} \\ &\subseteq \sum_{\mathbf{r}+\mathbf{s}<\mathbf{n}+\mathbf{m}} H^{\leq \mathbf{r}} \otimes H^{\leq \mathbf{s}} = (H \otimes H)^{<\mathbf{n}+\mathbf{m}}.\end{aligned} \tag{3.5}$$

Similarly, $(H \otimes H)^{<\mathbf{n}}(H \otimes H)^{\leq \mathbf{m}} \subseteq (H \otimes H)^{<\mathbf{n}+\mathbf{m}}$.

To prove the Lemma, let us put

$$\Sigma(\mathbf{n}) := \sum_{\mathbf{m}+\mathbf{m}'=\mathbf{n}} e_{\mathbf{m}} \otimes e_{\mathbf{m}'}\,.$$

Our goal is to show that $\Delta(e_{\mathbf{n}}) \in \Sigma(\mathbf{n}) + (H \otimes H)^{<\mathbf{n}}$. This certainly holds for $\mathbf{n} = \mathbf{0}$, since $e_{\mathbf{0}} = 1$ and $\Delta(1) = 1 \otimes 1 = \Sigma(\mathbf{0})$. Now assume that $\mathbf{n} \neq \mathbf{0}$ and proceed by induction on $|\operatorname{supp} \mathbf{n}|$. If $|\operatorname{supp} \mathbf{n}| = 1$, then $\mathbf{n} = n\delta_\lambda$ and $e_{\mathbf{n}} = e_{n\delta_\lambda} = \frac{e_\lambda^n}{n!}$ for some $\lambda \in \Lambda$, $n \in \mathbb{Z}_{>0}$. By Lemma 3.2.4, we know that $\Delta(e_\lambda) \in 1 \otimes e_\lambda + e_\lambda \otimes 1 + \sum_{i=1}^{|\lambda|-1} H_i \otimes H_{|\lambda|-i}$. The summand $H_i \otimes H_{|\lambda|-i}$ is spanned by the tensors $e_{\mathbf{j}} \otimes e_{\mathbf{j}}$ with $|\mathbf{i}| \leq i$ and $|\mathbf{j}| \leq |\lambda| - i$ by Lemma 3.2.3; so $|\mathbf{i} + \mathbf{j}| \leq |\lambda|$. In addition, $|\mathbf{i}|, |\mathbf{j}| < |\lambda|$. Thus, if $\mu \geq \lambda$, then $\mathbf{i}(\mu) = \mathbf{j}(\mu) = 0$, since $|\lambda| \leq |\mu|$. This shows that $\mathbf{i} + \mathbf{j} < \delta_\lambda$. Therefore, $\sum_{i=1}^{|\lambda|-1} H_i \otimes H_{|\lambda|-i} \subseteq (H \otimes H)^{<\delta_\lambda}$ and hence $\Delta(e_\lambda) \in 1 \otimes e_\lambda + e_\lambda \otimes 1 + (H \otimes H)^{<\delta_\lambda}$. Now we compute:

$$\begin{aligned}\Delta(e_{n\delta_\lambda}) = \frac{1}{n!}\Delta(e_\lambda)^n &\in \frac{1}{n!}\big(1 \otimes e_\lambda + e_\lambda \otimes 1 + (H \otimes H)^{<\delta_\lambda}\big)^n \\ &\subseteq \frac{1}{n!}(1 \otimes e_\lambda + e_\lambda \otimes 1)^n + (H \otimes H)^{<n\delta_\lambda},\end{aligned}$$

where the last inclusion is obtained by expanding the product and using (3.5) on all but the first summand. The binomial theorem further gives $\frac{1}{n!}(1 \otimes e_\lambda + e_\lambda \otimes 1)^n = \sum_{i=0}^{n} \frac{1}{i!(n-i)!}\, e_\lambda^i \otimes e_\lambda^{n-i} = \sum_{i=0}^{n} e_{i\delta_\lambda} \otimes e_{(n-i)\delta_\lambda} = \Sigma(n\delta_\lambda)$. Therefore,

$$\Delta(e_{n\delta_\lambda}) \in \Sigma(n\delta_\lambda) + (H \otimes H)^{<n\delta_\lambda}.$$

