ANALYTIC CONTINUATION OF NONANALYTIC VECTOR-VALUED EISENSTEIN SERIES

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
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May, 2006

ABSTRACT

ANALYTIC CONTINUATION OF NONANALYTIC VECTOR-VALUED EISENSTEIN SERIES

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We give an analytic continuation of a vector-valued nonanalytic Eisenstein series associated to a representation χ_ρ. The representation χ_ρ is induced from the representation ρ associated to a holomorphic vector-valued modular form.

ACKNOWLEDGEMENTS

I want to thank the Temple mathematical community past and present. In particular I would like to thank my advisor, Marvin Knopp, for his tremendous commitment to developing mathematicians. I also would like to acknowledge Professor Ehrenpreis for his support. Finally, I would like to acknowledge my office mates: Tadele Mengesha, Henok Mawi, Sheng Xiong. It was a pleasure sharing space with them.

To my parents,

Elmer and Jean,

with great love and appreciation.

TABLE OF CONTENTS

CHAPTER 1

Introduction

1.1 Background

Let $f(z)$ be a modular cusp form of weight k, $k > 0$, on a subgroup $\Gamma' \subseteq \Gamma = SL(2,\mathbb{Z})$. Let $f(z) = \sum_{n+\kappa>0}^{\infty} a_n e^{2\pi i \frac{(n+\kappa)z}{\lambda}}$. The classical (scalar) Rankin-Selberg method provides the estimate

$$a_n = O(n^{\frac{k}{2}-\frac{1}{5}}) \tag{1.1}$$

for Γ' a congruence subgroup. In [18] Selberg observes that by extending the Rankin-Selberg method to vector-valued modular cusp forms, defined below, with unitary representation, one obtains the estimate (1.1) for $f(z)$ a modular cusp form on an arbitrary subgroup $\Gamma' \subseteq \Gamma$ of finite index. Details are provided in the appendix.

In the Rankin-Selberg method, the zeta function, $\zeta_{\overrightarrow{F}}(s)$ associated to the vector-valued form $(\overrightarrow{F}, \rho)$ of level N, has the integral representation

$$\left(\frac{4\pi}{N}\right)^{-(s+k-1)} \Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \int_{\mathfrak{F}} \sum_{V\in<S^N>\backslash\Gamma} \Im(Vz)^s y^k \overline{\overrightarrow{F}}(z)\rho^t(V)\overline{\rho}(V)\overrightarrow{F}^t(z)\,\frac{dxdy}{y^2}. \tag{1.2}$$

If ρ is unitary, then (1.2) becomes

$$\left(\frac{4\pi}{N}\right)^{-(s+k-1)} \Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \int_{\mathcal{F}} E(z,s) y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}. \tag{1.3}$$

Here $E(z,s) = \frac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$ is the (scalar) nonanalytic Eisenstein series. The Eisenstein series is nonanalytic in the variable z. The series defines an analytic function in the s variable for $\Re s > 1$. It admits an analytic continuation (we use the expression analytic continuation even when the continuation is meromorphic) to the whole s-plane and satisfies the functional equation $E(z,s) = \phi(s)E(z,1-s)$, where $\phi(s) = \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$, see [4]. The analytic continuation and functional equation for $\zeta_{\overrightarrow{F}}(s)$ follow from the corresponding properties of the Eisenstein series. Thus the Rankin-Selberg method works if $(\overrightarrow{F}, \rho)$ is a vector-valued modular form with unitary representation. The question arises: Can the Rankin-Selberg method can be extended to $(\overrightarrow{F}, \rho)$ with arbitrary representation?

For arbitrary ρ satisfying $\rho(S^N)$ unitary, we are led to study the matrix-valued Eisenstein series

$$E_s(z,\rho) = \sum_{V \in <S^N>\backslash\Gamma} \Im(Vz)^s \rho^t(V)\overline{\rho}(V). \tag{1.4}$$

If we establish an analytic continuation and functional equation for the Eisenstein series (1.4) then the Rankin-Selberg method will work. We make assumptions on the representation ρ in order to handle the series (1.4). We consider representations ρ for which $\rho^t\overline{\rho}$ is diagonal. In fact we will assume ρ is monomial, a class of representations for which $\rho^t\overline{\rho}$ is diagonal.

Definition 1.1 *A representation $\rho : \Gamma \longrightarrow GL(p,\mathbb{C})$ is called monomial if $\rho(V)$ has exactly one non-zero entry in each column and row.*

The assumption that ρ is monomial leads to a vector-valued nonanalytic Eisenstein series $\overrightarrow{E}(z,s;\chi_\rho)$ where χ_ρ is a representation induced by ρ. The Rankin-Selberg method suggests we prove that $\overrightarrow{E}(z,s;\chi_\rho)$ admits an analytic continuation to the whole s-plane and that it satisfies a functional equation. In this thesis we modify a method due to Selberg [19] to obtain the analytic continuation of $\overrightarrow{E}(z,s;\chi_\rho)$.

This thesis is organized as follows. In chapter 2, we introduce a vector-valued Eisenstein series with representation induced from a monomial representation. We prove its basic properties and give its Fourier expansion. We also introduce a generalized Ramanujan sum and corresponding zeta function. In chapter 3 we discus the matrix resolvent kernel. We develop its Fourier expansion via the double coset decomposition. We also prove estimates needed in the sequel. In chapter 4 we prove $\overrightarrow{E}(z,s;\chi_\rho)$ has an analytic continuation to the whole s-plane. The appendix gives a proof of the Rankin-Selberg estimates in the unitary case.

1.2 Definitions

1.2.1 The Hyperbolic Plane

Let H denote the upper half-plane in the complex variable $z = x+iy, \quad y > 0$. The invariant area element on H, denoted $d\mu(z)$, is

$$d\mu(z) = \frac{dxdy}{y^2}.$$

The invariant line element is $dl = \frac{|dz|}{y}$. Let Γ denote the group of fractional linear transformations

$$\begin{aligned} g : H &\longrightarrow H \\ z &\longrightarrow \frac{az+b}{cz+d}; \quad a,b,c,d \in \mathbb{Z} \quad ad-bc=1. \end{aligned}$$

We have $\Gamma \equiv SL(2,\mathbb{Z})\backslash\{\pm I\}$. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$ represent the same fractional linear transformation, we use the convention

$$c \geq 0 \quad \text{and if } c = 0 \quad \text{then } d = +1. \tag{1.5}$$

We use $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for the standard generators of Γ. A fundamental domain for Γ is given by

$$\mathcal{F} = \{z \in H : |z| > 1 \quad |x| < \frac{1}{2}\}. \tag{1.6}$$

Next for Y large we decompose $\mathcal{F} = \mathcal{F}(Y) \bigcup \mathcal{F}_\infty(Y)$, where $\mathcal{F}(Y) = \{z \in \mathcal{F} : y \leq Y\}$ is a compact region and $\mathcal{F}_\infty(Y) = \{z \in \mathcal{F} : y > Y\}$ is called the cuspidal zone. In the analysis of the kernel we must handle summation over all of Γ. This is accomplished through the Bruhat or double coset decomposition [7] of Γ:

$$\Gamma = \Gamma_\infty \cup \bigcup_{c>0} \bigcup_{d \bmod c} \Omega_{cd}. \tag{1.7}$$

The union is disjoint. Here $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{Z}\}$, $\Omega_{cd} = \Gamma_\infty \omega_{cd} \Gamma_\infty$ and $\omega_{cd} = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ Therefore we can further decompose (1.7) as

$$\Gamma = \Gamma_\infty \cup \bigcup_{c>0} \bigcup_{d \bmod c} \bigcup_{n=-\infty}^{\infty} \bigcup_{m=-\infty}^{\infty} S^n \omega_{cd} S^m. \tag{1.8}$$

The hyperbolic laplacian is $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. In what follows, we use the point-pair invariant function

$$u(z, z') = \frac{|z - z'|^2}{4yy'}. \tag{1.9}$$

A point-pair invariant function $u(z, z')$ satisfies $u(\gamma z, \gamma z') = u(z, z') \quad \forall \gamma \in G$ [17].

Remark 1.1 *Given $g : \mathbb{R}^+ \longrightarrow \mathbb{C}$, we form the kernel function $\tilde{g}(z, z')$ defined on $H \times H$ by $\tilde{g}(z, z') = g(u(z, z'))$. By abuse of notation we denote $\tilde{g}(z, z')$ by $g(z, z')$ or, for emphasis, by $g(u(z, z'))$.*

The function $u(z, z')$ is related to the hyperbolic distance $d(z, z')$ by the formula [7]

$$\cosh d(z, z') = 1 + 2u(z, z'). \tag{1.10}$$

1.2.2 Vector-Valued Modular Forms

Let $k \in \mathbb{R}$. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index p in Γ. We let $A_1, \cdots, A_p$ denote a complete set of right coset representatives of Γ' in Γ. Let v be a multiplier system for the group Γ' and weight k. A function, $f(z)$, holomorphic on H is a modular cusp form with respect to (Γ', k, v) if, see [8],

i) $f(Vz) = v(V)(cz+d)^k f(z) \quad \forall \quad V = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma'$;

ii) at each cusp $q_j = A_j(\infty)$, $f(z)$ has the expansion

$$f(z) = \sigma_j(z) \sum_{n+\kappa_j > \mu_j} a_n(j) e^{2\pi i (n+\kappa_j)\frac{(A_j^{-1}z)}{N_j}} \qquad \mu_j > 0. \tag{1.11}$$

Here

$$\sigma_j(z) = \begin{cases} 1, & \text{if } q_j = \infty; \\ \frac{1}{z-q_j}, & \text{if } q_j < \infty. \end{cases} \tag{1.12}$$

Also κ_j is defined by $v(A_j S^{N_j} A_j^{-1}) = e^{2\pi i \kappa_j} \quad 0 \leq \kappa_j < 1$; N_j is the smallest positive integer such that $A_j S^{N_j} A_j^{-1} \in \Gamma'$.

In a series of papers [10],[11], and [12] Knopp and Mason developed a general theory of vector-valued modular forms analogous to the classical (scalar) case. The following definition is given in [12].

Definition 1.2 *A vector-valued modular cusp form $(\overrightarrow{F}, \rho)$ of real weight k on the modular group Γ is a p-tuple $\overrightarrow{F}(z) = (F_1(z), \ldots, F_p(z))$ of functions holomorphic in the complex upper half-plane H, together with a p-dimensional complex representation $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ satisfying the following:*

(a) For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$(F_1(z), \ldots, F_p(z))^t|_k V(z) = \rho(V)(F_1(z), \ldots, F_p(z))^t. \tag{1.13}$$

(b) Each component function $F_j(z)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(z) = \sum_{n \geq h_j} a_n(j) e^{\frac{2\pi i n z}{N_j}}, \tag{1.14}$$

with $h_j, N_j \in \mathbb{Z}^+$, and $q = e^{2\pi iz}$.

The slash operator $|_k V$ in (1.13) is defined by

$$F_j|_k V(z) = \overline{v}(V)(cz+d)^{-k} F_j(Vz) \tag{1.15}$$

with v a multiplier system with respect to Γ. It is assumed that v satisfies the nontriviality condition

$$v(-I) = (-1)^{-k}. \tag{1.16}$$

The space of vector-valued modular cusp forms is denoted by $\mathcal{S}(k.\rho, v)$. The level, N, of $\overrightarrow{F}$ is defined by

$$N = lcm\{N_1, \dots, N_p\} \quad N = N_j m_j. \tag{1.17}$$

Thus we can write,

$$F_j(z) = \sum_{n\geq 1} a_n(j) e^{\frac{2\pi i m_j n z}{N}}. \tag{1.18}$$

If we allow $h_j \in \mathbb{Z}$ in (1.14) then we get the space of vector-valued modular forms denoted $\mathcal{M}(k, \rho, v)$. If $(\overrightarrow{F}(z), \rho)$ is a nontrivial element of $\mathcal{M}(k, \rho, v)$ of level N, then (1.13) and (1.18) imply

$$\begin{aligned} \overrightarrow{F}^t(z) &= \overrightarrow{F}^t(S^N z) \\ &= v(S^N)\rho(S^N)\overrightarrow{F}^t(z). \end{aligned} \tag{1.19}$$

Therefore

$$[v(S)\rho(S)]^N = I. \tag{1.20}$$

In other words,

$$\rho(S^N) = \begin{pmatrix} e^{-2\pi i\kappa N} & 0 & \dots & 0 \\ 0 & e^{-2\pi i\kappa N} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{-2\pi i\kappa N} \end{pmatrix}. \tag{1.21}$$

Remark 1.2 *If $k \in \mathbb{Q}$, then there exist nontrivial nonunitary monomial representations of Γ. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index with positive genus. Then one can construct a parabolic multiplier system v on Γ'. That is, a multiplier system where $|v(M)|$ is not identically 1, $M \in \Gamma'$, and $v(P) = 1$ for P parabolic with trace 2. Knopp and Mason in [10] prove there exists a nontrivial $f(z)$ satisfying i), above, with v a parabolic multiplier system and ii), above, with $\mu_j \in \mathbb{R}$. This nontrivial parabolic generalized modular form can then be lifted to a nontrivial $(\overrightarrow{F}(z), \rho)$ on Γ with ρ monomial and nonunitary.*

Next we define vector-valued automorphic forms of Maass type. Fix a p-dimensional complex representation ρ.

Definition 1.3 *The p-tuple $\overrightarrow{F}(z) = (F_1(z), ..., F_p(z))$ is said to be a vector-valued automorphic function with respect to (Γ, ρ) if*

$$\overrightarrow{F}^t(Vz) = \rho(V)\overrightarrow{F}^t(z) \quad \forall \quad V \in \Gamma. \tag{1.22}$$

We denote the space of $\overrightarrow{F}(z)$ satisfying (1.22) by $\mathcal{A}(\Gamma \backslash H, \rho)$.

Definition 1.4 *$\overrightarrow{F}(z) \in \mathcal{A}(\Gamma \backslash H, \rho)$ is called a vector-valued automorphic form if each $F_j(z)$ is an eigenfunction of the laplacian*

$$(\Delta - s(s-1))F_j(z) = 0. \tag{1.23}$$

We denote this space by $\mathcal{A}_s(\Gamma \backslash H, \rho)$, thus

$$\mathcal{A}_s(\Gamma \backslash H, \rho) = \\ \{\overrightarrow{F}(z) \in \mathcal{A}(\Gamma \backslash H, \rho) : (\Delta - s(s-1))F_j(z) = 0 \quad 1 \leq j \leq p\}. \tag{1.24}$$

Remark 1.3 *By the general theory of the laplacian, $\overrightarrow{F}(z) \in \mathcal{A}_s(\Gamma \backslash H, \rho)$ implies each $F_j(z)$ is real analytic.*

We need a subspace with growth conditions, thus we define

$$\mathcal{B}'_\mu(\Gamma \backslash H, \rho) = \{\overrightarrow{F}(z) \in \mathcal{A}(\Gamma \backslash H, \rho) : F_j(z) \in C^\infty(H) \text{and} \\ F_j(z) = O(y^\mu) \text{ and } \frac{\partial F_j(z)}{\partial y} = O(y^\mu) \text{ for } y \text{ sufficiently large}\}. \tag{1.25}$$

1.3 Eichler's Estimate

Unitary representations have bounded entries. For nonunitary representations we rely on the Eichler estimate to bound the entries. For the modular group, Eichler's theorem is stated [9]:

Theorem 1.1 *If $V \in \Gamma$ consider a factorization of V into sections, $V = C_1 \cdots C_l$. Each section C_i is either a nonparabolic generator of Γ, i.e. T, or a power of a parabolic generator of Γ, i.e. $S^k \quad k \in \mathbb{Z}$. Then for any $V \in \Gamma$ the factorization can be carried out so that*

$$l \leq n_1 log\mu(V) + n_2, \tag{1.26}$$

where $n_1, n_2 > 0$ are independent of V and

$$\mu(V) = a^2 + b^2 + c^2 + d^2 \quad \text{if } V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \tag{1.27}$$

The following Lemma proved in [9] is extremely useful in the sequel.

Lemma 1.1 *For real numbers c, d and $z = x + iy$, we have*

$$\left(\frac{y^2}{1+4|z|^2}\right)(c^2+d^2) \leq |cz+d|^2 \leq 2(|z|^2 + y^{-2})(c^2+d^2). \tag{1.28}$$

The next estimate allows us to bound the Eichler length by a value which depends only on the values in the last row.

Lemma 1.2 *Given $\left(\begin{smallmatrix} a & b \\ m & n \end{smallmatrix}\right) \in \Gamma$ and $N \in \mathbb{Z}^+$ there exists $q \in \mathbb{Z}$ such that $S^{qN}\left(\begin{smallmatrix} a & b \\ m & n \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ m & n \end{smallmatrix}\right)$ and*

$$a'^2 + b'^2 \leq N^2(m^2 + n^2). \tag{1.29}$$

In particular

$$a'^2 + b'^2 + m^2 + n^2 \leq (N^2+1)(m^2+n^2). \tag{1.30}$$

Proof: Given $\left(\begin{smallmatrix} a & b \\ m & n \end{smallmatrix}\right) \in \Gamma$, we have

$$an - bm = 1. \tag{1.31}$$

Using the division algorithm, we write

$$a = mNq_a + l_a, \quad q_a \in \mathbb{Z} \quad 0 \le l_a < mN; \tag{1.32}$$
$$b = mNq_b + l_b, \quad q_b \in \mathbb{Z} \quad 0 \le l_b < mN.$$

Substitute (1.32) into (1.31) to get

$$(mNq_a + l_a)n - (mNq_b + l_b)m = 1,$$

therefore

$$mNq_b + l_b = \frac{(mNq_a + l_a)n - 1}{m}. \tag{1.33}$$

Now consider

$$\begin{pmatrix} 1 & -Nq_a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ m & n \end{pmatrix} = \begin{pmatrix} a-mNq_a & b-nNq_a \\ m & n \end{pmatrix}. \tag{1.34}$$

Then by (1.32)

$$(1.34) = \begin{pmatrix} l_a & mNq_b+l_b-nNq_a \\ m & n \end{pmatrix},$$

and by (1.33)

$$\begin{aligned} (1.34) &= \begin{pmatrix} l_a & \frac{(mNq_a+l_a)n-1}{m} - nNq_a \\ m & n \end{pmatrix} \\ &= \begin{pmatrix} l_a & \frac{l_an-1}{m} \\ m & n \end{pmatrix} \\ &= \begin{pmatrix} a' & b' \\ m & n \end{pmatrix}. \end{aligned}$$

We obtain

$$\begin{aligned} a'^2 + b'^2 &= l_a^2 + (\frac{l_an-1}{m})^2 \quad 0 \le l_a < Nm \\ &\le (N)m^2 + \frac{[(l_a+1)|n|)]^2}{m^2} \\ &\le (Nm)^2 + \frac{(Nm)^2n^2}{m^2} \\ &= N^2(m^2+n^2). \end{aligned} \tag{1.35}$$

Theorem 1.1 implies the following:

Proposition 1.1 *Let* $\omega_{cd} \in \Gamma$ *satisfy*

$$\mu(\omega_{cd}) \le (N^2+1)(c^2+d^2). \tag{1.36}$$

Then there exists α_χ *and* K'_χ *such that*

$$|\chi_{ij}(\omega_{cd})| \leq (N^2+1)^{\alpha_\chi} K'_\chi (c^2+d^2)^{\alpha_\chi}. \tag{1.37}$$

Proof: Define

$$K_\chi = \max_{1\leq l,m\leq p, 0\leq k\leq N-1} \{|\chi_{lm}(T)|, |\chi_{lm}(S^k)|\}. \tag{1.38}$$

Let

$$\omega_{cd} = V_1 \cdots V_l$$

be the factorization in Theorem 1.1. We have

$$\begin{aligned}
\chi_{ij}(\omega_{cd}) &= \chi_{ij}(V_1 \cdots V_l) \\
&= \sum_{k_1}^{p} \chi_{ik_1}(V_1)\chi_{k_1 j}(V_2 \cdots V_l) \\
&= \sum_{k_1}^{p} \sum_{k_2}^{p} \cdots \sum_{k_{l-1}}^{p} \chi_{ik_1}(V_1)\chi_{k_1 k_2}(V_2) \cdots \chi_{k_{l-1} j}(V_l).
\end{aligned}$$

Now if

$$\begin{aligned}
V &= S^n \\
&= S^{qN+k} \quad 0 \leq k \leq N-1,
\end{aligned}$$

then

$$\chi(S^{qN+k}) = \chi(S^{qN})\chi(S^k).$$

If χ satisfies (1.21), then

$$|\chi_{ij}(S^{qN+k})| = |\chi_{ij}(S^k)|.$$

Therefore

$$\begin{aligned}
|\chi_{ij}(\omega_{cd})| &\leq p^{l(\omega_{cd})-1} K_\chi^{l(\omega_{cd})} \\
&\leq (K_\chi p)^{n_1 \mu(\omega_{cd}) + n_2} p^{-1} \quad \text{by (1.26).}
\end{aligned}$$

Thus

$$|\chi_{ij}(\omega_{cd})| = \mu(\omega_{cd})^{\alpha_\chi} \frac{(K_\chi p)^{n_2}}{p}$$

where

$$\alpha_\chi = n_1 \log K_\chi p.$$

Using (1.36), we have

$$|\chi_{ij}(\omega_{cd})| \leq (N^2+1)^{\alpha_\chi} K'_\chi (c^2+d^2)^{\alpha_\chi}, \tag{1.39}$$

where $K'_\chi = \frac{(K_\chi p)^{n_2}}{p}$.

