THE COHEN-MACAULAY PROPERTY
OF MULTIPLICATIVE INVARIANTS

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ABSTRACT

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In this dissertation we study the following general problem:

*Suppose S is a Cohen-Macaulay ring and G is a finite group acting by automorphisms on S. When is the subring of G-invariants $S^G$ again Cohen-Macaulay?*

The Cohen-Macaulay condition is one of the central notions in commutative algebra; it will be discussed in detail in Chapter 2 below. Our main focus in this thesis is on the case where $S = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial algebra and $G$ acts by $k$-algebra automorphisms on $S$ in such a way that each variable $x_i$ is sent to a monomial $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ depending on $i$ and the acting group element from $G$. Actions of this type are known as multiplicative actions. (Occasionally, they are also called “purely monomial” or “exponential” actions; see, e.g., [Bou2].) Article [LP] initiated the investigation of the
Cohen-Macaulay property of multiplicative invariants. In Chapters 3 and 4, we give an account of [LP] with some simplifications. In Chapter 5, we use this material and direct computations to determine all multiplicative invariants in dimension 3 that are Cohen-Macaulay, with the exception of two cases that remain open. This classification is our main contribution to the Cohen-Macaulay problem for invariant rings.
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To My Parents
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CHAPTER 1

Introduction

When a finite group $G$ acts on a commutative Noetherian ring $S$, many properties of $S$ are inherited by the ring of invariant elements

$$S^G = \{ s \in S \mid g(s) = s \text{ for all } g \in G \} .$$

For example, if $S$ is a normal domain, then so is $S^G$. In this dissertation we try to understand to what extent the Cohen-Macaulay property passes from $S$ to $S^G$.

In the classical literature of invariant theory, $S$ is a polynomial algebra $k[x_1, \ldots, x_n]$ over a field $k$ and the group $G$ acts by sending each variable $x_i$ to a $k$-linear combination of all variables. This type of action is known as a linear action; these actions provide the setting of the ground-breaking works of D. Hilbert and E. Noether. A crucial feature of linear actions is that $S = k[x_1, \ldots, x_n]$ is equipped with a non-trivial grading (by “total degree”)
which is preserved under the $G$-action. This need not be the case for arbitrary

group actions. One such instance is when $G$ acts multiplicatively. For these

actions, $S$ is a Laurent polynomial algebra $k[x_1^\pm, \ldots, x_n^\pm]$ (which is Cohen-

Macaulay) and the elements of $G$ act by sending each variable $x_i$ to a monomial

$x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}$ depending on $i$ and the acting group element from $G$; see, e.g.,

Lorenz [Lo1, Lo2, Lo3] and Farkas [F1, F2, F3].

The definition of Cohen-Macaulay rings is rather technical. It involves
two constants, the height and the depth, each defined for an arbitrary ideal

$a$ of a commutative Noetherian ring $S$. The notion of height is essentially
topological; it is related to the dimension of the set

$$V(I) = \{ p \mid p \text{ is a prime ideal of } S \text{ with } p \supseteq a \},$$

endowed with the so-called Zariski topology. On the other hand, depth is

purely algebraic and measures the length of a maximal regular sequence con-
tained in $a$; for detailed definitions, see Chapter 2 below. Depth is always

bounded above by height. If equality holds for all ideals $a$, then the ring $S$ is
called Cohen-Macaulay. In 1916 Macaulay [M] showed that polynomial alge-
bras over a field are Cohen-Macaulay. Later in 1945, Cohen [Co] generalized
this result and proved that regular local rings are Cohen-Macaulay. Subse-
sequently, it turned out that Cohen-Macaulay rings form a rather wide class of
commutative Noetherian rings which nevertheless enjoys some very desirable
properties. The equality of an algebraic and a topological constant, as pos-
tulated in the definition of Cohen-Macaulay rings, allows in particular for a powerful dimension theory.