For the inductive step, let $\mathbf{n} \in M$ with $|\operatorname{supp} \mathbf{n}| > 1$. Put $\mu := \max \operatorname{supp} \mathbf{n}$, $n := \mathbf{n}(\mu)$, and $\mathbf{n}' := \mathbf{n} - n\delta_\mu$. Thus, $\operatorname{supp} \mathbf{n}' = \operatorname{supp} \mathbf{n} \setminus \{\mu\}$ and hence, by induction, $\Delta(e_{\mathbf{n}'}) \in \Sigma(\mathbf{n}') + (H \otimes H)^{<\mathbf{n}'}$. Furthermore,

$$e_{\mathbf{n}} = \prod_{\lambda \in \operatorname{supp} \mathbf{n}}^{<} \frac{1}{\mathbf{n}(\lambda)!} e_\lambda^{\mathbf{n}(\lambda)} = \Big(\prod_{\lambda \in \operatorname{supp} \mathbf{n}, \lambda \neq \mu}^{<} \frac{1}{\mathbf{n}(\lambda)!} e_\lambda^{\mathbf{n}(\lambda)} \Big) \frac{e_\mu^n}{n!} = e_{\mathbf{n}'}\, e_{n\delta_\mu} .$$

It follows that

$$\Delta(e_{\mathbf{n}}) = \Delta(e_{\mathbf{n}'})\Delta(e_{n\delta_\mu}) \in \big(\Sigma(\mathbf{n}') + (H \otimes H)^{<\mathbf{n}'}\big)\big(\Sigma(n\delta_\mu) + (H \otimes H)^{<n\delta_\mu}\big)$$

By (3.5), $(H \otimes H)^{<\mathbf{n}'}\Sigma(n\delta_\mu) \subseteq (H \otimes H)^{<\mathbf{n}'}(H \otimes H)^{\leq n\delta_\mu} \subseteq (H \otimes H)^{<\mathbf{n}}$ as well as $\Sigma(\mathbf{n}')(H \otimes H)^{<n\delta_\mu} \subseteq (H \otimes H)^{<\mathbf{n}}$ and $(H \otimes H)^{<\mathbf{n}'}(H \otimes H)^{<n\delta_\mu} \subseteq (H \otimes H)^{<\mathbf{n}}$. Finally, $\Sigma(\mathbf{n}')\Sigma(n\delta_\mu) = \Sigma(\mathbf{n})$. This completes the proof. $\square$

3.2.5 The convolution algebra

In this section, we fix an arbitrary $\Bbbk$-algebra R (associative, with 1) and consider the convolution algebra $\operatorname{Hom}_{\Bbbk}(H, R)$; this is a $\Bbbk$-algebra with convolution $*$ as multiplication:

$$(f * g)(h) = f(h_{(1)})g(h_{(2)}) \qquad (f, g \in \operatorname{Hom}_{\Bbbk}(H, R), h \in H).$$

We continue working under the standing assumptions that the Hopf algebra H is connected and $\operatorname{char} \Bbbk = 0$.

Minimal support elements

Let $M = \mathbb{Z}_+^{(\Lambda)}$ be the monoid of Section 3.2.2 and let $\mathscr{M}$ be the PBW basis of H as in Section 3.2.4; so $\mathscr{M} \cong M$ as sets via $e_{\mathbf{n}} \leftrightarrow \mathbf{n}$. Since every $f \in \operatorname{Hom}_{\Bbbk}(H, R)$ is determined by its values on the basis $\mathscr{M}$, which can be arbitrarily assigned elements of R, we have a bijection $\Phi\colon \operatorname{Hom}_{\Bbbk}(H, R) \xrightarrow{\sim} R^M$, the set of all functions $s\colon M \to R$; explicitly,

$$(\Phi f)(\mathbf{n}) = f(e_{\mathbf{n}}) \qquad (f \in \operatorname{Hom}_{\Bbbk}(H, R), \mathbf{n} \in M). \tag{3.6}$$