CHAPTER 2

Eisenstein Series

In this chapter we define the vector-valued nonanalytic Eisenstein series $\overrightarrow{E}(z, s; \chi_\rho)$.

2.1 Preliminaries

Proposition 2.1 *Let ρ be a monomial p dimensional representation, then $\rho^t\overline{\rho}$ is diagonal.*

Proof: $\rho(V)$ monomial implies there exists $\sigma_V \in S_p$, S_p is the symmetric group on p letters, and $\alpha_1(V), \ldots, \alpha_p(V) \in \mathbb{C}$ such that

$$\rho(V) = \alpha_1(V)E_{1\sigma_V(1)} + \alpha_2(V)E_{2\sigma_V(2)} + \ldots + \alpha_p(V)E_{p\sigma_V(p)}, \tag{2.1}$$

where $E_{ij} = (\delta_{ij}(k,l))_{1\le k,l\le p}$. Then
$\rho^t(V)\overline{\rho}(V)$

$$
\begin{aligned}
&= \left(\alpha_1(V)E_{1\sigma_V(1)} + \ldots + \alpha_p(V)E_{p\sigma_V(p)}\right)^t \left(\overline{\alpha}_1(V)E_{1\sigma_V(1)} + \ldots + \overline{\alpha}_p(V)E_{p\sigma_V(p)}\right)\\
&= \sum_{i,j=1}^{p} \alpha_i(V)\overline{\alpha}_j(V)E_{\sigma_V(i)i}E_{\sigma_V(j)j}\\
&= \sum_{i,j=1}^{p} \alpha_i(V)\overline{\alpha}_j(V)\delta_{ij}E_{\sigma_V(i)\sigma_V(j)}\\
&= \sum_{i=1}^{p} |\alpha_i(V)|^2 E_{\sigma_V(i)\sigma_V(i)}.
\end{aligned}
$$

Thus

$$
\rho^t(V)\overline{\rho}(V) = \begin{pmatrix} |\alpha_{\sigma^{-1}(1)}(V)|^2 & & & \\ & |\alpha_{\sigma^{-1}(2)}(V)|^2 & & \\ & & \ddots & \\ & & & |\alpha_{\sigma^{-1}(p)}(V)|^2 \end{pmatrix}.
$$

Proposition 2.2 *Let ρ be a monomial representation. Let $\alpha_i(V)$ and $\sigma_V(i)$ be defined by* (2.1)*. We have*

$$\sigma_{VW}(i) = \sigma_W(\sigma_V(i)) \tag{2.2}$$

$$\textit{and the cocycle condition} \quad \alpha_i(VW) = \alpha_i(V)\alpha_{\sigma_V(i)}(W). \tag{2.3}$$

Proof:

$$\begin{aligned}
\rho(VW) &= \sum_{i}^{p} \alpha_i(VW) E_{i\sigma_{VW}(i)} \\
&= \rho(V)\rho(W) \\
&= \sum_{i,j=1}^{p} \alpha_i(V)\alpha_j(W) E_{i\sigma_V(i)} E_{j\sigma_W(j)} \\
&= \sum_{i,j=1}^{p} \alpha_i(V)\alpha_j(W) \delta_{\sigma_V(i)j} E_{i\sigma_W(j)} \\
&= \sum_{i=1}^{p} \alpha_i(V)\alpha_{\sigma_V(i)}(W) E_{i\sigma_W(\sigma_V(i))}.
\end{aligned}$$

Thus $\alpha_i(VW)E_{i\sigma_{VW}(i)} = \alpha_i(V)\alpha_{\sigma_V(i)}(W)E_{i\sigma_W(\sigma_V(i))}$ and the result follows. $[v(S)\rho(S)]^N = I$ implies

$$\begin{aligned}
|\alpha_i(S^N)| &= 1 \\
\sigma_S^N(i) &= i.
\end{aligned} \tag{2.4}$$

Relations (2.2) and (2.3) imply

$$\begin{aligned}
i = \sigma_I(i) &= \sigma_{\gamma\circ\gamma^{-1}}(i) = \sigma_{\gamma^{-1}} \circ \sigma_\gamma(i) \\
&= \sigma_{\gamma^{-1}\gamma}(i) = \sigma_\gamma \circ \sigma_{\gamma^{-1}}(i).
\end{aligned}$$

Therefore

$$\sigma_\gamma^{-1} = \sigma_{\gamma^{-1}}. \tag{2.5}$$

Also,

$$\begin{aligned}
1 = \alpha_i(I) &= \alpha_i(\gamma\gamma^{-1}) \\
&= \alpha_i(\gamma)\alpha_{\sigma_\gamma(i)}(\gamma^{-1}) \qquad \text{by (2.3).}
\end{aligned}$$

Therefore

$$\alpha_{\sigma_\gamma(i)}(\gamma^{-1}) = \frac{1}{\alpha_i(\gamma)}. \tag{2.6}$$

Remark 2.1 *ρ a representation of $SL(2,\mathbb{Z})$ which restricts to Γ implies $\rho(-I) = \rho(I)$. This, in turn, implies $\sigma_{-V} = \sigma_V$ and $\alpha_i(-V) = \alpha_i(V)$.*

2.2 Eisenstein Series

We define the Eisenstein Series $\overrightarrow{E}(z,s;\chi_\rho)=(E_1(z,s;\chi_\rho),\cdots,E_p(z,s;\chi_\rho))$ by

$$E_i(z,s;\chi_\rho)=\sum_{M\in<S^N>\backslash\Gamma}|\alpha_{\sigma_M^{-1}(i)}(M)|^2\Im(Mz)^s. \tag{2.7}$$

We have

$$\begin{aligned}E_i(z,s;\chi_\rho)&=\sum_{V\in\Gamma_\infty\backslash\Gamma}\sum_{k=0}^{N-1}|\alpha_{\sigma_{S^kV}^{-1}(i)}(S^kV)|^2\Im(S^kVz)^s \\ &=\sum_{V\in\Gamma_\infty\backslash\Gamma}\sum_{k=0}^{N-1}|\alpha_{\sigma_{S^k}^{-1}\sigma_V^{-1}(i)}(S^k)\alpha_{\sigma_V^{-1}(i)}(V)|^2\Im(Vz)^s \\ &=\frac{1}{2}\sum_{(c,d)=1}\sum_{k=0}^{N-1}\frac{|\alpha_{\sigma_{S^k}^{-1}\sigma_V^{-1}(i)}(S^k)\alpha_{\sigma_V^{-1}(i)}(V)|^2y^s}{|cz+d|^{2s}}.\end{aligned} \tag{2.8}$$

Here $V=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$.

Remark 2.2 (2.7) *is well defined since*

$$\begin{aligned}|\alpha_{\sigma_{S^{jN}M}^{-1}(i)}(S^{jN}M)|^2&=|\alpha_{\sigma_{S^{jN}}^{-1}\circ\sigma_M^{-1}(i)}(S^{jN}M)|^2 \\ &=|\alpha_{\sigma_{S^{jN}}^{-1}\circ\sigma_M^{-1}(i)}(S^{jN})\alpha_{\sigma_{S^{jN}}\circ\sigma_{S^{jN}}^{-1}\circ\sigma_M^{-1}(i)}(M)|^2 \\ &=|\alpha_{\sigma_M^{-1}(i)}(M)|^2.\end{aligned}$$

Introduce the $p\times p$ matrix

$$\chi_\rho(\gamma)=\sum_{i=1}^{p}|\alpha_{\sigma_\gamma(i)}(\gamma^{-1})|^2E_{i\sigma_\gamma(i)}. \tag{2.9}$$

Using (2.6), we have

$$\chi_\rho(\gamma)=\sum_{i=1}^{p}\frac{1}{|\alpha_i(\gamma)|^2}E_{i\sigma_\gamma(i)}. \tag{2.10}$$

Proposition 2.3 *Let* $\chi_\rho(\gamma)=\sum_{i=1}^{p}\frac{1}{|\alpha_i(\gamma)|^2}E_{i\sigma_\gamma(i)}$. *Then* χ_ρ *is a representation.*

Proof:

$$\text{We have } \chi_\rho(VW) = \sum_{i=1}^{p} \frac{1}{|\alpha_i(VW)|^2} E_{i\sigma_{VW}(i)}$$
$$= \sum_{i=1}^{p} \frac{1}{|\alpha_i(V)|^2 |\alpha_{\sigma_V(i)}(W)|^2} E_{i\sigma_W \circ \sigma_V(i)}.$$

Also,

$$\chi_\rho(V)\chi_\rho(W) = \left(\sum_{i=1}^{p} \frac{1}{|\alpha_i(V)|^2} E_{i\sigma_V(i)}\right)\left(\sum_{j=1}^{p} \frac{1}{|\alpha_j(W)|^2} E_{j\sigma_W(j)}\right) \tag{2.11}$$
$$= \sum_{i,j=1}^{p} \frac{1}{|\alpha_i(V)|^2} \frac{1}{|\alpha_j(W)|^2} E_{i\sigma_V(i)} E_{j\sigma_W(j)}$$
$$= \sum_{i}^{p} \frac{1}{|\alpha_i(V)|^2} \frac{1}{|\alpha_j(W)|^2} \delta_{\sigma_V(i)j} E_{i\sigma_V(i)} E_{j\sigma_W(j)}$$
$$= \sum_{i}^{p} \frac{1}{|\alpha_i(V)|^2} \frac{1}{|\alpha_{\sigma_V(i)}(W)|^2} E_{i\sigma_W \circ \sigma_V(i)}.$$

Therefore $\chi_\rho(VW) = \chi_\rho(V)\chi_\rho(W)$. We call χ_ρ the representation induced by ρ.

Proposition 2.4 *$E_i(z, s; \chi_\rho)$ is absolutely convergent for $\Re s > 1 + \alpha$.*

Proof:

$$\text{Let } q_V = tr\left(\rho^t(V)\overline{\rho}(V)\right)$$
$$= \sum_{i=1}^{p} |\alpha_{\sigma_V^{-1}(i)}(V)|^2$$
$$= \sum_{i=1}^{p} |\alpha_i(V)|^2.$$

Thus, for $\sigma = \Re s$,

$$E_i(z,\sigma;\chi_\rho) \leq \sum_{V\in<S^N>\backslash\Gamma} q_V \Im(Vz)^\sigma \tag{2.12}$$

$$= \sum_{V\in\Gamma_\infty\backslash\Gamma} \sum_{k=0}^{N-1} q_{S^kV} \Im(S^kVz)^\sigma \tag{2.13}$$

$$= \sum_{V\in\Gamma_\infty\backslash\Gamma} \sum_{k=0}^{N-1} q_{S^kV} \Im(Vz)^\sigma \tag{2.14}$$

Now,

$$q_{VW} = \sum_{i=1}^{p} |\alpha_i(VW)|^2 = \sum_{i=1}^{p} |\alpha_i(V)|^2 |\alpha_{\sigma_V(i)}(W)|^2 \tag{2.15}$$

by (2.2). Let $V = V_1V_2\cdots V_l$ be the factorization given in Theorem 1.1. We have

$$q_{S^kV} = q_{S^kV_1V_2\cdots V_l} \tag{2.16}$$

$$= \sum_{i=1}^{p} |\alpha_i(S^k)||\alpha_{\sigma_{S^k(i)}}(V_1)|^2 |\alpha_{\sigma_{V_1}\circ\sigma_{S^k(i)}}(V_2)|^2 \cdots |\alpha_{\sigma_{V_{l-1}}\circ\sigma_{V_{l-2}}\circ\cdots\circ\sigma_{V_1}\circ\sigma_{S^k(i)}}(V_l)|^2.$$

Let

$$K_{\chi_\rho} = \max\left(|\alpha_j(T)|^2, |\alpha_j(S^k)|^2\right)_{1\leq k\leq N,\, 1\leq j\leq p}. \tag{2.17}$$

We have

$$\begin{aligned} q_{S^kV} &\leq K_{\chi_\rho}^{l+1} p \\ &\leq K_{\chi_\rho}^{n_1 log\mu(S^kV)} K_{\chi_\rho}^{n_2+1} p \quad \text{by (1.26)} \\ &= \mu(S^kV)^\alpha K_{\chi_\rho}^{n_2+1} p \qquad \text{where} \quad \alpha_{\chi_\rho} = n_1 log K_{\chi_\rho}. \end{aligned}$$

The proof of Lemma 1.2 implies we can choose a set $\mathcal{M}$ of left coset representatives of Γ_∞ in Γ such that

$$a^2 + b^2 \leq N^2(c^2+d^2) \quad \forall \quad V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}.$$

This implies

$$\mu(S^kV) \leq 2(k^2+N^2)(c^2+d^2) \tag{2.18}$$

$$\leq 4N^2(c^2+d^2) \quad \text{for } 0 \leq k \leq N-1 \tag{2.19}$$

Thus

$$\begin{aligned}
&\sum_{V\in\Gamma_\infty\backslash\Gamma}\sum_{k=0}^{N-1} q_{S^kV}\Im(Vz)^\sigma\\
=&\sum_{V\in\mathcal{M}}\sum_{k=0}^{N-1}\frac{q_{S^kV}y^\sigma}{|cz+d|^{2\sigma}}\\
\le&\sum_{V\in\mathcal{M}}\sum_{k=0}^{N-1}\frac{\mu(S^kV)^\alpha K_{\chi_\rho}^{n_2+1}py^\sigma}{|cz+d|^{2\sigma}}\\
\le&(4N^2)^\alpha K_{\chi_\rho}^{n_2+1}pN\sum_{V\in\mathcal{M}}\frac{(c^2+d^2)^\alpha y^\sigma}{|cz+d|^{2\sigma}}
\end{aligned}$$

From Lemma 1.1, we have $c^2+d^2\le\frac{1+4|z|^2}{y^2}|cz+d|^2$. It follows that

$$E_i(z,\sigma;\chi_\rho)\le(4N^2)^\alpha K_{\chi_\rho}^{n_2+1}pN(\frac{1+4|z|^2}{y})^\alpha\sum_{V\in\mathcal{M}}\frac{y^{\sigma-\alpha}}{|cz+d|^{2\sigma-2\alpha}} \tag{2.20}$$

which converges uniformly on compact subsets for $\sigma>1+\alpha$. Therefore the series (2.7) converges absolutely-uniformly for $\Re s>1+\alpha$.

Proposition 2.5

$$\overrightarrow{E}(\gamma z,s;\chi_\rho)=\chi_\rho(\gamma)\overrightarrow{E}(z,s;\chi_\rho)\qquad\forall\gamma\in\Gamma. \tag{2.21}$$

Proof: We must show

$$E_i(\gamma z,s;\chi_\rho)=|\alpha_{\sigma_\gamma(i)}(\gamma^{-1})|^2E_{\sigma_\gamma(i)}(z,s;\chi_\rho)\qquad\forall\gamma\in\Gamma. \tag{2.22}$$

We have by (2.8) $E_i(\gamma z,s;\chi_\rho)$

$$=\sum_{V\in\Gamma_\infty\backslash\Gamma}\sum_{k=0}^{N-1}|\alpha_{\sigma_{S^k}^{-1}\circ\sigma_V^{-1}(i)}(S^k)\alpha_{\sigma_V^{-1}(i)}(V)|^2(\Im(V\gamma z))^s.$$

Let $W=V\gamma$. Then $V=W\gamma^{-1}$, so that $E_i(\gamma z,s;\chi_\rho)$ is , by (2.2),

$$=\sum_{W\gamma^{-1}\in\Gamma_\infty\backslash\Gamma}\sum_{k=0}^{N-1}|\alpha_{\sigma_{S^k}^{-1}\circ\sigma_W^{-1}\circ\sigma_\gamma(i)}(S^k)\alpha_{\sigma_W^{-1}\circ\sigma_\gamma(i)}(W\gamma^{-1})|^2(\Im(Wz))^s.$$

Therefore, using (2.3), we have $E_i(\gamma z, s; \chi_\rho)$

$$= \sum_{W\gamma^{-1}\in\Gamma_\infty\backslash\Gamma} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma_W^{-1}}(S^k)\alpha_{\sigma_W^{-1}\circ\sigma_\gamma(i)}(W)\alpha_{\sigma_\gamma(i)}(\gamma^{-1})|^2(\Im(Wz))^s$$

$$= |\alpha_{\sigma_\gamma(i)}(\gamma^{-1})|^2 \sum_{W\gamma^{-1}\in\Gamma_\infty\backslash\Gamma} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma_W^{-1}\circ\sigma_\gamma(i)}(S^k)\alpha_{\sigma_W^{-1}\circ\sigma_\gamma(i)}(W)|^2(\Im(Wz))^s$$

$$= |\alpha_{\sigma_{\gamma^{-1}}(i)}(\gamma^{-1})|^2 E_{\sigma_\gamma(i)}(z, s; \chi_\rho).$$

Remark 2.3 *In particular,*

$$\begin{aligned} E_i(z+N, s; \chi_\rho) &= E_i(S^N z, s; \chi_\rho) \\ &= |\alpha_{\sigma_{S^N}(i)}(S^{-N})|^2 E_{\sigma_{S^N}(i)}(z, s; \chi_\rho) \\ &= E_i(z, s; \chi_\rho), \quad \textit{by } (2.4). \end{aligned}$$

Therefore $E_i(z, s; \chi_\rho)$ is periodic, with period N.

2.3 Fourier Expansion

Since $E_j(z+N, s; \chi_\rho) = E_j(z, s; \chi_\rho)$, it has a real Fourier expansion

$$\begin{aligned} E_j(z, s; \chi_\rho) &= \frac{1}{2} \sum_{(m.n)=1} \sum_{k=0}^{N-1} \frac{|\alpha_{\sigma_{S^k}^{-1}\circ\sigma_V^{-1}}(j)\alpha_{\sigma_V^{-1}(j)}(V)|^2 y^s}{|mz+n|^{2s}} \\ &= \sum_{-\infty}^{\infty} a_l^j(y, s; \chi_\rho) e^{\frac{2\pi i l x}{N}}. \end{aligned}$$

The derivation of the $a_l^j(y, s; \chi_\rho)$ follows Bump [2]. We have

$$a_l^j(y, s; \chi_\rho) = \frac{1}{2} \sum_{(m.n)=1} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma_V^{-1}}(j)\alpha_{\sigma_V^{-1}(j)}(V)|^2 y^s \int_0^N \frac{e^{-\frac{2\pi i l x}{N}}}{[(mx+n)^2+m^2y^2]^s}\, dx \tag{2.23}$$

Next we use $\alpha_i(V) = \alpha_i(-V)$ to get

$$a_l^j(y,s;\chi_\rho) = \left(\sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}}(j)(S^k)|^2\right) y^s \delta_0(l)$$
$$+ \sum_{\substack{m=1\\(m,n)=1}}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma_V^{-1}}(j)\alpha_{\sigma_V^{-1}(j)}(V)|^2 y^s \int_0^N \frac{e^{-\frac{2\pi i l x}{N}}}{[(mx+n)^2+m^2y^2]^s}\, dx. \tag{2.24}$$

Write $n = mNq + r, \quad 0 \le r < mN$; then $(n,m) = 1 \Leftrightarrow (r,m) = 1$. Therefore

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma_V^{-1}(j)}(S^k)\alpha_{\sigma_V^{-1}(j)}(V)|^2 y^s \int_0^N \frac{e^{-\frac{2\pi i l x}{N}}}{[(mx+n)^2+m^2y^2]^s}\, dx$$
$$= \sum_{m=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\substack{r=0\\(r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & mq+r\end{smallmatrix}\right)}(j)}(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & mq+r\end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & mq+r\end{smallmatrix}\right)|^2$$
$$\times\, y^s \int_0^1 \frac{e^{-\frac{2\pi i l x}{N}}}{[(m(x+q)+r)^2+m^2y^2]^s}\, dx. \tag{2.25}$$

Since $\left(\begin{smallmatrix} * & * \\ m & mNq+r\end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & Nq \\ 0 & 1\end{smallmatrix}\right)$, (2.25) becomes

$$\sum_{m=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\substack{r=0\\(r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq}\right]}(j)}(S^k)\alpha_{\sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq}\right]}}\left[\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq}\right]|^2$$
$$\times\, y^s \int_{Nq}^{(N+1)q} \frac{e^{-\frac{2\pi i l x}{N}}}{[(m(x+q)+r)^2+m^2y^2]^s}\, dx. \tag{2.26}$$

On the other hand,

$$\begin{aligned}
&|\alpha_{\sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq}\right]}(j)}(\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq})|^2\\
=&|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}\circ\sigma^{-1}_{S^{Nq}}(j)}(\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)S^{Nq})|^2\\
=&|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}\circ\sigma^{-1}_{S^{Nq}}(j)}(\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right))\alpha_{\sigma_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}\circ\sigma^{-1}_{S^{Nq}}(j)}(S^{Nq})|^2\\
=&|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}(j)}\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)|^2.
\end{aligned}$$