Returning to rings of invariants, if the order of $G$ is invertible in $S$ and $S$ is Cohen-Macaulay then $S^G$ is easily seen to be Cohen-Macaulay as well. Also, $\text{height}(\mathfrak{a}) = \text{height}(\mathfrak{a} \cap S^G)$ holds in this case, for any ideal $\mathfrak{a}$ in $S$. Thus, the Cohen-Macaulay property of invariant rings $S^G$ is problematic only when the order of $G$ is not invertible in $S$; this is called the modular case. Some depth estimates for modular actions were found by Ellingsrud and Skjebred [ES] (when $G$ is a $p$-group) and by G. Kemper [Ke].

The main problem addressed in this thesis, stated specifically, is as follows:

Suppose $S = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial algebra over a field $k$ of characteristic $p$ and $G$ is a finite group acting multiplicatively on $S$ such that $p$ divides the order $|G|$. When is the invariant algebra $S^G$ Cohen-Macaulay?

The dissertation is divided into 4 chapters. In Chapter 2, we give a brief survey of background material from commutative algebra and homological algebra. In Chapter 3, we define trace modules and obtain a height formula. The main result here are certain depth estimates; see in particular Lemma 3.4.2. Using this result, we give an alternative proof of some of Kemper’s results in Section 3.5. Chapter 4 is devoted specifically to multiplicative actions. Lemma 4.1.3 reduces the Cohen-Macaulay problem for multiplicative invariants to the special case of effective actions. This Lemma is as yet unpublished and is due to Lorenz. The material in Chapters 3 and 4 is a simplified version.
of [LP]. In Chapter 5 we have classified the multiplicative invariant rings of dimension 3 in Cohen-Macaulay and non-Cohen-Macaulay classes. We use the classification list from [T] and use the notation $W_i(n)$ for the group $W_i$ on the page $n$ in the following results:

If $\text{char } k \neq 2$ then all multiplicative invariants $k[A]^G$ in dimension 3 are Cohen-Macaulay. If $\text{char } k = 2$, then, with exception of the groups conjugate to one of the following

- $W_5(173)$ (order 2),
- $W_2(174)$ and $W_4(174)$ (both cyclic of order 4),
- $W_3(184)$ and $W_4(184)$ (both cyclic of order 6),

and the possible exception of the conjugates of

$$W_{14}(174) \simeq C_2 \times C_2 \quad \text{and} \quad W_{10}(191) \simeq A_4$$

all finite subgroups of $\text{GL}_3(\mathbb{Z})$ have Cohen-Macaulay multiplicative invariant algebras.
CHAPTER 2

Basic Concepts

In this chapter, we will develop the basic concepts and set up notations. Our main focus will be on the Cohen-Macaulay property of a module of invariant elements under the action of a finite group. Section 2.1 contains the necessary background from commutative algebra. In Section 2.2, we recall the definition of Cohen-Macaulay modules over a commutative Noetherian ring using the notion of depth and height. In Section 2.3, we consider actions of a finite group $G$ on an arbitrary commutative Noetherian ring $S$ and study the height and depth of modules over the associated skew group ring $S*G$. In Section 2.4, we also recall the fundamentals of spectral sequences, an important tool to be used in later chapters.
2.1 Background from commutative algebra

Throughout we assume that $S$ is a commutative Noetherian ring with identity and $M$ is a finitely generated unitary left $S$-module. Most of the material of this section can be found in [Bou1] and in [E]. We also follow terminology and notations used in these references, and make an explicit mention if any change is made.

**Definition 2.1.1.** Let $R$ be a subring of $S$. An element $s \in S$ is called integral over $R$ if $s$ satisfies a monic polynomial with coefficients in $R$. The set of all elements of $S$ that are integral over $R$ forms a subring $\bar{R}$ of $S$ which is called the integral closure of $R$ in $S$. If $S = \bar{R}$ then we say that $R \hookrightarrow S$ is an integral extension. On the other hand, if $R = \bar{R}$ then we say that $R$ is integrally closed in $S$. A normal domain is an integral domain which is integrally closed in its own quotient field.

If $S$ is finitely generated $R$-algebra which is integral over $R$, then it is easy to show that $S$ is a finitely generated module over $R$. The converse is also true, i.e. if $S$ is a finite $R$-module then the extension is integral. The following lemma is a collection of standard results on integral extensions.