Now assume that M is equipped with the well-order $\leq$ of Lemma 3.2.2. Note that the support $\operatorname{supp} \Phi f = \{\mathbf{n} \in M \mid f(e_{\mathbf{n}}) \neq 0\}$ need not be finite if $f \neq 0$, but it does have a smallest element for the well-order $\leq$. For $0 \neq f \in \operatorname{Hom}_{\Bbbk}(H, R)$, we may therefore define

$$\mathbf{f} := \min \operatorname{supp} \Phi f \in M \qquad \text{and} \qquad f_{\min} := f(e_{\mathbf{f}}) \in R \setminus \{0\}. \tag{3.7}$$

Lemma 3.2.6. *Let* $0 \neq f, g \in \operatorname{Hom}_{\Bbbk}(H, R)$ *and define* $\mathbf{f}, \mathbf{g} \in M$ *as in* (3.7). *Then:*

(a) $(f * g)(e_{\mathbf{n}}) = 0$ *for* $\mathbf{n} < \mathbf{f} + \mathbf{g}$ *and* $(f * g)(e_{\mathbf{f}+\mathbf{g}}) = f(e_{\mathbf{f}})\, g(e_{\mathbf{g}})$.

(b) *If* $f(e_{\mathbf{f}})\, g(e_{\mathbf{g}}) \neq 0$, *then* $f * g \neq 0$ *and* $(f * g)_{\min} = f(e_{\mathbf{f}})\, g(e_{\mathbf{g}})$.

Proof. Since part (b) is clear from (a), we will focus on (a).

Note that f vanishes on $H^{<\mathbf{f}}$, the subspace of H that is generated by the basis elements $e_{\mathbf{n}}$ with $\mathbf{n} < \mathbf{f}$; similarly, $g(H^{<\mathbf{g}}) = \{0\}$. For any $e_{\mathbf{i}} \otimes e_{\mathbf{j}} \in (H \otimes H)^{<\mathbf{f}+\mathbf{g}}$, we have either $\mathbf{i} < \mathbf{f}$ or $\mathbf{j} < \mathbf{g}$, because otherwise $\mathbf{i} + \mathbf{j} \geq \mathbf{f} + \mathbf{g}$ by Lemma 3.2.2(b). Hence, $f(e_{\mathbf{i}})g(e_{\mathbf{j}}) = 0$ and so

$$m \circ (f \otimes g)(H \otimes H)^{<\mathbf{f}+\mathbf{g}} = 0. \tag{3.8}$$

Now let $\mathbf{n} \in M$ be such that $\mathbf{n} \leq \mathbf{f} + \mathbf{g}$. By Lemma 3.2.5,

$$\Delta(e_{\mathbf{n}}) \in \sum_{\mathbf{m}+\mathbf{m}'=\mathbf{n}} e_{\mathbf{m}} \otimes e_{\mathbf{m}'} + (H \otimes H)^{<\mathbf{n}}.$$

Equation (3.8) gives us that $(f * g)(e_{\mathbf{n}}) = 0$ if $\mathbf{n} < \mathbf{f} + \mathbf{g}$, and

$$(f * g)(e_{\mathbf{n}}) = \sum_{\mathbf{m}+\mathbf{m}'=\mathbf{n}} f(e_{\mathbf{m}})g(e_{\mathbf{m}'})$$

if $\mathbf{n} = \mathbf{f} + \mathbf{g}$. Consider $(\mathbf{m}, \mathbf{m}') \in M \times M$ with $\mathbf{m} + \mathbf{m}' = \mathbf{n}$ as in the sum above. If $\mathbf{m} < \mathbf{f}$ or $\mathbf{m}' < \mathbf{g}$, then $f(e_{\mathbf{m}})g(e_{\mathbf{m}'}) = 0$. Therefore, the only contribution to the sum comes from the pair $(\mathbf{m}, \mathbf{m}') = (\mathbf{f}, \mathbf{g})$, proving the lemma. □

Subrings of the convolution algebra

The unit map $u = u_H \colon \Bbbk \to H$ gives rise to the following algebra map:

$$u^* \colon \operatorname{Hom}_{\Bbbk}(H, R) \twoheadrightarrow R, \quad f \mapsto f(1).$$

The theorem below is an adaptation of Lemma 3.1.2.