We have shown

$$|\alpha_{\sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) S^{Nq}\right]}}{}^{(j)}(\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) S^{Nq})|^2 = |\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}{}^{(j)} \left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) |^2. \tag{2.27}$$

Therefore,

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) S^{Nq}\right]}}{}^{(j)}(S^k) \alpha_{\sigma^{-1}_{\left[\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) S^{Nq}\right]}} \left[\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) S^{Nq}\right] |^2 \\
&\qquad \times y^s \int_{Nq}^{(N+1)q} \frac{e^{-\frac{2\pi i l x}{N}}}{\left[(m(x+q)+r)^2 + m^2 y^2\right]^s} \, dx \\
&= \sum_{m=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}{}^{(j)}(S^k) \alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}} \left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) |^2 \\
&\qquad \times y^s \int_{Nq}^{(N+1)q} \frac{e^{-\frac{2\pi i l x}{N}}}{\left[(m(x+q)+r)^2 + m^2 y^2\right]^s} \, dx \\
&= \sum_{m=1}^{\infty} \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}{}^{(j)}(S^k) \alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}} \left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) |^2 \\
&\qquad \times y^s \int_{-\infty}^{\infty} \frac{e^{-\frac{2\pi i l x}{N}}}{\left[(m(x+q)+r)^2 + m^2 y^2\right]^s} \, dx.
\end{aligned}$$

Under the substitution $x \longrightarrow x + \dfrac{r}{m}$, the above becomes

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}{}^{(j)}(S^k) \alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}} \left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right) |^2 \\
&\qquad\qquad \times y^s \int_{-\infty}^{\infty} \frac{e^{-\frac{2\pi i l x}{N}}}{(x^2 + y^2)^s} \, dx.
\end{aligned} \tag{2.28}$$

Now, see [2], for $\Re s > \frac{1}{2}$

$$y^s \int_{-\infty}^{\infty} \frac{e^{-\frac{2\pi i l x}{N}}}{(x^2+y^2)^s} \, dx = \begin{cases} \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s}, & \text{if } l = 0; \\ \frac{\pi^s}{\Gamma(s)} \left|\frac{l}{N}\right|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}\left(\frac{2\pi |l| y}{N}\right), & \text{if } l \neq 0. \end{cases} \tag{2.29}$$

Here $K_s(y)$ is the K-Bessel function defined by, see[2],

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(t+t^{-1})} t^s \,\frac{dt}{t}. \tag{2.30}$$

$K_s(y)$ satisfies the estimate

$$|K_s(y)| \le e^{-\frac{y}{2}} K_\sigma(2) \quad \text{if } y > 4;\ \sigma = \Re s. \tag{2.31}$$

Therefore

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sum_{\substack{r=0\\(r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)|^2 \\
&\qquad\qquad \times y^s \int_{-\infty}^{\infty} \frac{e^{-\frac{2\pi i l x}{N}}}{(x^2+y^2)^s}\,dx \\
&= \sum_{m=1}^{\infty} \frac{\sum_{\substack{r=0\\(r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)|^2}{m^{2s}} \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s}\delta_0(l) \\
&\quad + 2\sum_{m=1}^{\infty} \frac{\sum_{\substack{r=0\\(r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)|^2}{m^{2s}} \\
&\qquad\qquad \times \frac{\pi^s}{\Gamma(s)} \left(\frac{|l|}{N}\right)^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}\left(\frac{2\pi |l| y}{N}\right).
\end{aligned}$$

We have shown

$$\begin{aligned}
a_0^j(y,s,\chi_\rho) &= \left(\sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}}(j)(S^k)|^2\right) y^s \\
&+ \sum_{m=1}^{\infty} \frac{\sum_{\substack{r=0\\(r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r\end{smallmatrix}\right)|^2}{m^{2s}} \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s}
\end{aligned} \tag{2.32}$$

and

$$a_l^j(y,s,\chi_\rho) = \\ 2\sum_{m=1}^{\infty} \frac{\sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2}{m^{2s}} \\ \times \frac{\pi^s}{\Gamma(s)} \left(\frac{|l|}{N}\right)^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(\frac{2\pi|l|y}{N}). \quad (2.33)$$

Let

$$S_j(l,\chi_\rho;m) = \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2.$$

$S_j(l,\chi_\rho;m)$ is a type of generalized Ramanujan sum. Let

$$Z_j(l,\chi_\rho;s) = \sum_{m=1}^{\infty} \frac{S(l,\chi_\rho;m)}{m^{2s}}$$

be the associated zeta function.

Thus we have

$$E_j(z,s;\chi_\rho) = \sum_{l=-\infty}^{\infty} a_l^j(y,s,\chi_\rho) e^{\frac{2\pi i l x}{N}},$$

where

$$a_0^j(y,s;\chi_\rho) = \left(\sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}}(j)(S^k)|^2\right) y^s + Z_j(0,\chi_\rho;s)\pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \quad (2.34)$$

and

$$a_l^j(y,s;\chi_\rho) = 2\frac{\pi^s}{\Gamma(s)} Z_j(l,\chi_\rho;s) \left(\frac{|l|}{N}\right)^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(\frac{2\pi|l|y}{N}), \text{for } l \neq 0. \quad (2.35)$$

Next we estimate

$$S_j(l,\chi_\rho;m) = \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma_{S^k}^{-1}\circ\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}(j)(S^k)\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2. \quad (2.36)$$

With $q \in \mathbb{Z}$ as in Lemma 1.1 and by (2.27), we have

$$
\begin{aligned}
|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2 &= |\alpha_{\sigma^{-1}_{S^{qN}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(S^{qN}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)\right)|^2 \\
&= |\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} a' & b' \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} a' & b' \\ m & r \end{smallmatrix}\right)|^2 \\
&\leq q_{\left(\begin{smallmatrix} a' & b' \\ m & r \end{smallmatrix}\right)}, \quad \text{where } q_V = \sum_{j=1}^{p} |\alpha_j(V)|^2.
\end{aligned}
$$

Thus, by an argument similar to that found in the proof of Proposition 2.4,

$$
\begin{aligned}
|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2 &\leq K_{\chi_\rho}^{L\left(\begin{smallmatrix} a' & b' \\ m & r \end{smallmatrix}\right)} p \\
&\leq \mu\left(\begin{smallmatrix} a' & b' \\ m & r \end{smallmatrix}\right)^{\alpha} K_{\chi_\rho}^{n_2} p \\
&\leq (N^2+1)^{\alpha}(m^2+r^2)^{\alpha} K_{\chi_\rho}^{n_2} p, \quad \text{by Lemma 1.1.}
\end{aligned}
$$

Thus, since $r < m$,

$$
|\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2 \leq \left[2(N^2+1)\right]^{\alpha} m^{2\alpha} K_{\chi_\rho}^{n_2} p. \tag{2.37}
$$

Therefore

$$
\begin{aligned}
|S_j(l,\chi_\rho;m)| &= |\sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}(j)}(S^k) \alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2| \\
&\leq K_{\chi_\rho} \sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2 \quad \text{by (2.17).}
\end{aligned}
$$

Then, by (2.37)

$$
\begin{aligned}
&|\sum_{\substack{r=0 \\ (r,m)=1}}^{mN-1} e^{\frac{2\pi i l r}{mN}} \sum_{k=0}^{N-1} |\alpha_{\sigma^{-1}_{S^k} \circ \sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}(j)}(S^k) \alpha_{\sigma^{-1}_{\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)}}\left(\begin{smallmatrix} * & * \\ m & r \end{smallmatrix}\right)|^2| \\
&\qquad \leq N^2\left[2(N^2+1)\right]^{\alpha} K_{\chi_\rho}^{n_2} p m^{2\alpha+1}, \text{a bound independent of } l \text{ and } j.
\end{aligned} \tag{2.38}
$$

Therefore,

$$
\begin{aligned}
|Z_j(l,\chi_\rho;s)| &= |\sum_{m=1}^{\infty} \frac{S_j(l,\chi_\rho;m)}{m^{2s}}| \\
&\leq N^2[2(N^2+1)]^{\alpha} K_{\chi_\rho}^{n_2} p \sum_{m=1}^{\infty} \frac{1}{m^{2(Re(s)-\alpha)-1}}.
\end{aligned}
$$

Thus $Z_j(l,\chi_\rho;s)$ converges absolutely for $\Re s > 1+\alpha$.

CHAPTER 3

The Matrix Resolvent Kernel

3.1 Preliminaries

The following facts about the laplacian acting on automorphic functions can be found in [7] ,[14]. Let $G_s(z,z')$ denote the free space Green's function for Δ on H. We have

$$G_s(z,z') = \frac{1}{4\pi}\int_0^1 (\xi(1-\xi))^{s-1}(\xi+u)^{-s}\,d\xi. \tag{3.1}$$

$G_s(z,z')$ satisfies $G_s(u) = \frac{1}{4\pi}log\frac{1}{u}+O(1)$, as $u \longrightarrow 0$. More precisely $G_s(z,w) = \frac{-1}{2\pi}\log|z-w| + H_s(z,w)$, where $H_s(z,w) \in C^\infty(H\times H)$. From $G_s(z,z')$, the automorphic kernel, $K_s(z,z')$, is constructed by summing over all of Γ:

$$K_s(z,z') = \sum_{\gamma\in\Gamma} G_s(z,\gamma z'). \tag{3.2}$$

Let $\mathcal{H}_0 = H\times H - diag \pmod{\Gamma}$ where $diag \pmod{\Gamma} = \{(z,z')\in H\times H : z'\equiv z \pmod{\Gamma}\}$. The series (3.2) converges absolutely-uniformly on compact subsets of $\mathcal{H}_0$ for $\Re s > 1$. Let $-R_s$ be the integral operator with kernel $K_s(z,z')$, that is

$$(-R_s f)(z) = \int_{\mathcal{F}} K_s(z,z')f(z')\,d\mu(z'). \tag{3.3}$$

$-R_s$ is called the resolvent of Δ. It inverts $\Delta - s(s-1)$ on the space

$$B_\mu(\Gamma\backslash H) = \{f\in\mathcal{A}(\Gamma\backslash H) : f\in C^\infty(H) \text{ and } f = O(y^\mu) \text{ as } y\longrightarrow\infty\}. \tag{3.4}$$

Hejhal [6] generalizes the above to the laplacian acting on $\mathcal{A}(\Gamma\backslash H,\chi)$ with χ unitary. In this chapter, we construct a matrix kernel and matrix resolvent for arbitrary χ satisfying (1.21). We prove in Section 3.3 the following theorem.

Theorem 3.1 If $\overrightarrow{F}\in B_\mu(\Gamma\backslash H,\chi)$, then

$$(\Delta - s(s-1))R_s\overrightarrow{F}^t(z) = \overrightarrow{F}^t(z) \quad \sigma\geq\mu+1. \tag{3.5}$$

We define the matrix kernel in the same manner as Hejhal [6].

Definition 3.1

$$K_s(z,z';\chi)=\sum_{\gamma\in\Gamma}G_s(z,\gamma z')\chi(\gamma) \tag{3.6}$$

where χ is an arbitrary $p\times p$ representation satisfying (1.21).

Proposition 3.1 *The series* (3.6) *converges absolutely-uniformly on compact subsets of $\mathcal{H}_0$ for $\Re s>1+\alpha_\chi$.*

Proof: Let $K=E_1\times E_2\subset\mathcal{H}_0$ be compact. Here E_1 and E_2 are compact subsets of H such that $\gamma(E_1)\cap E_2=\emptyset$, $\forall\gamma\in\Gamma$. Let $w=(z,z')\in H\times H$ and $w_o=(z_o,z_o')\in H\times H$. Consider $f(w,w_o)=u(z,z')+u(z_o,z_o')$. We have $f(w,w_o)>0$ for $w\in E_1\times E_2$ and $w_o\in diag \pmod{\Gamma}$. Since $E_1\times E_2$ is compact and $diag \pmod{\Gamma}$ is closed there exists $\delta>0$ such that

$$u(z,z_o)+u(z',\gamma z_o)\geq\delta \quad \forall(z,z')\in E\times E' \tag{3.7}$$
$$\text{and } \forall(z_o,\gamma z_o)\in\mathcal{H}_0.$$

Given $(z,z')\in E_1\times E_2$, set $z_o=z$ then (3.7) becomes

$$u(z,\gamma z')>\delta \quad \forall(z,z')\in E_1\times E_2 \quad \gamma\in\Gamma. \tag{3.8}$$

We may assume for some $A>0$

$$-A\leq x,x'\leq A \quad ,\frac{1}{A}\leq y,y'\leq A \tag{3.9}$$

for all $(z,z')\in E_1\times E_2$. It follows from (3.1) that there exists $A_{\sigma,\delta}$ such that

$$|G_s(u(z,z'))|\leq\frac{A_{\sigma,\delta}}{(\frac{2}{4}+u(z,z'))^\sigma} \quad u(z,z')>\delta. \tag{3.10}$$

Following [19], we write

$$K_s(z,z';\chi) = \sum_{n=-\infty}^{\infty} G_s(z,S^n x')\chi^n(S) + {\sum_{M\in\Gamma}}' G_s(z,Mz')\chi(M) \tag{3.11}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{N-1} G_s(z,S^{Nn+k}x')\chi^{Nn+k}(S) + {\sum_{M\in\Gamma}}' G_s(z,Mz')\chi(M).$$

Using (1.21), we have

$$K_s(z,z';\chi) = \sum_{n=-\infty}^{\infty} e^{-2\pi i N n\kappa}\sum_{k=0}^{N-1} G_s(z,S^{Nn+k}x')\chi^k(S) + {\sum_{M\in\Gamma}}' G_s(z,Mz')\chi(M).$$

where ${\sum}'_{M\in\Gamma}$ means all powers of S are missing.
The ij entry is

$$(K_s)_{ij}(z,z';\chi) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{N-1} G_s(z,S^{Nn+k}z')e^{-2\pi i N n\kappa}\chi_{ij}(S^k)$$
$$+ {\sum_{M\in\Gamma}}' G_s(z,Mz')\chi_{ij}(M). \tag{3.12}$$

Now,by the definition of K_χ, (1.38),

$$\left|\sum_{n=-\infty}^{\infty} e^{-2\pi i N n\kappa}\sum_{k=0}^{N-1} G_s(z,S^{Nn+k}x')\chi^k(S)\right| \tag{3.13}$$
$$\leq K_\chi \sum_{n=-\infty}^{\infty} \left|G_s(z,S^n z')\right|.$$

Therefore $\sum_{n=-\infty}^{\infty} e^{-2\pi i N n\kappa}\sum_{k=0}^{N-1} G_s(z,S^{Nn+k}x')\chi^k(S)$ converges absolutely-uniformly on $K\subset\mathcal{H}_0$ compact since $\sum_{\gamma\in\Gamma} G_s(z,\gamma z')$ does.

In the second term we use the fact that $u(z,Mz')>\delta \quad \forall M\in\Gamma$ for $(z,z')\in K$. We have

$${\sum_{M\in\Gamma}}' G_s(z,Mz')\chi_{ij}(M)$$
$$= \sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G_s(z,S^n\omega_{cd}S^m z')\chi_{ij}(S^n\omega_{cd}S^m).$$

Here we used the double coset decomposition (1.8). Using the division algorithm, we write

$$\sum_{M\in\Gamma}{}' G_s(z,Mz')\chi_{ij}(M)$$
$$=\sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G_s(z,S^{Nn+k}\omega_{cd}S^{Nm+l}z')\chi_{ij}(S^{nN}S^k\omega_{cd}S^lS^{mN}).$$

Since χ is a representation satisfying (1.21)

$$\chi_{ij}(S^{nN}S^k\omega_{cd}S^lS^{mN})) = e^{-2\pi i(n+m)N\kappa}\chi_{ij}(S^k\omega_{cd}S^l). \tag{3.14}$$

Therefore

$$\sum_{M\in\Gamma}{}' G_s(z,Mz')\chi_{ij}(M)$$
$$=\sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\chi_{ij}(S^k\omega_{cd}S^l)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-2\pi i(n+m)N\kappa}G_s(z,S^{Nn+k}\omega_{cd}S^{Nm+l}z')$$

We have to estimate

$$\sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left|\chi_{ij}(S^k\omega_{cd}S^l)\right|\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_s(z,S^n(\omega_{cd}S^m)z')\right|. \tag{3.15}$$

Now

$$\chi(S^k\omega_{cd}S^l) = \chi(S^k)\chi(\omega_{cd})\chi(S^l), \tag{3.16}$$

therefore

$$\chi_{ij}(S^k\omega_{cd}S^l) = \sum_{t_1,t_2=1}^{p}\chi_{it_1}(S^k)\chi_{t_1t_2}(\omega_{cd})\chi_{t_2j}(S^l). \tag{3.17}$$

Using the definition of K_χ, (1.38), we have

$$\left|\chi_{ij}(S^k\omega_{cd}S^l)\right| \le K_\chi^2\sum_{t_1,t_2=1}^{p}\left|\chi_{t_1t_2}(\omega_{cd})\right|. \tag{3.18}$$

We now use the Eichler estimate to bound $|\chi_{t_1t_2}(\omega_{cd})|$. We may assume, by Lemma 1.2 and (1.21), that

$$\mu(\omega_{cd}) \le (N^2+1)(c^2+d^2).$$

Therefore by Proposition 1.1

$$|\chi_{ij}(S^k\omega_{cd}S^l)| \leq K''_\chi(c^2+d^2)^{\alpha_\chi}, \tag{3.19}$$

where

$$K''_\chi = p^2K_\chi^2K'_\chi(N^2+1)^{\alpha_\chi}. \tag{3.20}$$

Therefore

$$\begin{aligned}
&\sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left|\chi_{ij}(S^k\omega_{cd}S^l)\right|\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_s(z,S^{Nn+k}\omega_{cd}S^{Nm+l}z')\right| \\
&\leq K''_\chi\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_s(z,S^{Nn+k}\omega_{cd}S^{Nm+l}z')\right| \\
&= K''_\chi\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_s(z,S^n\omega_{cd}S^mz')\right|. \qquad (3.21)
\end{aligned}$$

Thus, we have to estimate

$$\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_s(z,S^n\omega_{cd}S^mz')\right|.$$

Let $M = \omega_{cd}S^m$, $x_M = \Re Mz$, and $y_M = \Im Mz$. We have

$$G_s(z,S^nMz') = G_s(z,Mz'+n).$$

By the Remark 1.1 this is

$$= G_s(u(z,Mz'+n)).$$

Now

$$\begin{aligned}
u(z,Mz'+n) &= \frac{(x-x'_M-n)^2+(y-y'_M)^2}{4yy'_M} \\
&= \frac{1}{4}\left(\frac{y}{y'_m}+\frac{y'_M}{y}-2+\frac{(x-x'_M-n)^2}{yy'_M}\right).
\end{aligned}$$

Therefore

$$G_s(z,S^nMz') = G_s(\frac{1}{4}\left(\frac{y}{y'_m}+\frac{y'_M}{y}-2+\frac{(x-x'_M-n)^2}{yy'_M}\right)). \tag{3.22}$$

Thus, by (3.10) and (3.7),

$$|G_s(z, S^n Mz')| \leq \frac{4^\sigma A_{\sigma,\delta}}{(\frac{y}{y'_M} + \frac{y'_M}{y} + \frac{(x-x'_M-n)^2}{yy_M})^\sigma}$$
$$\leq \frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y + \frac{(x-x'_M-n)^2}{y})^\sigma}. \tag{3.23}$$

Now

$$\left(y + \frac{(x-x'_M-n)^2}{y}\right)^\sigma = y^\sigma \left(1 + \left|\frac{x-x'_M-n}{y}\right|^2\right)^\sigma.$$

Next we apply Peetre's inequality, [3]: $\forall \xi, \eta \in R^n$ and $\sigma \in R$,

$$(1+|\xi|^2)^\sigma (1+|\eta|^2)^{-\sigma} \leq 2^{|\sigma|}(1+|\xi-\eta|^2)^{|\sigma|}. \tag{3.24}$$

We obtain

$$2^\sigma \left(1 + \left|\frac{n}{y} - \frac{(x-x'_M)}{y}\right|^2\right)^\sigma \geq \frac{(1+(\frac{n}{y})^2)^\sigma}{(1+\frac{(x-x'_M)^2}{y})^\sigma}, \text{ note } \sigma > 1.$$

Therefore

$$\sum_{n=-\infty}^{\infty} \frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma} \leq 2^\sigma A_{\sigma,\delta}(y'_M)^\sigma \sum_{n=-\infty}^{\infty} \frac{(1+\frac{(x-x'_M)^2}{y^2})^\sigma}{y^\sigma(1+(\frac{n}{y})^2)^\sigma}$$
$$< 22^\sigma A_{\sigma,\delta}(y'_M)^\sigma \sum_{n=0}^{\infty} \frac{(1+\frac{(x-x'_M)^2}{y^2})^\sigma}{(y+\frac{n^2}{y})^\sigma}.$$