Theorem 3.2.7. *Assume that* $\operatorname{char} \Bbbk = 0$. *Let* R *be a* $\Bbbk$*-algebra, let* H *be a connected Hopf* $\Bbbk$*-algebra, and let* $S \subseteq \operatorname{Hom}_{\Bbbk}(H, R)$ *be a subring such that* $u^*(S) = R$. *If* R *is prime (semiprime, a domain), then so is* S.

Proof. First assume that R is prime. Let $0 \neq s, t \in S$ be given and let $s_{\min}, t_{\min} \in R \setminus \{0\}$ be as in (3.7). Then $0 \neq s_{\min} r t_{\min}$ for some $r \in R$. By assumption, there exists an element $u \in S$ such that $u(1) = r$. Since $1 = e_{\mathbf{0}}$, we evidently have $\mathbf{0} = \min \operatorname{supp} \Phi u$ and $u_{min} = r$. It now follows from Lemma 3.2.6 that $s * u * t \neq 0$ and $(s * u * t)_{min} = s_{min} r t_{min}$. This proves that S is prime.

For the assertions where R is semiprime or a domain, take $s = t$ or $r = 1$, respectively. □

Of course, Theorem 3.2.7 applies with $S = \operatorname{Hom}_{\Bbbk}(H, R)$. Thus, we obtain the following corollary.

Corollary 3.2.8. *Let* H *be a connected Hopf algebra over a field* $\Bbbk$ *of characteristic* 0 *and let* R *be a* $\Bbbk$*-algebra that is prime (semiprime, a domain), then so is the convolution algebra* $\operatorname{Hom}_{\Bbbk}(H, R)$.

Proof of Theorem 3.2.1

We are now ready to give the proof of Theorem 3.2.1, which is also a consequence of Theorem 3.2.7.

Let A be an H-module algebra and let I be an arbitrary ideal of A. The core $(I : H)$ is the kernel of the map $\rho\colon A \longrightarrow \operatorname{Hom}_{\Bbbk}(H, A/I)$ that is defined by $\rho(a) = (h \mapsto h.a + I)$. So $\rho(A) \cong A/(I : H)$ as rings. Note that the composite map $u^* \circ \rho\colon A \to \operatorname{Hom}_{\Bbbk}(H, A/I) \to A/I$ is the canonical epimorphism, $a \mapsto a + I$ $(a \in A)$. So $u^*(\rho(A)) = A/I$. Therefore, Theorem 3.2.7 applies with $S = \rho(A)$ and $R = A/I$. We obtain that when A/I is prime (semiprime, a domain), then so is $\rho(A)$. Since $\rho(A) \cong A/(I : H)$, this is equivalent to the statement of Theorem 3.2.1. □

CHAPTER 4

JOINT WORK

4.1 Hopf algebra actions: stratification

Introduction

Let H be a Hopf algebra over a field $\Bbbk$ and let A be an arbitrary associative $\Bbbk$-algebra. An ***action*** of H on A is given by a $\Bbbk$-linear map $H \otimes A \to A$, $h \otimes a \mapsto h.a$, that makes A into a left H-module and satisfies the "measuring" conditions $h.(ab) = (h_1.a)(h_2.b)$ and $h.1 = \langle \epsilon, h \rangle 1$ for $h \in H$ and $a, b \in A$. Here, $\otimes = \otimes_{\Bbbk}$, $\Delta h = h_1 \otimes h_2$ denotes the comultiplication of H, and ϵ is the counit. We will write $H \circlearrowleft A$ to indicate such an action. Algebras equipped with an H-action are called left ***H-module algebras***. With algebra maps that are also H-module maps as morphisms, left H-module algebras form a category, ${}_H\mathsf{Alg}$. For example, an action of a group algebra $\Bbbk G$ on A amounts to the datum of a group homomorphism $G \to \operatorname{Aut} A$, the automorphism group of the algebra A. For the enveloping algebra $U\mathfrak{g}$ of a Lie $\Bbbk$-algebra $\mathfrak{g}$, an action $U\mathfrak{g} \circlearrowleft A$ is given by a Lie homomorphism $\mathfrak{g} \to \operatorname{Der} A$, the Lie algebra of all derivations of A. In both these prototypical cases, the acting Hopf algebra is cocommutative. We present here the results of investigating the effect of a given action $H \circlearrowleft A$ of an arbitrary cocommutative Hopf algebra H on the prime and semiprime ideals of A, partially generalizing prior work of Lorenz on rational actions of algebraic groups [30], [31], [32].

If $A \neq 0$ and the product of any two nonzero H-ideals of A is again nonzero, then A is said to be ***H-prime***. An H-ideal I of A is called ***H-prime*** if A/I is H-prime. It is easy to see that H-cores of prime ideals are H-prime. Denoting the collection of all H-primes of A by H-$\operatorname{Spec} A$, we thus obtain a map $\operatorname{Spec} A \to H\text{-}\operatorname{Spec} A$, $P \mapsto P{:}H$. The fibers

$$\operatorname{Spec}_I A \stackrel{\text{def}}{=} \{P \in \operatorname{Spec} A \mid P{:}H = I\}$$

are called the ***H-strata*** of $\operatorname{Spec} A$. The stratification $\operatorname{Spec} A = \bigsqcup_{I \in H\text{-}\operatorname{Spec} A} \operatorname{Spec}_I A$ was pioneered by Goodearl and Letzter [20] in the case of group actions or, equivalently, actions of group algebras. It has proven to be a useful tool for investigating $\operatorname{Spec} A$, especially for rational actions of a connected affine algebraic group G over an algebraically closed field $\Bbbk$. In this case, one has a description of each stratum $\operatorname{Spec}_I A$ in terms of the prime spectrum of a suitable ***commutative*** algebra [31, Theorem 9]. Our principal goal is to generalize this result to the context of cocommutative Hopf algebras.

Main result

To state our main result, we first make some observations. Let H be cocommutative and $\Bbbk$ algebraically closed. Assume that the action $H \circlearrowright A$ is locally finite, that is, $\dim_{\Bbbk} H.a < \infty$ for all $a \in A$. Then A becomes a right comodule algebra over the (commutative) finite dual H° of H:

$$\begin{array}{ccc} A & \longrightarrow & A \otimes H^\circ \\ a & \longmapsto & a_0 \otimes a_1 \end{array} \quad \text{with} \quad h.a = a_0\langle a_1, h\rangle \qquad (h \in H, a \in A). \tag{4.1}$$

The action $H \circlearrowright A$ will be called ***integral*** if the image of the map (4.1) is contained in $A \otimes \mathcal{O}$ for some Hopf subalgebra $\mathcal{O} \subseteq H^\circ$ that is an integral domain. This condition serves as a replacement for connectedness in the case of algebraic group actions. Assuming it to be satisfied, it follows that each $I \in H$-$\operatorname{Spec} A$ is in fact a prime ideal of A. Consequently, the extended center $\mathcal{C}(A/I) = \mathcal{Z}\,\mathrm{Q}(A/I)$ is a $\Bbbk$-field, where $\mathrm{Q}(A/I)$ denotes the symmetric ring of quotients of A/I. The action

$H \circlearrowright A/I$ extends uniquely to an H-action on $\mathrm{Q}(A/I)$ and this action stabilizes the center $\mathcal{C}(A/I)$. Furthermore, $\mathcal{O} \in {}_H\mathsf{Alg}$ via the "hit" action $\rightharpoonup$ that is given by $\langle h \rightharpoonup f, k\rangle = \langle f, kh\rangle$ for $f \in \mathcal{O}$ and $h, k \in H$. The actions $\rightharpoonup$ and $H \circlearrowright \mathcal{C}(A/I)$ combine to an H-action on the tensor product; so

$$C_I := \mathcal{C}(A/I) \otimes \mathcal{O} \in {}_H\mathsf{Alg}\,. \tag{4.2}$$

The algebra C_I is a commutative integral domain. We let $\mathrm{Spec}^H C_I$ denote the subset of $\mathrm{Spec}\, C_I$ consisting of all prime H-ideals of C_I .