Now

$$\begin{aligned} |x - x'_M| &= |x - x'_{\omega_{cd}S^m}| \\ &= |x - (x'+m)_{\omega_{cd}}| \\ &= |x - \frac{a}{c} + \frac{1}{c^2}\frac{x'+m+\frac{d}{c}}{(x'+m+\frac{d}{c})^2+(y')^2}| \\ &\leq |x| + |\frac{a}{c}| + max(1, \frac{1}{y'^2}), \end{aligned}$$

since

$$\frac{|x'+m+\frac{d}{c}|}{|x'+m+\frac{d}{c}|^2+y'^2} \leq \begin{cases} \frac{1}{y'^2}, & \text{if } |x'+m+\frac{d}{c}| \leq 1 \\ 1, & \text{if } |x'+m+\frac{d}{c}| \geq 1. \end{cases}$$

Our choice of representative ω_{cd} satisfies $|\frac{a}{c}| \le 1$, see the proof of Lemma 1.2. Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma} < 22^\sigma A_{\sigma,\delta} {y'_M}^\sigma \left(1+\frac{|x|+1+max(1,\frac{1}{y'^2})}{y^2}\right)^\sigma \sum_{n=0}^{\infty} \frac{1}{1+(\frac{n}{y})^2)^\sigma}. \tag{3.25}$$

Further, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1}{y+\frac{n^2}{y}}\right)^\sigma \\
&= \frac{1}{y^\sigma} + \sum_{n=1}^{\infty}\left(\frac{1}{y+\frac{n^2}{y}}\right)^\sigma \\
&\le \frac{1}{y^\sigma} + \int_0^\infty \left(\frac{1}{y+\frac{x^2}{y}}\right)^\sigma dx \\
&= \frac{1}{y^{\sigma-1}}\left(\frac{1}{y} + \int_0^\infty \frac{1}{(1+u^2)^\sigma}\, du\right) \\
&= \frac{1}{y^{\sigma-1}}\left(\frac{1}{y} + A_{3,\sigma}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma} \\
&< 22^\sigma A_{\sigma,\delta} {y'_M}^\sigma \left(1+\frac{|x|+1+max(1,\frac{1}{y'^2})}{y^2}\right)^\sigma \frac{1}{y^{\sigma-1}}\left(\frac{1}{y}+A_{3,\sigma}\right).
\end{aligned}$$

Therefore there exists a constant $C(\sigma, \delta, K)$, where K is our compact set, such that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma} &< C(\sigma,\delta,K){y'_M}^\sigma \\
&= C(\sigma,\delta,K)\frac{y'^\sigma}{|(c+m)z'+d|^{2\sigma}}.
\end{aligned} \tag{3.26}$$

Therefore,

$$\begin{aligned}
&\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|G_s(z,S^n\omega_{cd}S^m z')|\\
&\le K''_\chi\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|G_s(z,S^n\omega_{cd}S^m z')| \quad \text{by (3.21)}\\
&\le K''_\chi\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma} \quad \text{by (3.23)}\\
&\le K''_\chi C(\sigma,\delta,K)\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\frac{{y'}^\sigma}{|(c+m)z'+d|^\sigma}. \qquad (3.27)
\end{aligned}$$

Since $d \le c$, we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{4^\sigma A_{\sigma,\delta}(y'_M)^\sigma}{(y+\frac{(x-x'_M-n)^2}{y})^\sigma}\\
&\le K'''_\chi C(\sigma,\delta,K)\sum_{c=1}^{\infty}\sum_{d \bmod c}c^{2\alpha_\chi}\sum_{m=-\infty}^{\infty}\frac{{y'}^\sigma}{|(c+m)z'+d|^\sigma}. \qquad (3.28)
\end{aligned}$$

We now apply Lemma 1.1 to obtain

$$\begin{aligned}
|c(z'+m)+d|^2 &\ge \left(\frac{{y'}^2}{1+4|z'|^2}\right)(c^2+(cm+d)^2)\\
&\ge \left(\frac{c^2{y'}^2}{1+4|z'|^2}\right).
\end{aligned}$$

Thus

$$c^{2\alpha_\chi} \le \left(\frac{1+4|z'|^2}{y'}\right)^{\alpha_\chi}\frac{|c(z'+m)+d|^{2\alpha_\chi}}{{y'}^{\alpha_\chi}}. \qquad (3.29)$$

Therefore,

$$\begin{aligned}
&\sum_{c=1}^{\infty}\sum_{d \bmod c}(c^2+d^2)^{\alpha_\chi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|G_s(z,S^n\omega_{cd}S^m z')| \qquad (3.30)\\
&\le K'''_\chi C(\sigma,\delta,K)\frac{(1+|z'|^2)^{\alpha_\chi}}{{y'}^{\alpha_\chi}}\sum_{c=1}^{\infty}\sum_{d \bmod c}\sum_{m=-\infty}^{\infty}\frac{{y'}^{\sigma-\alpha_\chi}}{|(c+m)z'+d|^{2(\sigma-\alpha_\chi)}}.
\end{aligned}$$

The last series converges uniformly on compact domains for $\sigma > \alpha_\chi + 1$ see [19, pp. 638-639].

Proposition 3.2 *For $Re(s) > \alpha_\chi + 1$, $K_s(z, z'; \chi)$ has the following properties:*

(a) $K_s(z,z';\chi) = K_s(z',z;\chi^{-1})$

(b) $K_s(Vz,z';\chi) = \chi(V)K_s(z,z';\chi)$ *for* $V \in \Gamma(1)$

(c) $K_s(z,Vz';\chi) = K_s(z,z';\chi)\chi(V^{-1})$

(d) $(K_s)_{ij}(z,z';\chi) = \frac{\delta_{ij}}{4\pi} log\frac{1}{u} + O(1),$ *as* $u \longrightarrow 0.$

Properties $(a)-(c)$ follow directly from the definition. For (d) , fix $z \in \mathfrak{F}$. We have $\Gamma_z = I$. Since $\Gamma(1)$ acts properly discontinuously, there exists $\delta > 0$ and $U_z = \{w : u(z,w) < \delta\}$ such that $z' \in U_z$ and $\gamma z' \notin U_z$, i.e. $u(z,\gamma z') > \delta$, $\forall \gamma \in \Gamma(1), \gamma \neq I$. Therefore

$$(K_s)_{ij}(z,z';\chi) = G_s(z,z')\chi_{ij}(I) + \sum_{I \neq \gamma \in \Gamma(1)} G_s(z,\gamma z')\chi_{ij}(\gamma). \tag{3.31}$$

The second term is bounded near z, thus from the properties of $G_s(z,z')$,

$$\begin{aligned}(K)_{ij}(z,z';\chi) &= \frac{\chi_{ij}(I)}{4\pi} log\frac{1}{u} + O(1), \quad \text{as} \quad u \longrightarrow 0\\ &= \frac{\delta_{ij}}{4\pi} log\frac{1}{u} + O(1), \quad \text{as} \quad \longrightarrow 0.\end{aligned}$$

3.2 Double Coset Expansion of $K_s(z,z';\chi)$

In this section we apply the double coset decomposition to the kernel $K_s(z,z';\chi)$ to obtain its Fourier expansion. This is done for the scalar case in [19] and [7]. The Fourier expansion affords us growth estimates for the kernel $K_s(z,z';\chi)$.

$$\begin{aligned}K_s(z,z';\chi) &= \sum_{n=-\infty}^{\infty} G_s(z,z'+n)\chi(S^n) \qquad (3.32)\\ &+ \sum_{c=1}^{\infty} \sum_{d \mod c} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G_s(z,S^{-n}\omega_{cd}S^{-m}z')\chi(S^{-n}\omega_{cd}S^{-m})\\ &= K_s^0(z,z';\chi) + \sum_{c=1}^{\infty} K_s^c(z,z';\chi).\end{aligned}$$

We have

$$K_s^0(z,z';\chi) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} G_s(z,z'+Nn+k)\chi(S^{Nn+k}). \tag{3.33}$$

By (1.21), we have

$$K_s^0(z,z';\chi) = \sum_{n=-\infty}^{\infty} e^{-2\pi i\kappa nN} I \sum_{k=0}^{N-1} G_s(z,z'+Nn+k)\chi(S^k). \tag{3.34}$$

Therefore

$$(K_s^0)_{ij} = \sum_{n=-\infty}^{\infty} e^{-2\pi i\kappa nN} \sum_{k=0}^{N-1} \chi_{ij}(S^k) G_s(z,z'+Nn+k). \tag{3.35}$$

Let

$$f_k^{i,j}(n) = \chi_{ij}(S^k) G_s(z,z'+Nn+k) \tag{3.36}$$

$$= \chi_{ij}(S^k) G_s(u(z,z'+Nn+k)). \tag{3.37}$$

Now

$$u(z,z'+Nn+k) = \frac{(x-x'-Nn-k)^2+(y-y')^2}{4yy'}. \tag{3.38}$$

Therefore

$$f_k^{i,j}(n) = \chi_{ij}(S^k) G_s(\frac{(x-x'-Nn-k)^2+(y-y')^2}{4yy'}). \tag{3.39}$$

Let

$$f^{i,j}(n) = \sum_{k=0}^{N-1} f_k^{i,j}(n). \tag{3.40}$$

Then

$$(K_s^0)_{ij} = \sum_{n=-\infty}^{\infty} e^{-2\pi i\kappa nN} f^{i,j}(n). \tag{3.41}$$

We apply the Poisson Summation Formula to obtain

$$(K_s^0)_{ij} = \sum_{n=-\infty}^{\infty} \hat{f}^{i,j}(n+\kappa N) \tag{3.42}$$

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} \hat{f}_k^{i,j}(n+\kappa N). \tag{3.43}$$

Next we calculate $\hat{f}_k^{i,j}$. We have

$$\hat{f}_k^{i,j}(n) = \int_{-\infty}^{\infty} e^{2\pi inu} \hat{f}_k^{i,j}(u)\, du \tag{3.44}$$

$$= \int_{-\infty}^{\infty} e^{2\pi inu} \chi_{ij}(S^k) G_s(\frac{(x-x'-Nu-k)^2+(y-y')^2}{4yy'})\, du \tag{3.45}$$

$$= \chi_{ij}(S^k) \int_{-\infty}^{\infty} e^{2\pi inu} G_s(\frac{(x-x'-Nu-k)^2+(y-y')^2}{4yy'})\, du \tag{3.46}$$

Let $-\xi = x - x' - Nu - k$, then

$$\hat{f}_k^{i,j}(n) = \frac{1}{N} e^{-\frac{2\pi ink}{N}} \chi_{ij}(S^k) e^{\frac{2\pi in(x-x')}{N}} \int_{-\infty}^{\infty} e^{\frac{2\pi in\xi}{N}} G_s(\xi + iy, iy')\, d\xi. \tag{3.47}$$

Following [7], we define $P_n(y, y')$ by

$$P_n(y, y') = \int_{-\infty}^{\infty} e^{2\pi i\xi\eta} G_s(iy + \xi, iy')\, d\xi. \tag{3.48}$$

Finally, we obtain

$$(K_s^0)_{ij} = \frac{1}{N} \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{N-1} e^{-\frac{2\pi i(n+\kappa N)k}{N}} \chi_{ij}(S^k) \right) e^{\frac{2\pi i(n+\kappa N)(x-x')}{N}} P_{\frac{n+\kappa N}{N}}(y, y'). \tag{3.49}$$

Now develop the expansion for $(K_s^c)_{ij}(z, z'; \chi)$.

$$(K_s^c)_{ij}(z, z'; \chi) = \sum_{d \bmod c} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G_s(z, S^{-n}\omega_{cd}S^{-m}z')\chi_{ij}(S^{-n}\omega_{cd}S^{-m})$$

$$= \sum_{d \bmod c} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G_s(z, S^{-Nn-k}\omega_{cd}S^{-Nm-l}z')\chi_{ij}(S^{-nN}S^{-k}\omega_{cd}S^{l}S^{-mN}). \tag{3.50}$$

Thus, using (3.14), we have

$$(K_s^c)_{ij}(z,z';\chi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{2\pi i(n+m)N\kappa}$$
$$\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \chi_{ij}(S^{-k}\omega_{cd}S^l)G_s(z+Nn+k,\omega_{cd}(z'-Nn-l)). \quad (3.51)$$

Let

$$f_{k,l,d}^{i,j}(m,n) = \chi_{ij}(S^{-k}\omega_{cd}S^l)G_s(u(z+Nn+k,\omega_{cd}(z'-Nn-l))). \quad (3.52)$$

Now

$$u(z+Nn+k,\omega_{cd}(z'-Nn-l)) = \frac{\left(x+Nn+k-\frac{a}{c}-\frac{(-x'+Nn+l-\frac{d}{c})}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}\right)^2+\left(y-\frac{y'}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}\right)^2}{\frac{4yy'}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}}. \quad (3.53)$$

Let

$$f^{i,j}(m,n) = \sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f_{k,l,d}^{i,j}(m,n). \quad (3.54)$$

Then

$$(K_s^c)_{ij}(z,z';\chi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{2\pi i(n+m)N\kappa} f^{i,j}(m,n). \quad (3.55)$$

Again, we apply Poisson summation formula to obtain

$$(K_s^c)_{ij}(z,z';\chi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \hat{f}^{i,j}(m+\kappa N, n+\kappa N). \quad (3.56)$$

Here

$$\hat{f}_{k,l,d}^{i,j}(m,n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(mu+nv)} f_{k,l,d}^{i,j}(u,v)\,dudv. \quad (3.57)$$

Thus

$$\hat{f}^{i,j}_{k,l,d}(m,n) = \chi_{ij}(S^{-k}\omega_{cd}S^{l}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(mu+nv)} \times$$

$$G_s\left(\frac{\left(x+Nn+k-\frac{a}{c}-\frac{(-x'+Nn+l-\frac{d}{c})}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}\right)^2+\left(y-\frac{y'}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}\right)^2}{\frac{4yy'}{c^2|x'-Nn-l+\frac{d}{c}+iy'|^2}}\right) dudv. \tag{3.58}$$

Let $\xi = x + Nu + k - \frac{a}{c}$ and $\eta = -x' + Nv + l - \frac{d}{c}$. Then

$$\hat{f}^{i,j}_{k,l,d}(m,n) = \chi_{ij}(S^{-k}\omega_{cd}S^{l}) e^{2\pi i\left(\frac{m(-x+\frac{a}{c}-k)}{N}+\frac{n(x'+\frac{d}{c}-l)}{N}\right)} \times$$
$$\frac{1}{N^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i\frac{(\xi n+\eta m)}{N}} G_s(iy+\xi, \frac{-1}{c^2(iy'-\eta)})\, d\xi d\eta. \tag{3.59}$$

Again following [7], we define

$$P_{n,m}(y,y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(\xi n+\eta m)} G_s(iy+\xi, \frac{-1}{iy'-\eta})\, d\xi d\eta. \tag{3.60}$$

Therefore,

$$(K^c_s)_{ij}(z,z';\chi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \hat{f}^{i,j}(m+\kappa N, n+\kappa N) \tag{3.61}$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \hat{f}^{i,j}_{k,l,d}(m+\kappa N, n+\kappa N).$$

Thus

$$(K^c_s)_{ij}(z,z';\chi) = \frac{1}{N^2c^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}$$
$$\times \sum_{d \bmod c}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \chi_{ij}(S^{-k}\omega_{cd}S^{l}) e^{2\pi i\left(\frac{(m+\kappa N)(-x+\frac{a}{c}-k)}{N}+\frac{(n+\kappa N)(x'+\frac{d}{c}-l)}{N}\right)} P_{\frac{n+\kappa N}{N},\frac{m+\kappa N}{c^2 N}}. \tag{3.62}$$

We introduce a generalized Kloosterman sum associated to χ.

Definition 3.2

$$S_{ij}(m,n,c;\chi) = \sum_{d \bmod c} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \chi_{ij}(S^{-k}\omega_{cd}S^{l}) e^{2\pi i\left(\frac{(m+\kappa N)(\frac{a}{c}-k)}{N} + \frac{(n+\kappa N)(\frac{d}{c}-l)}{N}\right)} \tag{3.63}$$

Lemma 1.2, Proposition 1.1 and $d \leq c$ imply the Kloosterman sums have the bound

$$|S_{ij}(m+\kappa N, n+\kappa N, c;\chi)| \leq c^{2\alpha_\chi+1}. \tag{3.64}$$

Using the Kloosterman sums (3.63), we have

$$(K_s^c)_{ij}(z,z';\chi) = \frac{1}{c^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S_{ij}(m+\kappa N, n+\kappa N, c;\chi) \times e^{2\pi i\left(\frac{(m+\kappa N)(-x)}{N} + \frac{(n+\kappa N)(x')}{N}\right)} P_{\frac{n+\kappa N}{N}, \frac{m+\kappa N}{c^2 N}}. \tag{3.65}$$

We use the following results found in [7]: For $Re(s) > 1$ and $y' > y$

$$P_0(y,y') = \frac{1}{2s-1} y^s {y'}^{1-s} \quad \text{and} \tag{3.66}$$

$$P_n(y,y') = \frac{1}{4\pi|n|} V_s(iny) W_s(iny') \quad n \neq 0. \tag{3.67}$$

For $y' > \frac{1}{y}$

$$P_{0,0}(y,y') = \frac{\pi^{\frac{1}{2}}}{2s-1} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} (yy')^{1-s}, \tag{3.68}$$

$$P_{0,m}(y,y') = \frac{\pi^s}{(2s-1)\Gamma(s)} \left(\frac{y}{|m|}\right)^{1-s} W_s(imy'), \quad m \neq 0, \tag{3.69}$$

$$P_{n,0}(y,y') = \frac{\pi^s}{(2s-1)\Gamma(s)} \left(\frac{y'}{|n|}\right)^{1-s} W_s(iny'), \quad n \neq 0, \tag{3.70}$$

and

$$P_{n,m}(y,y') = \frac{1}{2|mn|^{\frac{1}{2}}} W_s(iny) W_s(imy') \begin{cases} J_{2s-1}(4\pi\sqrt{mn}) & \text{for } mn > 0 \\ I_{2s-1}(4\pi\sqrt{|mn|}) & \text{for } mn < 0. \end{cases} \tag{3.71}$$

Where

$$W_s(z) = 2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi y) e^{2\pi i x} \tag{3.72}$$

is the Whittaker function. The definition is extended to the lower half plane by

$$W_s(z) = W_s(\overline{z}). \tag{3.73}$$

$K_s(y)$ is the K-Bessel function given by

$$K_s(y) = \frac{1}{2} \int_0^{\infty} e^{-\frac{y}{2}(t+\frac{1}{t})} t^{-s-1}\, dt \quad y > 0. \tag{3.74}$$

where $\Re s > -\frac{1}{2}$. $K_s(y)$ satisfies the estimate, see [2],

$$|K_s(y)| \leq C_s e^{-\frac{y}{2}} \quad y > 4. \tag{3.75}$$

Also,

$$V_s(z) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e^{2\pi i x} \tag{3.76}$$

extends as $W_s(z)$. We use the following estimates found in [7], $n \neq 0$

$$\begin{aligned} W_s(nz) &<< |n| y^{\frac{1}{2}} e^{-\pi|n|y} \quad 2\pi y > 4 \\ V_s(nz) &<< |n| y^{\frac{1}{2}} e^{\pi|n|y} \\ I_{2s-1}(y) &<< \min\{y^{2\sigma-1}, y^{-\frac{1}{2}}\} e^{y} \\ J_{2s-1}(y) &<< \min\{y^{2\sigma-1}, y^{-\frac{1}{2}}\} \end{aligned} \tag{3.77}$$

Next, we plug in (3.66) and (3.67) into (3.49), to get for $\Re s > 1$, $y' > y$ and $y' > \frac{1}{y}$

$$\begin{aligned} (K_s^0)_{ij} = {} & \delta_0(n+\kappa N) \frac{1}{N} \sum_{k=0}^{N-1} \chi_{ij}(S^k) \frac{y^s {y'}^{1-s}}{2s-1} \\ & + \sum_{\substack{n=-\infty \\ n+\kappa N \neq 0}}^{\infty} e^{2\pi i \frac{(n+\kappa N)}{N}(-x+x')} \frac{1}{4\pi|n+\kappa N|} V_s(i(n+\kappa N)y) W_s(i(n+\kappa N)y'). \end{aligned} \tag{3.78}$$

Also, using (3.63),(3.68)-(3.71),and (3.61), we get for $\Re s > \alpha_\chi + 1$
$y' > y$ and $y' > \frac{1}{y}$

$$(K_s^c)_{ij}(z,z';\chi) = \delta_0(n+\kappa N)\delta_0(m+\kappa N)\frac{\pi^{\frac{1}{2}}}{2s-1}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}(yy')^{1-s}\frac{S_{ij}(0,0,c;\chi)}{c^2}$$
$$+\delta_0(n+\kappa N)\frac{\pi^s y^{1-s}}{(2s-1)\Gamma(s)}\sum_{\substack{m=-\infty\\ m+\kappa N\neq 0}}^{\infty}\frac{S_{ij}(m+\kappa N,0,c;\chi)}{c^{2s}}e^{2\pi i 2\pi i(m+\kappa N)x'}\frac{W_s(i(m+\kappa N)y')}{|m+\kappa N|^{1-s}}$$
$$+\delta_0(n+\kappa N)\frac{\pi^s {y'}^{1-s}}{(2s-1)\Gamma(s)}\sum_{\substack{n=-\infty\\ n+\kappa N\neq 0}}^{\infty}\frac{S_{ij}(0,n+\kappa N,c;\chi)}{c^{2s}}e^{2\pi i(n+\kappa N)(-x)}\frac{W_s(i(n+\kappa N)y)}{|n+\kappa N|^{1-s}}$$
$$+\sum_{\substack{m=-\infty\\ m+\kappa N\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n+\kappa N\neq 0}}^{\infty}\frac{S_{ij}(m+\kappa N,n+\kappa N,c;\chi)}{c^{2s}}e^{2\pi i(n+\kappa N)(-x)+(m+\kappa N)x'}$$
$$\times\frac{W_s(i(n+\kappa N)y)W_s(i(m+\kappa N)y')}{2|(n+\kappa N)(m+\kappa N)|^{\frac{1}{2}}}$$
$$\times\begin{cases} J_{2s-1}(\frac{4\pi}{c}\sqrt{(n+\kappa N)(m+\kappa N)}) & \text{for } (n+\kappa N)(m+\kappa N)>0\\ I_{2s-1}(\frac{4\pi}{c}\sqrt{|n+\kappa N||m+\kappa N|}) & \text{for } (n+\kappa N)(m+\kappa N)<0.\end{cases}$$

We introduce the following functions:

Definition 3.3 *Let*

$$\varphi^{ij}(s) = \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{c=1}^{\infty}\frac{S_{ij}(0,0,c;\chi)}{c^{2s}}, \tag{3.79}$$
$$\varphi^{ij}_{n+\kappa N}(s) = \frac{\pi^s}{\Gamma(s)}\frac{1}{|n+\kappa N|^{1-s}}\sum_{c=1}^{\infty}\frac{S_{ij}(0,n+\kappa N,c;\chi)}{c^{2s}},$$
$$Z_s(m+\kappa N,n+\kappa N) = \frac{1}{\sqrt{|m+\kappa N||n+\kappa N|}}\sum_{c=1}^{\infty}\frac{S_{ij}(m+\kappa N,n+\kappa N,c;\chi)}{c}$$
$$\times\begin{cases} J_{2s-1}(\frac{4\pi}{c}\sqrt{(n+\kappa N)(m+\kappa N)}) & \textit{for } (n+\kappa N)(m+\kappa N)>0\\ I_{2s-1}(\frac{4\pi}{c}\sqrt{|n+\kappa N||m+\kappa N|}) & \textit{for } (n+\kappa N)(m+\kappa N)<0,\end{cases}$$

and the "Eisenstein series"

$$E_{ij}(z,s) = \delta_{ij}\delta_0(n+\kappa N)y^s + \delta_0(m+\kappa N)\delta_0(n+\kappa N)\varphi^{ij}(s)y^{1-s} \quad (3.80)$$
$$+ \delta_0(n+\kappa N)\sum_{\substack{n=-\infty \\ n+\kappa N\neq 0}}^{\infty} \varphi^{ij}_{n+\kappa N}(s)\overline{W}_{\overline{s}}((n+\kappa N)z).$$

The bound (3.64) implies $\varphi^{ij}(s)$ and $\varphi^{ij}_{n+\kappa N}(s)$ are analytic for $\sigma > \alpha_\chi + 1$. We have the bounds

$$|\varphi^{ij}(s)| \leq C_s, \quad (3.81)$$
$$|\varphi^{ij}_{n+\kappa N}(s)| \leq \frac{C_s}{|n+\kappa N|^{1-\sigma}}$$

and

$$|Z_s(m+\kappa N, n+\kappa N)| \leq C_s e^{4\pi\sqrt{|n+\kappa N||m+\kappa N|}}.$$

$Z_s(m+\kappa N, n+\kappa N)$ is entire and $E_{ij}(z,s)$ is analytic for $\sigma > \alpha_\chi + 1$. Therefore for $\Re s > 1 \quad y' > y$,

$$[K_s]_{ij}(z,z';\chi) =$$
$$\frac{{y'}^{1-s}}{2s-1}\{\delta_{ij}\delta_0(n+\kappa N)y^s + \delta_0(m+\kappa N)\delta_0(n+\kappa N)\varphi^{ij}(s)y^{1-s}$$
$$+ \delta_0(n+\kappa N)\sum_{\substack{n=-\infty \\ n+\kappa N\neq 0}}^{\infty} \varphi^{ij}_{n+\kappa N}(s)\overline{W}_{\overline{s}}((n+\kappa N)z)\}$$
$$+ \delta_{ij}\sum_{\substack{n=-\infty \\ n+n_j\neq 0}}^{\infty} \frac{1}{4\pi|n+n_j|}\overline{V}_{\overline{s}}((n+n_j)z)W_s((n+n_j)z')$$
$$+ \delta_0(m+\kappa N)\frac{y^{1-s}}{2s-1}\sum_{\substack{m=-\infty \\ m+\kappa N\neq 0}}^{\infty} \varphi^{ij}_{m+\kappa N}(s)W_s((m+\kappa N)z')$$
$$+ \sum_{\substack{m=-\infty \\ m+\kappa N}}^{\infty}\sum_{\substack{n=-\infty \\ n+\kappa N\neq 0}}^{\infty} Z_s(m+\kappa N, n+\kappa N)\overline{W}_{\overline{s}}((n+\kappa N)z)W_s((m+\kappa N)z').$$
$$(3.82)$$

3.3 The Resolvent

Let χ be an irreducible representation and

$$K_s(z,z';\chi) = \sum_{\gamma\in\Gamma(1)} G_s(z,\gamma z')\chi(\gamma)$$

the corresponding matrix kernel. We define a matrix integral operator, $-R_s$ with kernel $K_s(z,z';\chi)$, as

$$-(R_s\overrightarrow{F})(z) = \int\limits_{\mathcal{F}} K_s(z,z';\chi)\overrightarrow{F}(z')d\mu(z'). \tag{3.83}$$

Looking at the i^{th} entry, we have

$$\begin{aligned}-(R_s\overrightarrow{F})_i(z) &= -\sum_{j=1}^{p}(R_s)_{ij}F_j(z)\\ &= -\sum_{i=1}^{p}\int_{\mathcal{F}}(K_j)_{ij}(z,z';\chi)F_j(z')d\mu(z').\end{aligned}$$

Theorem 3.1 *If $\overrightarrow{F} \in \mathcal{B}'_\mu(\Gamma\backslash H,\chi)$, then*

$$(\Delta + s(1-s))R_s\overrightarrow{F}(z) = \overrightarrow{F}(z) \quad \sigma \geq \mu + 1. \tag{3.84}$$

Thus R_s inverts $(\Delta+s(s-1))$ on the space $\mathcal{B}'_\mu(\Gamma\backslash H,\chi)$ We assume the following lemma which [7] proves using the invariance of the laplacian.

Lemma 3.1 *If $\overrightarrow{F} \in B_\mu(\Gamma\backslash H,\chi)$, then*

$$-(\Delta + s(1-s))R_s\overrightarrow{F}(z) = \int\limits_{\mathcal{F}} K_s(z,z';\chi)(\Delta + s(1-s))\overrightarrow{F}(z')d\mu(z'). \tag{3.85}$$

Proof of Theorem 3.1: Given $z \in \mathcal{F}$ and $\epsilon > 0$, write $\mathcal{F} = (\mathcal{F} - B_\epsilon(z))\bigcup B_\epsilon(z)$. We have,

$$\begin{aligned}&\int\limits_{\mathcal{F}} K_s(z,z';\chi)(\Delta + s(1-s))\overrightarrow{F}(z')d\mu(z')\\ &\quad= \int\limits_{\mathcal{F}-B_\epsilon(z)} K_s(z,z';\chi)(\Delta + s(1-s))\overrightarrow{F}(z')d\mu(z')\\ &\qquad+ \int\limits_{B_\epsilon(z)} K_s(z,z';\chi)(\Delta + s(1-s))\overrightarrow{F}(z')d\mu(z').\end{aligned} \tag{3.86}$$

In the first integral use Green's Formula to write

$$\int\limits_{\mathcal{F}-B_\epsilon(z)} \left(K_s(z,z';\chi)(\Delta+s(1-s))\overrightarrow{F}(z')-(\Delta+s(1-s))K_s(z,z';\chi)\overrightarrow{F}(z')\right)d\mu(z')$$
$$=\int\limits_{\mathcal{F}-B_\epsilon(z)} \left(K_s(z,z';\chi)\Delta_e\overrightarrow{F}(z')-\Delta_e K_s(z,z';\chi)\overrightarrow{F}(z')\right)dx\,dy$$
$$=\int\limits_{\partial(\mathcal{F}-B_\epsilon(z))} \left(K_s(z,z';\chi)\frac{\partial\overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl$$
$$=\int\limits_{\partial(\mathcal{F}-B_\epsilon(z))} \left(K_s(z,z';\chi)y\frac{\partial\overrightarrow{F}(z')}{\partial n}-y\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)\frac{dl}{y}. \tag{3.87}$$

Here dl denotes euclidean arc length and Δ_e the euclidean laplacian. In the last line we have rewritten the integrand for convenience since $y\frac{\partial}{\partial n}$ and $\frac{dl}{y}$ are invariant under Γ. Since $(\Delta-s(1-s))K_s(z,z',\chi)=0, \quad |z-z'|\geq\epsilon$, we have

$$\int\limits_{mathscrF-B_\epsilon(z)} K_s(z,z';\chi)(\Delta+s(1-s))\overrightarrow{F}(z')d\mu(z')$$
$$=\int\limits_{\partial\mathcal{F}} \left(K_s(z,z';\chi)\frac{\partial\overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl$$
$$-\int\limits_{|z-z'|=\epsilon} \left(K_s(z,z';\chi)\frac{\partial\overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl.$$

We shall show

1) $\lim_{\epsilon\to 0}\int\limits_{B_\epsilon(z)} K_s(z,z';\chi)(\Delta+s(1-s))\overrightarrow{F}(z')d\mu(z')=0,$

2) $\int\limits_{\partial\mathcal{F}} \left(K_s(z,z';\chi)\frac{\partial\overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl=0,$

3) $-\lim_{\epsilon\to 0}\int\limits_{|z-z'|=\epsilon} \left(K_s(z,z';\chi)\frac{\partial\overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl=\overrightarrow{F}(z).$

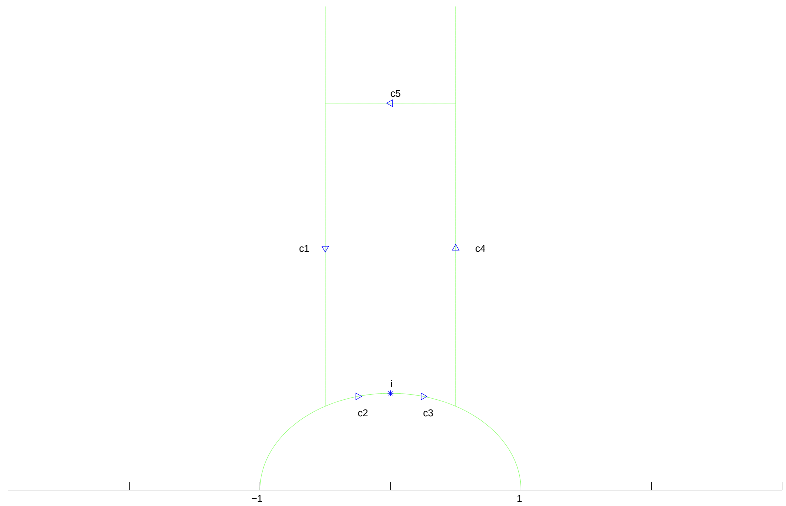

Figure 3.1: Path of integration

For 2) we have

$$\int\limits_{\partial\mathcal{F}} \left(K_s(z,z';\chi) \frac{\partial \overrightarrow{F}(z')}{\partial n} - \frac{\partial K_s(z,z';\chi)}{\partial n} \overrightarrow{F}(z') \right) dl \tag{3.88}$$

$$= \lim_{Y\to\infty} \int\limits_{\partial\mathcal{F}} \left(K_s(z,z';\chi) \frac{\partial \overrightarrow{F}(z')}{\partial n} - \frac{\partial K_s(z,z';\chi)}{\partial n} \overrightarrow{F}(z') \right) dl$$

$$= \lim_{Y\to\infty} \left(\int_{c1} + \int_{c2} + \int_{c3} + \int_{c4} + \int_{c5} \right) \left(K_s(z,z';\chi) \frac{\partial \overrightarrow{F}(z')}{\partial n} - \frac{\partial K_s(z,z';\chi)}{\partial n} \overrightarrow{F}(z') \right) dl.$$

Note that

$$\begin{aligned} &\int_{c4} K_s(z,z';\chi)y'\frac{\partial \overrightarrow{F}(z')}{\partial n}\,\frac{dl(z')}{y'} \qquad (3.89)\\ = &-\int_{c1} K_s(z,Sz';\chi)(y\frac{\partial}{\partial n})^S\overrightarrow{F}(Sz')\,\frac{dl(Sz')}{\Im Sz'}\\ = &-\int_{c1} K_s(z,z';\chi)\chi(S^{-1})(\frac{\partial}{\partial n})\overrightarrow{F}(Sz')\,\frac{dl(z')}{y'}\\ = &-\int_{c1} K_s(z,z';\chi)\chi(S^{-1})\chi(S)\frac{\partial \overrightarrow{F}(z')}{\partial n}\,dl(z')\\ &-\int_{c1} K_s(z,z';\chi)\frac{\partial \overrightarrow{F}(z')}{\partial n}\,dl(z'). \end{aligned}$$

Therefore

$$\int_{c1}+\int_{c4}\left(K_s(z,z';\chi)\frac{\partial \overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl=0.$$

Similarly,

$$\int_{c2}+\int_{c3}\left(K_s(z,z';\chi)\frac{\partial \overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl=0.$$

Therefore

$$\begin{aligned} &\int\limits_{\partial\mathcal{F}}\left(K_s(z,z';\chi)\frac{\partial \overrightarrow{F}(z')}{\partial n}-\frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl \qquad (3.90)\\ =\lim_{Y\to\infty}&\int\limits_0^1\left(K_s(z,x'+iY;\chi)\frac{\partial \overrightarrow{F}(x'+Y)}{\partial y'}-\frac{\partial K_s(z,x'+Y;\chi)}{\partial y'}\overrightarrow{F}(x'+iY)\right)dl. \end{aligned}$$

Now $\overrightarrow{F}\in\mathcal{B}'_\mu(\Gamma\backslash H,\chi)$ implies $|F_j(z')|\leq y'^\mu$ and $|\frac{\partial F_j(z')}{\partial y'}|\leq y'^\mu$. Also by (3.82), we have $|(K_s)_{ij}(z,z';\chi)|\leq {y'}^{1-\sigma}$ and $|\frac{\partial (K_s)_{ij}(z,z';\chi)}{\partial y'}|\leq y'^{-\sigma}$, therefore $\int_{c5}\longrightarrow 0$

as $Y \longrightarrow \infty$ if $\sigma > \mu$. Next, for any $(K_s)_{ij}$ and any F_k

$$
\begin{aligned}
&|\int\limits_{B_\epsilon(z)} [K_s]_{ij}(z,z';\chi)(\Delta + s(1-s))F_k(z')d\mu(z')| \qquad (3.91)\\
&\text{if } i \neq j\\
&\leq C|B_\epsilon(z)| \longrightarrow \quad \text{as} \quad \epsilon \longrightarrow 0\\
&\text{if } i = j\\
&C\int\limits_{B_\epsilon(z)} log|z-z'| \quad \longrightarrow 0 \quad \text{as}\epsilon \longrightarrow 0.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int\limits_{|z-z'|=\epsilon}\sum_{k=1}^{k=p}\left([K_s]_{jk}(z,z';\chi)\frac{\partial\overrightarrow{F_k}(z')}{\partial n} - \frac{\partial K_s(z,z';\chi)}{\partial n}\overrightarrow{F}(z')\right)dl \qquad (3.92)\\
=&\lim_{\epsilon\to 0} -\frac{\delta_{jk}}{2\pi}\int\limits_{|z-z'|=\epsilon}\frac{\partial log|z-z'|}{\partial n}F_k(z')\,dl\\
=&\lim_{\epsilon\to 0} -\frac{\delta_{jk}}{2\pi}\int\limits_{|z-z'=\epsilon|}\frac{\partial logr}{\partial r}|_{r=\epsilon}F_k(z+\epsilon e^{i\theta})\,d\theta + \lim_{\epsilon\to 0}\int_{|z-z'|=\epsilon} O(1)\,dl\\
=&\lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int\limits_0^{2\pi} F_k(z+\epsilon e^{i\theta})\,d\theta\\
=&F_j(z).
\end{aligned}
$$

Thus we have proved Theorem 3.1.

CHAPTER 4

Analytic Continuation

Let $\overrightarrow{E}(z,s;\chi_\rho)$ be the Eisenstein series defined in (2.7). In this section we prove the analytic continuation of $\overrightarrow{E}(z,s;\chi_\rho)$; we follow [7] closely. $\overrightarrow{E}(z,s;\chi_\rho) \in \mathcal{B}'_\mu(\Gamma\backslash H,\rho)$ follows from (2.34), (2.35) and (2.21). Fix $a \geq \sigma+1$. Apply (3.84) to

$$\begin{aligned}\overrightarrow{F}(z) &= (\Delta + a(1-a))\overrightarrow{E}(z,s;\chi_\rho) \\ &= (a(1-a)-s(1-s))\overrightarrow{E}(z,s;\chi_\rho).\end{aligned} \tag{4.1}$$

We have

$$-\overrightarrow{E}(z,s;\chi_\rho) = (a(1-a)-s(1-s))\int\limits_{\mathcal{F}} K_a(z,z';\chi_\rho)\overrightarrow{E}(z',s;\chi_\rho)d\mu(z'). \tag{4.2}$$

Thus $\overrightarrow{E}(z,s;\chi_\rho)$. is the solution to homogeneous singular Fredholm system of the second kind with parameter. The goal is to modify the kernel K so that a modified E solves a Fredholm equation with a constructable resolvent kernel. In this case the modified E has an integral representation which gives the analytic continuation. We modify the kernel in steps. First we eliminate the singularities on the diagonal by taking the difference

$$K_{ab}(z,z';\chi_\rho) = K_a(z,z';\chi_\rho) - K_b(z,z';\chi_\rho)$$

for fixed $a > b > 2\alpha + 1$. Using (4.2), we get a new Fredholm system

$$\overrightarrow{E}(z, s; \chi_\rho) = \lambda_{ab} \int\limits_{\mathcal{F}} K_{ab}(z, z'; \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z'). \tag{4.3}$$

Here $\lambda_{ab} = \frac{(a-s)(a+s-1)(b-s)(b+s-1)}{(b-a)(a+s-1)}$ and K_{ab} is continuous in $\mathcal{F} \times \mathcal{F}$. Next we define the truncated kernel on $\mathcal{F} \times \mathcal{F}$:

$$K_{ab}{}^{Y}(z, z'; \chi_{\prime}) = \begin{cases} K_{ab}(z, z'; \chi_\rho) \quad z' \in \mathcal{F}(Y); \\ K_{ab}(z, z'; \chi_\rho) - \frac{1}{(2a-1)} y'^{1-a} [E](z, a; \chi) \\ + \frac{1}{(2b-1)} y'^{1-b} [E](z, b; \chi) \quad z' \in \mathcal{F}(Y) \end{cases} \tag{4.4}$$

$[E](z, s; \chi)$ is defined by (3.80). Therefore we have

$$\begin{aligned} -\nu_{ab} \overrightarrow{E}(z, s; \chi_\rho) &= \int\limits_{\mathcal{F}} K_{ab}(z, z'; \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z') \qquad (4.5) \\ &= \int\limits_{\mathcal{F}(Y)} K_{ab}(z, z'; \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z') \\ &+ \int\limits_{\mathcal{F}_\infty(Y)} K_{ab}(z, z'; \chi_\rho) \overrightarrow{E}(z, s; \chi_\rho) d\mu(z') \\ &= \int\limits_{\mathcal{F}} K^{Y}{}_{ab}(z, z'; \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z') \\ &+ \frac{1}{2a-1} \int\limits_{\mathcal{F}_\infty(Y)} y'^{1-a} [E](z, a, \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z') \\ &- \frac{1}{2b-1} \int\limits_{\mathcal{F}_\infty(Y)} y'^{1-b} [E](z, b, \chi_\rho) \overrightarrow{E}(z', s; \chi_\rho) d\mu(z'). \end{aligned}$$

Remark 4.1 *Here* $-\nu_{ab} = \frac{1}{\lambda_{ab}}$. $[E](z, s, \chi)$ *is a matrix defined for any representation* χ *by* (3.80). $\overrightarrow{E}(z, s; \chi_\rho)$ *is defined by* (2.7)*; it is vector valued with representation* χ_ρ.