Theorem 4.1.1. *Let H be a cocommutative Hopf algebra over an algebraically closed field $\Bbbk$ and let A be a $\Bbbk$-algebra that is equipped with an integral action $H \circlearrowright A$. Then, for any $I \in H\text{-}\mathrm{Spec}\, A$, there is a bijection*

$$c\colon\ \mathrm{Spec}_I\, A \overset{\sim}{\to} \mathrm{Spec}^H C_I$$

having the following properties, for any $P, P' \in \mathrm{Spec}_I\, A$:

(i) $c(P) \subseteq c(P')$ *if and only if* $P \subseteq P'$*, and*

(ii) $\mathrm{Fract}(C_I/c(P)) \cong \mathcal{C}((A/P) \otimes \mathcal{O})$.

Examples

As a first example, let G be an affine algebraic $\Bbbk$-group and let $\mathcal{O} = \mathcal{O}(G)$ be the algebra of polynomial functions on G. Then $\mathcal{O} \subseteq H^\circ$ with $H = \Bbbk G$. A rational G-action on A, by definition, is a locally finite action $H \circlearrowright A$ such that the image of (4.1) is contained in $A \otimes \mathcal{O}$. If G is connected, then $\mathcal{O}$ is an integral domain and so the action is integral. In this setting, Theorem 4.1.1 is covered by [31, Theorem 9].

Next, let $\mathfrak{g}$ be a Lie $\Bbbk$-algebra acting by derivations on A and assume that every $a \in A$ is contained in some finite-dimensional $\mathfrak{g}$-stable subspace of A. With $H = U\mathfrak{g}$, we then have a locally finite action $H \circlearrowright A$ and hence a map (4.1). If $\mathrm{char}\, \Bbbk = 0$, then the convolution algebra H^* is a power series algebra over $\Bbbk$ and hence H^* is a commutative domain. Since H° is a subalgebra of H^*, we may take $\mathcal{O} = H^\circ$ and so we have an integral action. Theorem 4.1.1 appears to be new in this case.

Outlook

In future work, we hope to pursue the general theme of the work described above for Hopf algebras that are not necessarily cocommutative. In particular, we plan to address "rationality" of prime ideals and explore the Dixmier-Mœglin equivalence in the context of Hopf algebra actions, generalizing the work on group actions in [30], [31].

4.2 Adjoint action of Hopf algebras

Locally finite parts

Let H be a Hopf algebra over a field $\Bbbk$, with comultiplication $\Delta h = h_{(1)} \otimes h_{(2)}$, antipode S, and counit ϵ. The left ***adjoint action*** of H on itself is defined by

$$k.h = k_{(1)} h\, \mathrm{S}(k_{(2)}) \qquad (h, k \in H). \tag{4.3}$$

This action makes H a left H-module algebra that will be denoted by H_{ad}. Our main interest is in the locally finite part,

$$H_{\text{ad fin}} = \{h \in H \mid \dim_{\Bbbk} H.h < \infty\}.$$

Of course, if H is finite dimensional, then $H_{\text{ad fin}} = H$; so we are primarily concerned with infinite-dimensional Hopf algebras.

Finitely generating subalgebras

Locally finite parts may of course be defined for arbitrary representations as the sum of all finite-dimensional subrepresentations: for any left module V over a $\Bbbk$-algebra R,

$$V_{\text{fin}} \stackrel{\text{def}}{=} \{v \in V \mid \dim_{\Bbbk} R.v < \infty\}.$$

This gives a functor $\,\boldsymbol{.}_{\text{fin}}$ on the category ${}_R\mathsf{Mod}$ of left R-modules. If R is finitely generated as right module over some subalgebra T, then $V_{\text{fin}} = \{v \in V \mid \dim_{\Bbbk} T.v < \infty\}$ for any $V \in {}_R\mathsf{Mod}$. Adopting group-theoretical terminology, we will call a

Hopf algebra H ***virtually*** of type $\mathcal{C}$, where $\mathcal{C}$ is a given class of Hopf algebras, if H is finitely generated as right module over some Hopf subalgebra $K \in \mathcal{C}$.