Now

$$\left(\frac{1}{2a-1}\int\limits_{\mathcal{F}_\infty(Y)} {y'}^{1-a}[E](z,a,\chi_\rho)\overrightarrow{E}(z',s;\chi_\rho)d\mu(z')\right)_j \tag{4.6}$$

$$=\sum_{k=1}^{p}\frac{E_{jk}(z,a,\chi_\rho)}{2a-1}\int\limits_{\mathcal{F}_\infty(Y)} {y'}^{1-a}E_k(z',s;\chi_\rho)d\mu(z')$$

$$=\sum_{k=1}^{p}\frac{E_{jk}(z,a,\chi_\rho)}{2a-1}\int\limits_0^1\int\limits_Y^\infty {y'}^{1-a}\left({y'}^s+\varphi_k(s){y'}^{1-s}+\cdots\right)\frac{dx'dy'}{{y'}^2}$$

$$=\sum_{k=1}^{p}\frac{E_{jk}(z,a,\chi_\rho)}{2a-1}\left\{\frac{Y^{s-a}}{a-s}+\varphi_k(s)\frac{Y^{1-s-a}}{s+a-1}\right\}. \tag{4.7}$$

Therefore

$$\begin{aligned}-\nu_{ab}\overrightarrow{E}(z',s;\chi_\rho) &= \int\limits_{\mathcal{F}} K_{ab}(z,z';\chi_\rho)\overrightarrow{E}(z,s;\chi_\rho)d\mu(z')\\ &+\frac{Y^{s-a}}{(2a-1)(a-s)}\overrightarrow{[E]}(z,a;\chi_\rho)+\frac{Y^{1-s-a}}{s+a-1}[E](z,a,\chi_\rho)\overrightarrow{\varphi}(s)\\ &-\frac{Y^{s-b}}{(2b-1)(b-s)}\overrightarrow{[E]}(z,b;\chi_\rho)+\frac{Y^{1-s-b}}{s+b-1}[E](z,b,\chi_\rho)\overrightarrow{\varphi}(s);\end{aligned} \tag{4.8}$$

where

$$\overrightarrow{[E]}(z,s,\chi_\rho)=\begin{pmatrix}\sum_{k=1}^p E_{1k}(z,s;\chi_\rho)\\ \sum_{k=1}^p E_{2k}(z,s;\chi_\rho)\\ \vdots\\ \sum_{k=1}^p E_{pk}(z,s;\chi_\rho)\end{pmatrix} \tag{4.9}$$

and

$$\overrightarrow{\varphi}(s)=\begin{pmatrix}\varphi_1(s)\\ \vdots\\ \varphi_p(s)\end{pmatrix}. \tag{4.10}$$

Next choose A_Y, A_{2Y}, A_{4Y} such that $\overrightarrow{\varphi}(s)$ is eliminated in the Fredholm equation with kernel $A_YK_{ab}^Y+A_{2Y}K_{ab}^{2Y}+A_{4Y}K_{ab}^{4Y}$. After simplification we get the equation

$$\overrightarrow{h}(z)=\overrightarrow{f}(z)+\lambda\int\limits_{\mathcal{F}}[H](z,z',\chi_\rho)\overrightarrow{h}(z')d\mu(z') \tag{4.11}$$

where

$$\begin{aligned}
\lambda &= \lambda_{ab}(s), \qquad (4.12)\\
\overrightarrow{h}(z) &= \overrightarrow{h}(z; s, a, b)\\
&= \frac{-\nu_{ab}(2^{s+a-1}-1)(2^{s+b-1}-1)\overrightarrow{E}(z, s, \chi_\rho)}{2^{2s-1}-1},\\
[H](z, z', \chi_\rho) &= [H](z, z'; \chi_\rho, s, a, b)\\
&= \frac{K^Y_{ab}(z, z') - (2^{a+s-1} + 2^{b+s-1})K^{2Y}_{ab}(z, z') + 2^{s+a-1}2^{s+b-1}K^{4Y}_{ab}(z, z')}{(2^{s+a-1}-1)(2^{s+b-1}-1)},
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{f}(z) &= \overrightarrow{f}(z; s, a, b)\\
&= \frac{(2^{2s-a+b-1}-1)Y^{s-a}\overrightarrow{[E]}(z, a, \chi_\rho)}{(2a-1)(a-s)}\\
&- \frac{(2^{2s+a-b-1}-1)Y^{s-b}\overrightarrow{[E]}(z, b, \chi_\rho)}{(2b-1)(b-s)}.
\end{aligned}$$

Now

$$f_j(z) << C_s y^a \qquad (4.13)$$

by (3.80) and C_s is bounded on $1 - c \leq \Re s \leq c$ so

$$f_j(z) << y^a \qquad (4.14)$$

uniformly for s such that $1-c \leq \Re s \leq c$. To estimate H_{ij} we note that for $\Re(s) > \alpha+1$ and $y' > y > Y$,

$$[K_{ab}^Y]_{ij}(z,z') = \delta_{ij} \sum_{\substack{n=-\infty \\ n+m_j\neq 0}}^{\infty} \frac{1}{4\pi|n+m_j|} \overline{V}_{\overline{a}}((n+m_j)z) W_a((n+m_j)z')$$
$$+ \delta_{m_i 1} \frac{y^{1-a}}{2s-1} \sum_{\substack{m=-\infty \\ m-m_j\neq 0}}^{\infty} \varphi_{m-m_j}^{ij}(a) W_a((m-m_j)z')$$
$$+ \sum_{\substack{m=-\infty \\ m-m_j}}^{\infty} \sum_{\substack{n=-\infty \\ n-m_i\neq 0}}^{\infty} Z_a(m-m_j, n-m_i) \overline{W}_{\overline{a}}((n-m_i)z) W_a((m-m_j)z')$$
$$- \delta_{ij} \sum_{\substack{n=-\infty \\ n+m_j\neq 0}}^{\infty} \frac{1}{4\pi|n+m_j|} \overline{V}_{\overline{b}}((n+m_j)z) W_b((n+m_j)z')$$
$$+ \delta_{m_i 1} \frac{y^{1-b}}{2s-1} \sum_{\substack{m=-\infty \\ m-m_j\neq 0}}^{\infty} \varphi_{m-m_j}^{ij}(b) W_b((m-m_j)z')$$
$$+ \sum_{\substack{m=-\infty \\ m-m_j}}^{\infty} \sum_{\substack{n=-\infty \\ n-m_i\neq 0}}^{\infty} Z_b(m-m_j, n-m_i) \overline{W}_{\overline{b}}((n-m_i)z) W_b((m-m_j)z').$$

Therefore for $y' > y > Y$,

$$[K_{ab}^Y]_{ij}(z,z') << e^{-\frac{\pi}{2}\{y'-y\}}. \tag{4.15}$$

For $\Re(s) > \alpha+1 \quad y > y' > Y$

$$\begin{aligned} & K_{ab}^Y(z,z';\chi) \qquad (4.16) \\ &= K_{ab}(z,z';\chi) - \frac{{y'}^{1-a}}{2a-1}[E](z,a;\chi) + \frac{{y'}^{1-b}}{2b-1}[E](z,b;\chi) \\ &= K_{ab}(z',z;\chi^{-1}) - \frac{{y'}^{1-a}}{2a-1}[E](z,a;\chi) + \frac{{y'}^{1-b}}{2b-1}[E](z,b;\chi) \\ &= K_{ab}^Y(z',z;\chi^{-1}) + \frac{{y'}^{1-a}}{2a-1}[E](z,a;\chi^{-1}) - \frac{{y'}^{1-b}}{2b-1}[E](z,b;\chi^{-1}) \\ &- \frac{{y'}^{1-a}}{2a-1}[E](z,a;\chi) + \frac{{y'}^{1-b}}{2b-1}[E](z,b;\chi). \end{aligned}$$

Therefore for $y > y' > Y$

$$[K_{ab}^Y]_{ij}(z,z') << y^a. \tag{4.17}$$

Therefore, for $y, y' > 4Y$

$$[H]_{ij}(z,z') << y^a e^{-\frac{\pi}{2}\max\{y'-y,0\}} \tag{4.18}$$

uniformly for $1-c \leq \Re s \leq c$. To get a bounded kernel we multiply (4.11) by $\eta(z) = e^{-\eta y}$ where $0 < \eta < \frac{\pi}{2}$:

$$\begin{aligned} \eta(z)\overrightarrow{h}(z) &= \eta(z)\overrightarrow{f}(z) + \lambda \int_{\mathcal{F}} \eta(z)[H](z,z',\chi_\rho)\overrightarrow{h}(z')d\mu(z') \\ &= \eta(z)\overrightarrow{f}(z) + \lambda \int_{\mathcal{F}} \eta(z)[H](z,z',\chi_\rho)\eta(z')^{-1}\eta(z')\overrightarrow{h}(z')d\mu(z') \end{aligned} \tag{4.19}$$

The jth equation in the above system is

$$\eta(z)h_j(z) = \eta(z)f_j(z) + \lambda\sum_{k=1}^{p}\int_{\mathcal{F}} \eta(z)[H]_{jk}(z,z',\chi_\rho)\eta(z')^{-1}\eta(z')h_k(z')d\mu(z'). \tag{4.20}$$

This is a Fredholm system with bounded kernel $\eta(z)[H](z,z',\chi_\rho)\eta(z')^{-1}$.

4.1 Fredholm Theory

Plemelj ,[15], solves the Fredholm system (4.20) by lifting it to a scalar equation on $\bigoplus_{k=1}^{p}\mathbb{C}$. Let $\mathcal{F}_j = \{0\} \oplus \cdots \oplus \overset{j}{\mathcal{F}} \oplus \cdots \{0\}$, and $\mathcal{F}^{\mathcal{L}} = \bigcup_{j=1}^{p}\mathcal{F}_j$. Define h^l, f^l, and H^L on $\mathcal{F}^{\mathcal{L}}$ and $\mathcal{F}^{\mathcal{L}} \times \mathcal{F}^{\mathcal{L}}$ as follows:

$$\begin{aligned} h^l(z_j^l) &= \eta(z_j)h_j(z_j) \\ f^l(z_j^l) &= \eta(z_j)f_j(z_j) \\ H^L(z_i^l, {z'}_i^l) &= \eta(z_j)h_j(z_j) = H_{ij}(z,z'). \end{aligned}$$

We can now write the system (4.20) in the scalar form

$$h^l(z^l) = f^l(z^l) + \lambda\int_{\mathcal{F}^{\mathcal{L}}} H^L(z^l, {z'}^l)h^l({z'}^l)\,dz'. \tag{4.21}$$

We have

1) $H^L(z,z')$ is continuous on $\mathcal{F}^{\mathcal{L}} \times \mathcal{F}^{\mathcal{L}}$ since $H_{ij}(z,z')$ is continuous on $\mathcal{F} \times \mathcal{F}$.

2) $H_s^L(z, z')$ is bounded on $\mathcal{F}^{\mathcal{L}} \times \mathcal{F}^{\mathcal{L}}$ uniformly in s on compact subsets of $\mathcal{S} = \{s \in \mathbb{C} : -c + 1 < \Re s < c\}$.

3) $\lambda(s)$ is an entire function of s.

4) $h_s^l(z)$ is meromorphic in s for $\Re(s) > \alpha + 1$;its poles,if any, occur at the roots of $\lambda(s)$.

5) f_s^l is meromorhpic with at most simple poles at $s = a$ and $s = b$.

6) $H_s^L(z, z')$ is meromorphic in s with at most simple poles at $s = 1 - a$ and $s = 1 - b$.

The equation (4.21) means given $f^l(z^l)$ solve for $h^l(z^l)$. The solution is obtained by constructing the resolvent kernel,$R_\lambda(z, z')$; the solution is given by

$$h^l(z^l) = f^l(z^l) + \lambda \int_{\mathcal{F}^{\mathcal{L}}} R_\lambda(z^l, z'^l) f^l(z'^l)\, dz'. \tag{4.22}$$

When the following conditions are satisfied

1) $Vol(\mathcal{F}^{\mathcal{L}}) < \infty$;

2) $H^L(z, z')$ is continuous and bounded on $\mathcal{F}^{\mathcal{L}} \times \mathcal{F}^{\mathcal{L}}$.

then the Fredholm construction produces the resolvent kernel in the form

$$R_\lambda(z^l, z'^l) = \frac{D_\lambda(z^l, z'^l)}{D(\lambda)}. \tag{4.23}$$

Here $D_\lambda(z^l, z'^l)$ and $D(\lambda)$ are given by power series in λ :

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m \tag{4.24}$$

$$D_\lambda(z^l, z'^l) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m(z^l, z'^l). \tag{4.25}$$

Here $C_m(s)$ and $C_m(z^l, z'^l; s)$ are defined by

$$C_m(s) = \int_D \cdots \int_D H^L \begin{pmatrix} \tau_1 & \dots & \tau_m \\ \tau_1 & \dots & \tau_m \end{pmatrix} d\mu(\tau_1) \cdots d\mu(\tau_m) \quad \tau_j = z^{lj} \tag{4.26}$$

and

$$C_m(z^l, {z'}^l; s) = \int\limits_D \cdots \int\limits_D H_s^L \begin{pmatrix} z & \tau_1 & \dots & \tau_m \\ z' & \tau_1 & \dots & \tau_m \end{pmatrix} d\mu(\tau_1) \cdots d\mu(\tau_m). \qquad (4.27)$$

Here

$$H_s^L \begin{pmatrix} \tau_1 & \dots & \tau_m \\ \omega_1 & \dots & \omega_m \end{pmatrix} = det\left(H_s^L(\tau_i, \omega_j)\right). \qquad (4.28)$$

We show that $D(\lambda)$ and $D_\lambda(z^l, {z'}^l)$ are entire in λ and analytic in s for $s \in \mathcal{S}$. We observe that λ is a polynomial in s hence entire in s. We also observe that $C_m(s)$ and $C_m(z^l, {z'}^l; s)$ are analytic for $s \in \mathcal{S}$, see Remark 4.2 below. Let K be a compact subset of $\mathcal{S}$. Let M and λ_0 be the uniform bound of $H_s^L(z, z')$ and $|\lambda|$, respectively, on K.

To bound $det\left(H_s^L(\tau_i, \omega_j)\right)$ apply Hadamard's inequality

$$|det(a_{ij})|^2 \leq \prod_{j=1}^m \left(\sum_{i=1}^m |a_{ij}|^2\right) \qquad (4.29)$$

to obtain

$$H_s^L \begin{pmatrix} \tau_1 & \dots & \tau_m \\ \omega_1 & \dots & \omega_m \end{pmatrix}^2 \leq \prod_{j=1}^m \left(\sum_{i=1}^m |H_s^L(\tau_i, \omega_j)|^2\right) \qquad (4.30)$$

$$\leq m^m M^{2m}. \qquad (4.31)$$

Therefore

$$|H^L \begin{pmatrix} \tau_1 & \dots & \tau_m \\ \omega_1 & \dots & \omega_m \end{pmatrix}| \leq (\sqrt{m} M)^m. \qquad (4.32)$$

Hence

$$|C_m(s)| \leq \left(\sqrt{m} M Vol(\mathcal{F}^{\mathcal{L}})\right)^m \quad Vol(\mathcal{F}^{\mathcal{L}}) = pVol(\mathcal{F}) < \infty. \qquad (4.33)$$

Similarly, we have

$$|C_m(z^l, {z'}^l; s)| \leq \left(\sqrt{m+1} M\right)^{m+1} Vol(\mathcal{F}^{\mathcal{L}})^m. \qquad (4.34)$$

We use the inequality, derived from Stirling's formula,

$$n! > n^n e^{-n} \tag{4.35}$$

to obtain the bound

$$\frac{(\sqrt{m}|\lambda_0|MpV)^m}{m!} \leq \left(\frac{|\lambda_0|Mp}{e^{(\frac{logm}{2}-1)}}\right)^m. \tag{4.36}$$

Pick m_0 such that $\frac{|\lambda_0|Mp}{e^{(\frac{logm}{2}-1)}} < \frac{1}{2}, \quad m > m_0$. Therefore, by the Weierstrass M-test, $C_m(s)$ is analytic for $s \in \mathcal{S}$.

Remark 4.2 *$C_m(s)$ is analytic for $s \in \mathcal{S}$. To see this, consider $m = 1$,by* (4.26)*,*

$$C_1(s) = \int_{\mathcal{F}^{\mathcal{L}}} H_s^L(\tau_1, \tau_1) d\mu(\tau). \tag{4.37}$$

Then, by (4.12) *and the definition of $\mathcal{F}^{\mathcal{L}}$,*

$$C_1(s) = \sum_{i=1}^{p} \int_{\mathcal{F}} [H_s]_{ii}(z, z) d\mu(z) \tag{4.38}$$

$$= \frac{1}{(2^{s+a-1}-1)(2^{s+b-1}-1)} \sum_{i=1}^{p} \int_{\mathcal{F}} [K_{ab}]_{ii}^{Y}(z, z') d\mu(z) \tag{4.39}$$

$$- \frac{(2^{a+s-1}+2^{b+s-1})}{(2^{s+a-1}-1)(2^{s+b-1}-1)} \sum_{i=1}^{p} \int_{\mathcal{F}} [K_{ab}]_{ii}^{2Y}(z, z') d\mu(z)$$

$$+ \frac{2^{s+a-1}2^{s+b-1}}{(2^{s+a-1}-1)(2^{s+b-1}-1)} \sum_{i=1}^{p} \int_{\mathcal{F}} [K_{ab}]_{ii}^{4Y}(z, z') d\mu(z).$$

Thus $C_1(s)$ is meromorphic with at most simple poles at $s = 1-a$ and $s = 1-b$. If

$$\omega_1 = \frac{1}{(2^{s+a-1}-1)(2^{s+b-1}-1)},$$
$$\omega_2 = \frac{(2^{a+s-1}+2^{b+s-1})}{(2^{s+a-1}-1)(2^{s+b-1}-1)}, \quad \textit{and}$$
$$\omega_3 = \frac{2^{s+a-1}2^{s+b-1}}{(2^{s+a-1}-1)(2^{s+b-1}-1)},$$

then $C_{m(s)}$ *is a polynomial in* $\omega_1, \omega_2,$ *and* ω_3. *Thus* $C_m(s)$ *is meromorphic with at most poles of order* m *at* $s = 1 - a$ *and* $s = 1 - b$. *In particular* $C_m(s)$ *is analytic for* $s \in \mathcal{S}$. $C_m(z^l, z'^l; s)$ *has a similar form except the order of the poles is at most* $m + 1$.

It follows that $R_\lambda(z^l, z'^l)$ is meromorphic for $s \in \mathcal{S}$. Thus the RHS of (4.22) gives the meromorphic continuation of h_s^l to $s \in \mathcal{S}$. Since c is arbitrary, we have a meromorphic continuation of h_s^l to the whole s-plane. Thus $\eta(z)(h_s)_j(z)$ has a meromorphic continuation to the whole s-plane. Therefore

$$E_j(z, s, \chi_\rho) = \frac{2^{2s-1} - 1)\lambda\eta(z)(h_s)_j(z)}{(2^{s+a-1} - 1)(2^{s+b-1} - 1)} \tag{4.40}$$

s-plane. We have proved the following

Theorem 4.1 $\overrightarrow{E}(z, s, \chi_\rho)$ *admits an analytic continuation to the whole s-plane.*

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NOTATION INDEX

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APPENDIX A

Rankin-Selberg for Unitary Vector-Valued Modular Forms

In the introduction, we noted that Selberg [18] introduced vector-valued modular forms with unitary representation to extend the $O(n^{\frac{k}{2}-\frac{1}{5}})$ estimate on cusp forms to arbitrary subgroups of finite index in $\Gamma(1)$. However no details are given. Here we will extend Rankin's method [16] to get estimates of Fourier coefficients of vector-valued modular cusp forms. (The detailed proof we present is in the nature of a public service.)

A.1 Definitions

Let $k \in \mathbb{R}$. Let $\Gamma' \subset \Gamma(1)$ be a subgroup of finite index μ in $\Gamma(1)$. We let $A_1, \cdots, A_q$ denote a complete set of right coset representatives of Γ' in $\Gamma(1)$. Let v be a multiplier system for the group Γ' and weight k. A function, $f(z)$, meromorphic on H is a modular form with respect to (Γ', k, v) if, see [8],

i) $f(Vz) = v(V)(cz+d)^k f(z) \quad \forall \quad V = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma'$;

ii) at each cusp $q_j = A_j(\infty)$, f(z) has the expansion

$$f(z) = \sigma_j(z) \sum_{n \geq h_j} a_n(j) e^{2\pi i (n+\kappa_j) \frac{(A_j^{-1} z)}{N_j}}. \tag{A.1}$$

Here

$$\sigma_j(z) = \begin{cases} 1, & \text{if } q_j = \infty; \\ \frac{1}{z-q_j}, & \text{if } q_j < \infty. \end{cases} \tag{A.2}$$

Also κ_j is defined by $v(A_j S^{N_j} A_j^{-1}) = e^{2\pi i \kappa_j} \quad 0 \leq \kappa_j < 1$; N_j is the smallest positive integer such that $A_j S^{N_j} A_j^{-1} \in \Gamma'$. $f(z)$ is a modular cusp form if $h_j + \kappa_j > 0 \quad 1 \leq j \leq \mu$.

Let $(\overrightarrow{F}, \rho)$ be a vector-valued modular form of real weight k on the modular group $\Gamma(1)$ with respect to a unitary representation. That is $(\overrightarrow{F}, \rho)$ is a p-tuple $\overrightarrow{F}(z) = (F_1(z), \ldots, F_p(z))$ of functions holomorphic in the complex upper half-plane H, together with a p-dimensional unitary complex representation $\rho : \Gamma(1) \longrightarrow GL(p, C)$ satisfying the following;

(a) For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ we have

$$(F_1(z), \ldots, F_p(z))^t|_k V(z) = \rho(V)(F_1(z), \ldots, F_p(z))^t, \tag{A.3}$$

$$\text{i.e } F_j(Vz) = v(V)(cz+d)^k \sum_{m=1}^{p} \rho_{jm}(V) F_m(z).$$

(b) Each component function $F_j(z)$ has an expansion convergent in H and meromorphic at ∞:

$$F_j(z) = \sum_{n \geq h_j} a_n(j) e^{\frac{2\pi i n z}{N_j}}, \tag{A.4}$$

with $h_j \in \mathbb{Z}$ and $N_j \in \mathbb{Z}^+$.

We assume $\overrightarrow{F}(z)$ is cuspidal, i.e. $h_j > 0 \quad 1 \leq j \leq p$. Let

$$N = lcm\{N_1, \ldots, N_p\} \quad N = N_j m_j \tag{A.5}$$

Thus we can write,

$$F_j(z) = \sum_{n\geq 1} a_n(j) e^{\frac{2\pi i m_j n z}{N}}. \tag{A.6}$$

Also we write $\|\overrightarrow{F}(z)\| = \sqrt{\sum_{j=1}^{p} \|F_j(z)\|^2}$.

Remark A.1 *Let $f(z)$ be a modular form with respect to (Γ', k, v), Γ' a subgroup of finite index μ in $\Gamma(1)$. We attach to $f(z)$ a vector-valued modular form,*

$$\overrightarrow{F}^t = \begin{pmatrix} F_1(z) \\ \vdots \\ F_\mu(z) \end{pmatrix}, \tag{A.7}$$

on all of $\Gamma(1)$. Here $F_j(z)$ is defined be

$$F_j(z) = (f|_k A_j)(z) = (\gamma_j z + \delta_j)^{-k} f(A_j z). \tag{A.8}$$

If w is any multiplier system on $\Gamma(1)$, then $\left(\overrightarrow{F}^t|_k^w V\right)(z) = \rho(V)\overrightarrow{F}^t(z)$ where ρ is both unitary and monomial.

We prove the following

Theorem A.1 . *Let $(\overrightarrow{F}, \rho)$ be a vector-valued modular form of real weight k on the modular group $\Gamma(1) = SL(2, Z)$ with respect to a unitary representation. If*

$$F_j(z) = \sum_{n\geq 1} a_n(j) e^{\frac{2\pi i m_j n z}{N}}, \tag{A.9}$$

then

$$a_n(j) = O(n^{\frac{k}{2} - \frac{1}{5}}). \tag{A.10}$$

Corollary A.1 *If $f(z)$ is a modular form with respect to (Γ', k, v), $\Gamma' \subset \Gamma$ of finite index, then*

$$a_n = O(n^{\frac{k}{2} - \frac{1}{5}}). \tag{A.11}$$

A.2 Basic Estimates

The $a_n(j)$ are the Fourier coefficients of the given $F_j(z)$. In the sequel there arises $b_n(j)$ and $c_n(j)$ related to $a_n(j)$ as follows:

$$b_n(j) = \sum_{d^2|n} |a_{\frac{n}{d^2}}(j)|^2 d^{2k-2} \tag{A.12}$$

$$c_n(j) = b_n(j)n^{1-k} = \sum_{d^2m=n} |a_m(j)|^2 m^{1-k}. \tag{A.13}$$

In this section we prove basic estimates for the asymptotics $\sum_{n\le x} |a_n(j)|^2$, $\sum_{n\le x} b_n(j)$, and $\sum_{n\le x} c_n(j)$.

Proposition A.1 *Let $(\overrightarrow{F}, \rho)$ be a unitary vector-valued cusp form of weight k. Then we have the Hecke estimate*

$$\|\overrightarrow{F}(z)\| \le C y^{-\frac{k}{2}}. \tag{A.14}$$

Proof:

$$F_j(z) = \sum_{n\ge 1} a_n(j) e^{\frac{2\pi i n z}{N_j}}.$$

implies

$$|F_j(z)| \le C e^{\frac{-2\pi y}{N_j}} \quad y > y_j. \tag{A.15}$$

Let $\varphi(z) = y^{\frac{k}{2}} \|\overrightarrow{F}(z)\|$; then $\varphi(z)$ is continuous on H and invariant, since ρ is unitary, under $\Gamma(1)$. We show that $\varphi(z)$ is bounded on $\mathcal{F}$. (A.15) implies there exists y_j such that $y^k |F_j(z)|^2 < \frac{1}{p} \quad y > y_j$. Let $Y_0 = max\{y_1, \dots, y_p\}$, then $y^{\frac{k}{2}} \|\overrightarrow{F}(z)\| \le 1$ on $\mathcal{F}_\infty(Y_0)$. Let $M = \sup_{z\in\mathcal{F}(Y_0)} \varphi(z)$ and $C = \max\{M, 1\}$, then $|\varphi(z)| \le C$, for $z \in \mathcal{F}$. This implies $|\varphi(z)| \le C$, for $z \in H$, since φ is invariant under $\Gamma(1)$. Thus $\|\overrightarrow{F}(z)\| \le C y^{-\frac{k}{2}}$, for $z \in H$. It follows that $|F_j(z)| \le C y^{-\frac{k}{2}}$.

Remark A.2 $\|\overrightarrow{F}(z)\| = O(e^{\frac{-2\pi y}{N}})$ *as* $y \longrightarrow \infty$ *and* $\|\overrightarrow{F}(x+iy)\| = O(y^{-\frac{k}{2}})$ *uniformly as* $y \longrightarrow 0$ *implies*

$$\iint_S y^{s+k} \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2} < \infty, \quad \textit{for } Re(s) > 1.$$

Here $S = \{z \in H : |\Re z| < \frac{1}{2}\}$.

Proposition A.2 $a_n(j) = O(n^{\frac{k}{2}})$

Proposition A.3 $\sum_{n\leq x} |a_n(j)|^2 = O(x^k)$.

Remark A.3 *Proposition A.2 follows immediately from Proposition A.3.*

Proof of Proposition A.3: We have the Hecke estimate

$$|F_j(z)| \leq Cy^{-\frac{k}{2}}.$$

Now, since $n \leq x$,

$$\begin{aligned}
\sum_{n\leq x} |a_n(j)|^2 e^{-\frac{4\pi xy}{N_j}} &\leq \sum_{n\leq x} |a_n(j)|^2 e^{-\frac{4\pi ny}{N_j}} \\
&\leq \sum_{n=1}^{\infty} |a_n(j)|^2 e^{-\frac{4\pi ny}{N_j}} \\
&= \frac{1}{N_j} \int_{-\frac{N_j}{2}}^{\frac{N_j}{2}} |F_j(x+iy)|^2 \, dx \qquad \text{by Parseval's theorem.}
\end{aligned}$$

Applying the Hecke estimate, then we obtain

$$\sum_{n\leq x} |a_n(j)|^2 e^{-\frac{4\pi xy}{N_j}} \leq C^2 y^k. \tag{A.16}$$

Set $y = \frac{1}{x}$ to derive the desired estimate $\sum_{n\leq x} |a_n(j)|^2 = O(x^k)$. Next we apply Abel's partial summation [1]:

Theorem A.2 *Let $\{g_n\}$ be a sequence of real numbers. For $x \geq 0$, define $G(x) = \sum_{n\leq x} g_n = \sum_{n=1}^{[x]} g_n = G([x])$. Let $f \in C^1([1,x])$, then the following formulas hold:*

(a) $\sum_{n\leq x} g_n f(n) = \sum_{n\leq x} G(n)(f(n) - f(n+1)) + G([x])(f([x]+1))$,

(b) $\sum_{n\leq x} g_n f(n) = -\int_1^x G(y) f'(y)\, dy + G(x) f(x)$.

Proposition A.4 *If $b_n(j) = \sum_{d^2|n} |a_{\frac{n}{d^2}}(j)|^2 d^{2k-2}$, then*

$$\sum_{n\leq x} b_n(j) = O(x^k). \tag{A.17}$$

Proof:

$$\sum_{n\le x} b_n(j) = \sum_{n\le x}\sum_{md^2=n} |a_m(j)|^2 d^{2k-2}$$
$$= \sum_{md^2\le x} |a_m(j)|^2 d^{2k-2},$$

where $\sum_{md^2\le x}$ is a sum over all lattice points under the hyperbola $md^2 = x$. Thus $\sum_{md^2\le x} = \sum_{d\le\sqrt{x}}\sum_{m\le\frac{x}{d^2}}$ and we have

$$\sum_{n\le x} b_n(j) = \sum_{md^2\le x} |a_m(j)|^2 d^{2k-2}$$
$$= \sum_{d\le\sqrt{x}} d^{2k-2} \sum_{m\le\frac{x}{d^2}} |a_m(j)|^2$$
$$\le C \sum_{d\le\sqrt{x}} d^{2k-2}\frac{x^k}{d^{2k}}$$
$$= Cx^k \sum_{d\le\sqrt{x}} \frac{1}{d^2}$$
$$\le Cx^k\zeta(2)$$
$$= O(x^k).$$

Proposition A.5 *Let* $c_n(j) = \sum_{d^2m=n} |a_m(j)|^2 m^{1-k}$. *Then*

$$\sum_{n\le x} c_n(j) = O(x).$$

Proof: Apply Abel's summation with $g_n = b_n(j)$ and $f(x) = x^{1-k}$:

$$\sum_{n\le x} c_n(j) = \sum_{n\le x} b_n(j) n^{1-k}$$
$$= -(1-k)\int_1^x \left(\sum_{n\le y} b_n(j)\right) y^{-k}\,dy + \left(\sum_{n\le x} b_n(j)\right) x^{1-k}.$$

Therefore

$$|\sum_{n\le x} c_n(j)| \le C\int_1^x y^k y^{-k}\,dy + Cx^k x^{1-k} \tag{A.18}$$
$$\le Cx.$$

Therefore $\sum_{n\leq x} c_n(j) = O(x)$.

Remark A.4 *Note also that* $c_n(j) = O(x)$.

A.3 Functional Equation

Let

$$k\alpha = \frac{3}{\Gamma(k)}(\frac{4\pi}{N})^k \iint\limits_{\mathcal{F}} y^k \| \overrightarrow{F}(z) \|^2 \frac{dxdy}{y^2}.$$

Let

$$\zeta_{\overrightarrow{F}}(s) = \sum_{j=1}^{p} \sum_{n=1}^{\infty} \frac{|a_n(j)|^2}{(m_j n)^{s+k-1}} \qquad \text{(A.19)}$$

where the $m_j's$ are defined in (A.5). $\zeta_{\overrightarrow{F}}(s)$ is the Rankin-Selberg zeta function corresponding to $\overrightarrow{F}$. As in the scalar case, [16], we prove

Theorem A.3 . *The function* $\zeta_{\overrightarrow{F}}(s)$ *defined by* (A.19) *has the properties:*

i) *The series* (A.19) *is absolutely convergent for* $Re(s) > 1$ *and absolutely-uniformly convergent for* $Re(s) > 1 + \epsilon, \quad \epsilon > 0$.

ii) $\zeta_{\overrightarrow{F}}(s)$ *may be continued as a meromorphic function over the whole plane.*

iii) $\zeta_{\overrightarrow{F}}(s)$ *has a simple pole of residue* $k\alpha$ *at* $s = 1$.

iv) $\zeta_{\overrightarrow{F}}(s)$ *satisfies the functional equation*

$$\psi(s) = \psi(1 - s),$$

where

$$\psi(s) = \pi^{-s}(\frac{4\pi}{N})^{1-s}\Gamma(s)\Gamma(s + k - 1)\zeta(2s)\zeta_{\overrightarrow{F}}(s).$$

v) $\psi(s)$ *is regular over the whole plane except for simple poles at the points* $s = 1$ *and* $s = 0$.

Proof: Let

$$\zeta_j(s) = \sum_{n=1}^{\infty} \frac{|a_n(j)|^2}{n^{s+k-1}}, \tag{A.20}$$

As before, we have for $y > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n(j)|^2 e^{-\frac{4\pi n y}{N_j}} &= \frac{1}{N_j} \int_{-\frac{N_j}{2}}^{\frac{N_j}{2}} |F_j(x+iy)|^2 \, dx \\ &= \frac{1}{N} \int_{-\frac{N}{2}}^{\frac{N}{2}} |F_j(x+iy)|^2 \, dx, \end{aligned} \tag{A.21}$$

since $N = m_j N_j$. But,

$$\Gamma(s+k-1) = \int_0^{\infty} e^{-u} u^{s+k-1} \frac{du}{u} = (\frac{4\pi n}{N_j})^{s+k-1} \int_0^{\infty} e^{-\frac{4\pi n y}{N_j}} y^{s+k-1} \frac{dy}{y}, \quad \text{for } Re(s) > 1-k.$$

Therefore, for $Re(s) > 1 - k$,

$$\begin{aligned} (\frac{4\pi}{N_j})^{-(s+k-1)} \Gamma(s+k-1) \zeta_j(s) &= \sum_{n=1}^{\infty} |a_n(j)|^2 \int_0^{\infty} e^{-\frac{4\pi n y}{N_j}} y^{s+k-1} \frac{dy}{y} \\ &= \int_0^{\infty} y^{s+k-1} \sum_{n=1}^{\infty} |a_n(j)|^2 e^{-\frac{4\pi n y}{N_j}} \frac{dy}{y} \\ &= \frac{1}{N} \int_0^{\infty} \int_{-\frac{N}{2}}^{\frac{N}{2}} y^{s+k} |F_j(x+iy)|^2 \, dx \, \frac{dy}{y^2}, \end{aligned}$$

by (A.21). Therefore,

$$(\frac{4\pi}{N})^{-(s+k-1)} \Gamma(s+k-1) \sum_{n=1}^{\infty} \frac{|a_n(j)|^2}{(m_j n)^{s+k-1}} = \frac{1}{N} \int_0^{\infty} \int_{-\frac{N}{2}}^{\frac{N}{2}} y^{s+k} |F_j(x+iy)|^2 \, dx \, \frac{dy}{y^2}, \tag{A.22}$$

for $Re(s) > 1-k$. Now sum over j to obtain

$$\left(\frac{4\pi}{N}\right)^{-(s+k-1)}\Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \frac{1}{N}\int_{-\frac{N}{2}}^{\frac{N}{2}}\int_{0}^{\infty} y^s y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2} \tag{A.23}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{0}^{\infty} y^s y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}.$$

The last equality uses the fact that, while each F_j has period N, $y^k\|\overrightarrow{F(z)}\|^2$ has period 1, since $\rho(S)$ is unitary. Therefore,

$$\left(\frac{4\pi}{N}\right)^{-(s+k-1)}\Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \iint_S y^{s+k}\|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}, \tag{A.24}$$

Continuing, we apply the unfolding trick and use the invariance of $y^k\|\overrightarrow{F(z)}\|^2$ under all of $\Gamma(1)$, to obtain

$$(\frac{4\pi}{N})^{-(s+k-1)}\Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \iint_{\mathcal{F}} \sum_{\sigma\in\Gamma_\infty\backslash\Gamma(1)} \Im(\sigma z)^s y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}$$

$$= \iint_{\mathcal{F}} \sum_{\substack{m,n\in Z\\(m,n)=1}} \frac{y^s}{|mz+n|^{2s}} y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}$$

$$= \iint_{\mathcal{F}} E(z,s) y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}$$

where $E(z,s) = \sum_{\substack{m,n\in Z\\(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}$. That is

$$(\frac{4\pi}{N})^{-(s+k-1)}\Gamma(s+k-1)\zeta_{\overrightarrow{F}}(s) = \iint_{\mathcal{F}} E(z,s) y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}. \tag{A.25}$$

At this point,we want to use the functional equation for

$$\mathcal{E}(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2}E(z,s).$$

In fact, we need the following found in Bump [2]

Theorem A.4 *$\mathcal{E}(z,s)$, originally defined for $Re(s) > 1$, has meromorphic continuation to all s; it is analytic except at $s = 1$ and $s = 0$, where it has simple poles. The residue at $s = 1$ is the constant function $z = \frac{1}{2}$. The Eisenstein series satisfies the functional equation*

$$\mathcal{E}(z,s) = \mathcal{E}(z, 1-s).$$

We have

$$\mathcal{E}((x+iy), s) = O(y^{\sigma}) \quad as\ y \longrightarrow \infty,$$

where $\sigma = max(Re(s), 1 - Re(s))$.

Multiply both sides of (A.25) by $\frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)$, we get

$$\pi^{-s}\left(\frac{4\pi}{N}\right)^{-(s+k-1)}\frac{1}{2}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\zeta_{\overrightarrow{F}}(s) = \iint\limits_{\mathcal{F}} y^k \|\overrightarrow{F}(z)\|^2 \mathcal{E}(z,s)\,\frac{dxdy}{y^2}. \tag{A.26}$$

Therefore

$$\begin{aligned}\psi(s) &= \pi^{-s}(\frac{4\pi}{N})^{1-s}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\zeta_{\overrightarrow{F}}(s) \qquad \text{(A.27)}\\ &= 2(\frac{4\pi}{N})^k \iint\limits_{\mathcal{F}} y^k \|\overrightarrow{F}(z)\|^2 \mathcal{E}(z,s)\,\frac{dxdy}{y^2}.\end{aligned}$$

It follows from Theorem A.4 that (A.27) defines a meromorphic continuation of $\psi(s)$ to all of s; it is analytic except for simple poles at $s = 1$ and $s = 0$. Furthermore the functional equation (A.4) implies that $\zeta_{\overrightarrow{F}}(s)$ satisfies the function equation

$$\psi(s) = \psi(1-s). \tag{A.28}$$

Let us see what (A.27) tell us about the analytic continuation of $\zeta_{\overrightarrow{F}}(s)$. Solving, we have

$$\zeta(2s)\zeta_{\overrightarrow{F}}(s) = \frac{2\pi^s(\frac{4\pi}{N})^{s+k-1}}{\Gamma(s)\Gamma(s+k-1)} \iint\limits_{\mathcal{F}} \mathcal{E}(z,s) y^k \|\overrightarrow{F}(z)\|^2\,\frac{dxdy}{y^2}. \tag{A.29}$$

Therefore $\zeta(2s)\zeta_{\overrightarrow{F}}(s)$ is analytic in the whole plane except for at most a simple pole at and $s = 1$;$\frac{1}{\Gamma(s)}$ cancels the pole of $\psi(s)$ at $s = 0$. It follows that $\zeta_{\overrightarrow{F}}(s)$

is a meromorphic function having a simple pole at $s = 1$ with residue

$$k\alpha = \frac{3}{\Gamma(k)}(\frac{4\pi}{N})^k \iint\limits_{\mathcal{F}} y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}.$$

Also, $\zeta_{\overrightarrow{F}}(s)$ may have poles at the complex zeros of $\zeta(2s)$.

A.4 Landau's Theorem

Let $\{c_n\}$ be a sequence of non-negative numbers. In this section we use Landau's Theorem [13] to estimate the asymptotic $B(x) = \sum_{n\leq x} c_n$. We use the following abbreviated form of Landau's theorem.

Theorem A.5 (Landau's Theorem) *Let $\beta, \beta_1, \beta_2, \delta_1, \delta_2 > 0$ be such that*

$$\beta_1 + \beta_2 = \delta_1 + \delta_2. \tag{A.30}$$

Let $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{R}$ be such that

$$\eta \doteq \gamma_1 + \gamma_2 - \alpha_1 - \alpha_2 > \frac{1}{2}. \tag{A.31}$$

Let $\{e_n\}$, $e_n \in \mathbb{C}$ and $\{\lambda_n\}$, $0 < \lambda_n < \lambda_{n+1}$ be infinite sequences. If the following conditions are satisfied:

i) $Z(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ is absolutely convergent for $Re(s) > \beta$ for our purposes $c_n \geq 0$.

ii) $Z(s)$ admits a meromorphic continuation to the entire plane, with finitely many poles in each vertical strip.

iii) The series $\sum_{n=1}^{\infty} e_n\lambda_n^s$ is absolutely convergent for $Re(s) < 0$.

iv) For $Re(s) < 0$

$$\Gamma(\alpha_1 + \beta_1 s)\Gamma(\alpha_2 + \beta_2 s)Z(s) = \Gamma(\gamma_1 - \delta_1 s)\Gamma(\gamma_2 - \delta_2 s)\sum_{n=1}^{\infty} e_n\lambda_n^s.$$

v) $Z(s) = O(e^{\gamma|t|})$ in vertical strips, for some $\gamma > 0$.

vi) There exists $A \geq 0$, such that

$$\sum_{\lambda_n \leq x} |e_n| \lambda_n^{\beta} = O(x^{\beta} log^A x).$$

Then if $\chi = \beta \frac{2\eta - 1}{2\eta + 1}$, *p is the order of the pole of* $Z(s)$ *at* $s = 1$ *and* $g = max(A, p-1)$, *then it follows that*

$$B(x) = \sum_{n \leq x} c_n = R(x) + O(x^{\chi} log^g x).$$

Here $R(x) = \sum_{\rho} Res \left\{ \frac{x^s Z(s)}{s}, \rho \right\}$, *where the* ρ *are the poles of* $Z(s)$ *such that,* $\chi \leq Re(\rho) \leq \beta$.