Main theorem

The locally finite part A_{fin} of any left H-module algebra A is a subalgebra that contains the algebra of H-invariants, $A^H = \{a \in A \mid h.a = \langle \epsilon, h \rangle a$ for all $h \in H\}$. For $A = H_{\text{ad}}$, the invariant algebra coincides with the center of H [33, Lemma 10.1]. The center is rarely a Hopf subalgebra of H, even if H is a group algebra, and $H_{\text{ad fin}}$ need not be a Hopf subalgebra either in general, for example when H is a quantized enveloping algebra; see [4, Example 2.8] or [29]. However, we have the following result. Recall that a left coideal subalgebra of H is a subalgebra C that is also a left coideal of H, i.e., $\Delta(C) \subseteq H \otimes C$.

Theorem 4.2.1. (a) *$H_{\text{ad fin}}$ is always a left coideal subalgebra of H.*

(b) *If H is virtually cocommutative, then $H_{\text{ad fin}}$ is a Hopf subalgebra of H.*

For a quantized enveloping algebra of a complex semisimple Lie algebra, $H = U_q(\mathfrak{g})$, part (a) is due to Joseph and Letzter: it follows from [23, Theorem 4.10] that $U_q(\mathfrak{g})_{\text{ad fin}}$ is a left coideal subalgebra and this fact is also explicitly stated as [29, Theorem 5.1]. Part (b) extends an earlier result of Bergen [4, Theorem 2.18] to arbitrary characteristics. Bergen's proof is based on his joint work with Passman [7], which determines $H_{\text{ad fin}}$ explicitly for group algebras and for enveloping algebras of Lie algebras in characteristic 0.

Finite parts of tensor products

Some of our work is in the context of (virtually) pointed Hopf algebras. Over an algebraically closed field, all cocommutative Hopf algebras are pointed [52, Lemma 8.0.1]. However, many pointed Hopf algebras of interest are not necessarily cocommutative. Examples include the algebras of polynomial functions of solvable connected affined algebraic groups (over an algebraically closed base field) and quantized enveloping algebras of semisimple Lie algebras. It turns out that $\bullet_{\text{fin}}$ is a tensor functor for virtually pointed Hopf algebras:

Theorem 4.2.2. *If H is virtually pointed, then $(V \otimes W)_{\text{fin}} = V_{\text{fin}} \otimes W_{\text{fin}}$ for any $V, W \in {}_H\mathsf{Mod}$.*

Dietzmann's lemma

A standard group-theoretic fact, known as Dietzmann's Lemma, states that any finite subset of a group that is stable under conjugation and consists of torsion elements generates a finite subgroup ([13] or [26, §53]). Our final result is the following version of Dietzmann's Lemma for arbitrary Hopf algebras.

Proposition 4.2.3. *Let $C_1, \dots, C_k$ be finite-dimensional left coideal subalgebras of H and assume that $C = \sum_{i=1}^{k} C_i$ is stable under the adjoint action of H. Then C generates a finite-dimensional subalgebra of H.*

Of course, the subalgebra that is generated by C is also a left coideal subalgebra, stable under the adjoint H-action, and it is contained in $H_{\text{ad fin}}$. Our proof of Proposition 4.2.3 will show that if all C_i are in fact sub-bialgebras of H, then it suffices to assume that C is stable under the adjoint actions of all C_i.

Outlook

As was mentioned earlier, the finite part $H_{\text{ad fin}}$ of a group algebra $H = \Bbbk G$ is the subgroup algebra $\Bbbk G_{\text{fin}}$ of the FC-center G_{fin}. In this case, it is known that $\Bbbk G$ is prime if and only if G_{fin} is torsion-free abelian [43]. For Hopf algebras, an analogous primeness criterion would be highly desirable. This is another possible future project.

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