A.4.1 Verification of Hypotheses

In this section, we use the results of Theorem A.3 to verify the hypotheses of Landau's theorem. Define $Z(s)$ by

$$Z(s) = \zeta(2s)\zeta_{\overrightarrow{F}}(s). \tag{A.32}$$

$Z(s)$ is the product of the Dirichlet series $\zeta(2s)$ and $\zeta_{\overrightarrow{F}}(s)$ which converge absolutely for $s > \frac{1}{2}$ and $s > 1$ respectively. Therefore Z(s) can be represented by a Dirichlet series

$$Z(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}, \tag{A.33}$$

absolutely convergent for $Re(s) > 1$. The sequence $\{c_n\}$ is defined be (A.33). Next, we calculate the c_n. We have for $\Re s > 1$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{c_n}{n^s} &= \zeta(2s)\zeta_{\overrightarrow{F}}(s) \\ &= \sum_{j=1}^{p} \sum_{n=1}^{\infty} \frac{|a_n(j)|^2}{(m_j n)^{s+k-1}} \\ &= \sum_{j=1}^{p} \frac{1}{m_j^{s+k-1}} \zeta(2s)\zeta_j(s). \end{aligned} \tag{A.34}$$

Now

$$\zeta(2s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \sum_{n=1}^{\infty} \frac{f_n}{n^s} \tag{A.35}$$

where

$$f_n = \begin{cases} 1, & \text{if } n = m^2 \text{ for some m;} \\ 0, & \text{otherwise.} \end{cases} \tag{A.36}$$

Also,

$$\zeta_j(s) = \sum_{n=1}^{\infty} \frac{g_n(j)}{n^s} \tag{A.37}$$

where

$$g_n(j) = \frac{|a_n(j)|^2}{n^{k-1}}. \tag{A.38}$$

Thus if we define $c_n(j)$ by

$$\zeta(2s)\zeta_j(s) = \sum_{n=1}^{\infty} \frac{c_n(j)}{n^s}, \tag{A.39}$$

then $c_n(j)$ is given by the Dirichlet convolution

$$\begin{aligned} c_n(j) &= \sum_{d|n} f_d g_{\frac{n}{d}}(j) \\ &= \sum_{d^2|n} |a_{\frac{n}{d^2}}(j)|^2 \left(\frac{n}{d^2}\right)^{1-k} \\ &= \sum_{d^2 m = n} |a_m(j)|^2 m^{1-k}. \end{aligned} \tag{A.40}$$

Continuing the calculation of c_n we have, by (A.34) and (A.39),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{c_n}{n^s} &= \sum_{j=1}^{p} \frac{1}{m_j^{s+k-1}} \sum_{n=1}^{\infty} \frac{c_n(j)}{n^s} \\ &= \sum_{j=1}^{p} \frac{1}{m_j^{k-1}} \sum_{n=1}^{\infty} \frac{c_n(j)}{(m_j n)^s} \\ &= \sum_{j=1}^{p} \frac{1}{m_j^{k-1}} \sum_{n=1}^{\infty} \frac{\tilde{c}_n(j)}{n^s}. \end{aligned}$$

Here

$$\tilde{c_n}(j) = \begin{cases} c_{\frac{n}{m_j}}(j) & \text{if } m_j|n; \\ 0 & \text{otherwise.} \end{cases} \tag{A.41}$$

Finally, we have

$$\sum_{n=1}^{\infty}\frac{c_n}{n^s}=\sum_{j=1}^{p}\frac{1}{m_j^{k-1}}\sum_{n=1}^{\infty}\frac{\tilde{c}_n(j)}{n^s} \tag{A.42}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{j=1}^{p}\frac{\tilde{c}_n(j)}{m_j^{k-1}}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\substack{j=1\\ m_j|n}}^{p}\frac{c_{\frac{n}{m_j}}(j)}{m_j^{k-1}}.$$

Therefore

$$c_n=\sum_{\substack{j=1\\ m_j|n}}^{p}\frac{c_{\frac{n}{m_j}}(j)}{m_j^{k-1}},$$

where $c_n(j)$ is given in (A.40). We have shown that condition i) is satisfied with $\beta=1$.

For condition ii) we note that $Z(s)$ has a meromorphic continuation, given by (A.29), analytic in the whole plane except for simple poles at $s=1$. The residue at $s=1$ is

$$Res\{Z(s),1\}=\frac{\pi^2}{6}k\alpha. \tag{A.43}$$

Next we want to define $e_n,\lambda_n,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,$ and $,\delta_2$ which appear in conditions iii) and iv). These are determined by the functional equation satisfied by $Z(s)$. (A.27),(A.28) and (A.32) imply

$$\pi^{-s}(\frac{4\pi}{N})^{1-s}\Gamma(s)\Gamma(s+k-1)Z(s)=\pi^{s-1}(\frac{4\pi}{N})^{s}\Gamma(1-s)\Gamma(k-s)Z(1-s). \tag{A.44}$$

We want to put the above in the form

$$\Gamma(\alpha_1+\beta_1 s)\Gamma(\alpha_2+\beta_2 s)Z(s)=\Gamma(\gamma_1-\delta_1 s)\Gamma(\gamma_2-\delta_2 s)\sum_{n=1}^{\infty}e_n\lambda_n^s.$$

$$\Gamma(s)\Gamma(s+k-1)Z(s)=\left(\frac{4\pi^2}{N}\right)^{2s-1}\Gamma(1-s)\Gamma(k-s)Z(1-s). \tag{A.45}$$

Therefore

$$\alpha_1 = 0 \qquad \beta_1 = 1$$
$$\alpha_2 = k-1 \qquad \beta_2 = 1$$
$$\gamma_1 = 1 \qquad \delta_1 = 1$$
$$\gamma_2 = k \qquad \delta_2 = 1.$$

Note the above implies $\eta = 2$. Also, for $\Re s < 0$, Z(1-s) is represented by its Dirichlet series, that is

$$Z(1-s) = \sum_{n=1}^{\infty} \frac{c_n}{n^{1-s}}. \tag{A.46}$$

Therefore

$$\sum_{n=1}^{\infty} e_n \lambda_n^s = \left(\frac{4\pi^2}{N}\right)^{2s-1} Z(1-s) \tag{A.47}$$
$$= \sum_{n=1}^{\infty} \frac{4\pi^2 c_n}{Nn} (\frac{(4\pi^2)^2 n}{N^2})^s.$$

Thus $e_n = \frac{4\pi^2 c_n}{Nn}$ and $\lambda_n = \frac{(4\pi^2)^2 n}{N^2}$. Hence conditions iii) and iv) are satisfied.

For condition v) we estimate $Z(s)$. Let $\sigma_1 < \sigma_2$; we want to show $Z(s) = O(e^{\gamma|t|})$, uniformly in σ for $\sigma_1 \leq \sigma \leq \sigma_2$. By (A.29) we have,

$$Z(s) = \frac{2\pi^s (\frac{4\pi}{N})^{s+k-1}}{\Gamma(s)\Gamma(s+k-1)} \iint_{\mathcal{F}} \mathcal{E}(z,s) y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}. \tag{A.48}$$

Theorem A.4 gives the estimate

$$\mathcal{E}((x+iy),s) = O(y^\sigma) \quad \text{as } y \longrightarrow \infty,$$

where $\sigma = max(Re(s), 1-Re(s))$. Therefore if $\gamma = \max(1-\sigma_1, \sigma_2)$ then

$$\mathcal{E}((x+iy),s) = O(y^\gamma) \quad \text{as } y \longrightarrow \infty, \tag{A.49}$$

uniformly in σ for $\sigma_1 \leq \sigma \leq \sigma_2$. We need the following lemma,

Lemma A.1 *Let (F,ρ) be a unitary cuspidal vector-valued modular form of weight k on $\Gamma(1)$. Then*

$$\iint_{\mathcal{F}} y^{\gamma}\|\overrightarrow{F}(z)\|^2\, dxdy < \infty \quad \textit{for any } \gamma \in R.$$

Proof:

$$\begin{aligned}|F_j(z)| &= |\sum_{n=1}^{\infty} a_n(j) e^{\frac{2\pi i n z}{N_j}}| \\ &\leq \sum_{n=1}^{\infty} |a_n(j)| e^{-\frac{2\pi n y}{N_j}}.\end{aligned}$$

By Proposition A.2, this is

$$\begin{aligned}&\leq C \sum_{n=1}^{\infty} n^{\frac{k}{2}} e^{-\frac{2\pi n y}{N}} \\ &\leq C_k \sum_{n=1}^{\infty} e^{(1-\frac{2\pi y}{N})n}.\end{aligned}$$

For $y > N$, this

$$\begin{aligned}&= C_k \frac{e^{(1-\frac{2\pi y}{N})}}{1 - e^{(1-\frac{2\pi y}{N})}}, \\ &\leq C_k' e^{(1-\frac{2\pi y}{N})} \\ &\leq C_k'' e^{-\frac{\pi y}{N}}.\end{aligned}$$

Lemma A.1 follows.

Therefore we have

$$|Z(s)| = |\frac{2\pi^s(\frac{4\pi}{N})^{s+k-1}}{\Gamma(s)\Gamma(s+k-1)} \iint_{\mathcal{F}} \mathcal{E}(z,s) y^k \|\overrightarrow{F}(z)\|^2 \frac{dxdy}{y^2}| \tag{A.50}$$

$$\leq \frac{C_{\sigma_1,\sigma_2,k}}{|\Gamma(s)\Gamma(s+k-1)|} \iint_{\mathcal{F}} y^{\gamma+k-2} \|\overrightarrow{F}(z)\|^2\, dxdy. \tag{A.51}$$

By the proof of Lemma A.1 this is

$$\leq \frac{C_{\gamma,k}}{|\Gamma(s)\Gamma(s+k-1)|}. \tag{A.52}$$

Now use Stirling's formula:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} \quad \sigma_1 \le \sigma \le \sigma_2 \quad \text{as} \quad |t| \longrightarrow \infty. \tag{A.53}$$

Thus,

$$\begin{aligned} |Z(s)| &\le \frac{C_{\sigma_1,\sigma_2,k}}{|\Gamma(s)\Gamma(s+k-1)|} \\ &\le C \frac{1}{e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}|t|^{\sigma+k-\frac{3}{2}}} \\ &\le C|t|^{2|\sigma_2|-k-2} e^{\pi|t|} \\ &\le Ce^{2\pi|t|}, \quad \text{for } |t| \text{ sufficiently large.} \end{aligned}$$

Finally to verify condition vi), since $\beta = 1$ and $e_n \ge 0$, we estimate $\sum_{\lambda_n \le x} e_n \lambda_n$. We have

$$\sum_{\lambda_n \le x} e_n \lambda_n = \sum_{\frac{(4\pi^2)^2 n}{N^2} \le x} \frac{4\pi^2 c_n}{Nn} \frac{(4\pi^2)^2 n}{N^2}. \tag{A.54}$$

Let $x' = \frac{N^2}{(4\pi^2)^2}$, then

$$\begin{aligned} \sum_{\lambda_n \le x} e_n \lambda_n &= \left(\frac{(4\pi^2)^2}{N^2}\right)^3 \sum_{n \le x'} c_n \qquad \text{(A.55)} \\ &= \left(\frac{(4\pi^2)^2}{N^2}\right)^3 \sum_{n \le x'} \sum_{\substack{j=1 \\ m_j | n}}^{p} \frac{c_{\frac{n}{m_j}}(j)}{m_j^{k-1}} \\ &= \left(\frac{(4\pi^2)^2}{N^2}\right)^3 \sum_{n \le x'} \sum_{\substack{j=1 \\ m_j | n}}^{p} \frac{c_{\frac{n}{m_j}}(j)}{m_j^{k}} m_j. \end{aligned}$$

Since $k > 0$ and $1 \leq m_j \leq N$, this is

$$\leq \frac{(4\pi^2)^3}{N^2} \sum_{n \leq x'} \sum_{\substack{j=1 \\ m_j | n}}^{p} c_{\frac{n}{m_j}}(j)$$

$$= \frac{(4\pi^2)^3}{N^2} \sum_{n \leq x'} \sum_{\substack{j=1 \\ m_j d = n}}^{p} c_d(j)$$

$$= \frac{(4\pi^2)^3}{N^2} \sum_{j=1}^{p} \sum_{1 \leq d \leq \frac{x'}{m_j}} c_d(j)$$

$$= \frac{(4\pi^2)^3}{N^2} \sum_{j=1}^{p} O(\frac{x'}{m_j}) \quad \text{by Proposition A.5.}$$

Condition vi) follows with $A = 0$.

Now $\eta = 2$ and $\beta = 1$; therefore $\chi = \beta\frac{2\eta-1}{2\eta+1} = \frac{3}{5}$. $Z(s)$ has a simple pole at $s = 1$, so $p = 1$. Thus $g = max(A, p-1) = 0$, and Landau's theorem gives:

$$B(x) = \sum_{n \leq x} c_n = R(x) + O(x^{\frac{3}{5}}), \tag{A.56}$$

where $R(x) = \sum_\rho Res\left\{\frac{x^s Z(s)}{s}, \rho\right\}$ and ρ is a pole of $Z(s)$, $\frac{3}{5} < \rho \leq 1$. By the comments above, $s = 1$ is the only pole of $Z(s)$; the residue at $s = 1$ is given by (A.43). Therefore

$$R(x) = Res\{\frac{x^s Z(s)}{s}, 1\} = \frac{\pi^2}{6} k\alpha x.$$

Therefore we have the asymptotic estimate

$$\sum_{n \leq x} c_n = \sum_{n \leq x} \sum_{\substack{j=1 \\ m_j | n}}^{p} \frac{c_{\frac{n}{m_j}}(j)}{m_j^{k-1}} = \frac{\pi^2}{6} k\alpha x + O(x^{\frac{3}{5}}).$$

A.4.2 Proof that $a_n(j) = O(n^{\frac{k}{2}-\frac{1}{5}})$

We prove $a_n(j) = O(n^{\frac{k}{2}-\frac{1}{5}})$ in two steps. First we introduce auxiliary $\{a_n\}$ and prove these a_n satisfy $a_n = O(n^{\frac{k}{2}-\frac{1}{5}})$.Second we relate the $a_n(j)$ to the a_n

and deduce the estimate for the $a_n(j)$ from this relationship. We define a_n by the equation

$$|a_n|^2 = \sum_{d^2|n} b_{\frac{n}{d^2}} \mu(d) d^{2k-2}.$$

where

$$b_n = c_n n^{1-k}. \tag{A.57}$$

Then, following Rankin [16], we deduce the estimate $a_n = O(n^{\frac{k}{2}-\frac{1}{5}})$ from the estimate

$$\sum_{n\leq x} c_n = \sum_{n\leq x} b_n n^{1-k} = \frac{\pi^2}{6} k\alpha x + O(x^{\frac{3}{5}}).$$

Note the following proof only holds for $k \geq \frac{2}{5}$.

Proposition A.6 $\sum_{n\leq x} b_n = \frac{\pi^2}{6}\alpha x^k + O(x^{k-\frac{2}{5}})$

Proof:

$$\sum_{n\leq x} b_n = \sum_{n\leq x} b_n n^{1-k} n^{k-1}.$$

Apply Theorem A.2 (b) with $g_n = b_n n^{1-k}$ and $f(x) = x^{k-1}$. Therefore

$$\begin{aligned}
\sum_{n\leq x} b_n &= \sum_{n\leq x} b_n n^{1-k} n^{k-1} \qquad \text{(A.58)} \\
&= \int_1^x \sum_{n\leq y} b_n n^{1-k} (k-1) y^{k-2}\, dy + \left(\sum_{n\leq x} b_n n^{1-k} \right) x^{k-1} \\
&= -(k-1) \int_1^x \left(\frac{\pi^2}{6} k\alpha y + O(y^{\frac{3}{5}}) \right) y^{k-2}\, dy + \left(\frac{\pi^2}{6} k\alpha x + O(x^{\frac{3}{5}}) \right) x^{k-1} \\
&= -(k-1) \frac{\pi^2}{6} k\alpha \frac{y^k}{k}\Big|_1^x + O(x^{k-\frac{2}{5}}) + \frac{\pi^2}{6} k\alpha x^k + O(x^{k-\frac{2}{5}}) \\
&= \frac{\pi^2}{6}\alpha x^k + O(1) + O(x^{k-\frac{2}{5}}) \\
&= \frac{\pi^2}{6}\alpha x^k + O(x^{k-\frac{2}{5}}) \quad \text{for } k \geq \frac{2}{5}.
\end{aligned}$$

Proposition A.7 $\sum_{n\leq x} |a_n|^2 = \alpha x^k + O(x^{k-\frac{2}{5}})$.

Proof: We have

$$|a_n|^2 = \sum_{d^2|n} b(\frac{n}{d^2})\mu(d)d^{2k-2}.$$

Therefore,

$$\begin{aligned}
\sum_{n\leq x}|a_n|^2 &= \sum_{n\leq x}\sum_{d^2|n} b(\frac{n}{d^2})\mu(d)d^{2k-2} \\
&= \sum_{d\leq\sqrt{x}} \mu(d)d^{2k-2} \sum_{m\leq\frac{x}{d^2}} b(m) \\
&= \sum_{d\leq\sqrt{x}} \mu(d)d^{2k-2}\left\{\frac{\pi^2}{6}\alpha(\frac{x}{d^2})^k + O((\frac{x}{d^2})^{k-\frac{2}{5}})\right\} \\
&= \sum_{d\leq\sqrt{x}} \mu(d)\left\{\frac{\pi^2}{6}\alpha x^k d^{-2} + O(x^{k-\frac{2}{5}}d^{-\frac{6}{5}})\right\} \\
&= \frac{\pi^2}{6}\alpha\left(\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2}x^k + \sum_{d\leq\sqrt{x}}\mu(d)O(x^{k-\frac{2}{5}}d^{-\frac{6}{5}})\right).
\end{aligned}$$

Now,

$$\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O(\frac{1}{\sqrt{x}}) \quad [1,\ \text{p61}].$$

Therefore,

$$\sum_{n\leq x}|a_n|^2 = \alpha x^k + O(x^{k-\frac{1}{2}}) + \sum_{d\leq\sqrt{x}}\mu(d)O(x^{k-\frac{2}{5}}d^{-\frac{6}{5}})$$

and

$$|\sum_{d\leq\sqrt{x}}\mu(d)O(x^{k-\frac{2}{5}}d^{-\frac{6}{5}})| \leq Cx^{k-\frac{2}{5}}\sum_{d=1}^{\infty}\frac{1}{d^{\frac{6}{5}}} = Cx^{k-\frac{2}{5}}.$$

Therefore,

$$\begin{aligned}
\sum_{n\leq x}|a_n|^2 &= \alpha x + O(x^{k-\frac{1}{2}}) + O(x^{k-\frac{2}{5}}) \qquad \text{(A.59)} \\
&= \alpha x + O(x^{k-\frac{2}{5}}).
\end{aligned}$$

Finally, we have

$$|a_n|^2 = \sum_{j=1}^{n} |a_j|^2 - \sum_{j=1}^{n-1} |a_j|^2 \tag{A.60}$$
$$= \alpha(n^k - (n-1)^k) + O(n^{k-\frac{2}{5}})$$
$$= O(n^{k-1})(n^{k-\frac{2}{5}})$$
$$= O(n^{k-\frac{2}{5}}).$$

Therefore,

$$a_n = O(n^{\frac{k}{2}-\frac{1}{5}}).$$

Next we relate the $a_n(j)$ to the a_n. We have

$$c_n = \sum_{\substack{j=1 \\ m_j|n}}^{p} \frac{c_{\frac{n}{m_j}}(j)}{m_j^{k-1}}$$

and

$$b_n = c_n n^{k-1}, \tag{A.61}$$

therefore,

$$b_n = \sum_{\substack{j=1 \\ m_j|n}}^{p} b_{\frac{n}{m_j}}(j).$$

Therefore,

$$
\begin{aligned}
|a_n|^2 &= \sum_{d^2|n} b_{\frac{n}{d^2}} \mu(d) d^{2k-2} \qquad \text{(A.62)} \\
&= \sum_{d^2|n} \sum_{\substack{j=1 \\ m_j|\frac{n}{d^2}}}^{p} b_{\frac{n}{m_j d^2}}(j) \mu(d) d^{2k-2} \\
&= \sum_{\substack{j=1 \\ m_j|n}}^{p} \sum_{d^2|\frac{n}{m_j}} b_{\frac{n}{m_j d^2}}(j) \mu(d) d^{2k-2} \\
&= \sum_{\substack{j=1 \\ m_j|n}}^{p} |a_{\frac{n}{m_j}}(j)|^2.
\end{aligned}
$$

Finally for the estimate

$$a_n(j) = O(n^{\frac{k}{2}-\frac{1}{5}}),$$

we note that

$$|a_n(j)|^2 \leq \sum_{\substack{l=1 \\ m_l|nm_j}}^{p} |a_{\frac{nm_j}{m_l}}(l)|^2 = |a_{nm_j}|^2, \qquad \text{(A.63)}$$

by (A.62). Therefore,

$$|a_n(j)|^2 = O((nm_j)^{k-\frac{2}{5}}) = O(n^{k-\frac{2}{5}}). \qquad \text{(A.64)}$$