**Lemma 2.1.2.** Suppose $R \hookrightarrow S$ is integral extension. Then the following statements hold:

(a) (Lying over) Given a prime $\mathfrak{p} \subset R$, there exist a prime ideal $\mathfrak{p} \subset S$ such
that \( \mathfrak{P} \cap R = p \). [Bou1, ch V.2.2, Th.1, p. 328].

(b) (Incomparability) If \( \mathfrak{P} \subseteq \Omega \) are two prime ideals of \( S \) lying over a prime \( p \) of \( R \), then \( \mathfrak{P} = \Omega \) [E, cor. 4.18 on p. 131].

(c) (Going up) Suppose \( b \) is an ideal of \( S \) and \( a = b \cap R \). Then for any prime ideal \( p \subset R \) containing \( a \), there exists a prime ideal \( \mathfrak{P} \subset S \) lying over \( p \) which contains \( b \). [Bou1, ch. V.2.1, cor. 2, p. 328].

(d) (Going down) If, in addition, \( S \) is an integral domain and \( R \) is normal, then given prime ideals \( p \subset q \) in \( R \) and a prime \( \Omega \) in \( S \) lying over \( q \), there exists a prime \( \mathfrak{P} \subset \Omega \) such that \( \mathfrak{P} \cap R = p \). [E, Theorem 13.9, p. 204].

The annihilator of \( M \) is an ideal in \( S \) which is denoted by \( \text{ann}_S(M) \). So \( \text{ann}_S(M) = \{ a \in S | aM = 0 \} \). If \( R \) is a subring of \( S \), then any \( S \)-module is also an \( R \)-module. It is clear that \( \text{ann}_R(M) = \text{ann}_S(M) \cap R \). The following lemma is an easy consequence of the Cayley-Hamilton theorem [E, Theorem 4.3.7]. We give a quick self-contained proof.

**Lemma 2.1.3.** For any ideal \( a \) of \( S \), \( a + \text{ann}_S(M) = S \) iff \( aM = M \).

**Proof.** If \( a + \text{ann}_S(M) = S \), then \( M = SM = aM + \text{ann}_S(M)M = aM \). To prove the converse, suppose \( aM = M \). Choose a set of generators \( m_1, \ldots, m_n \) of \( M \). Now for each \( j \), we have \( m_j = \sum_{i=1}^{n} \lambda_{ij} m_i \), \( \lambda_{ij} \in a \). Therefore
\[(1 - \lambda_{nn})m_n = \sum_{i=1}^{n-1} \lambda_{in}m_i.\] Thus \((1 - \lambda_{nn})M\) has less than \(n\) generators. Inductively we can find \(a \in \mathfrak{a}\) such that \((1-a)M = 0\). Hence \(1-a \in \text{ann}_S(M)\). Therefore \(\mathfrak{a} + \text{ann}_S(M) = S.\)

**Definition 2.1.4.** The height of a prime ideal \(\mathfrak{p}\) of \(S\) is the maximal integer \(r\) such that there exist a strictly ascending chain of prime ideals \(\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}\). For an arbitrary ideal \(\mathfrak{a}\), \(\text{height}(\mathfrak{a}) = \inf\{\text{height}(\mathfrak{p}) \mid \mathfrak{p}\text{ is a prime containing }\mathfrak{a}\}\). (If \(\mathfrak{a} = S\) then \(\text{height}(\mathfrak{a}) = \infty\).) The height of an ideal \(\mathfrak{a}\) in a module \(M\) is denoted by \(\text{height}(\mathfrak{a}, M)\); it is defined to be the height of the ideal \((\mathfrak{a} + \text{ann}_S(M))/\text{ann}_S(M)\) in the ring \(S/\text{ann}_S(M)\). Thus, by Lemma 2.1.3, if \(\mathfrak{a}M = M\) then \(\text{height}(\mathfrak{a}, M) = \infty\).

**Definition 2.1.5.** A prime ideal \(\mathfrak{p}\) of \(S\) is called an associated prime of the module \(M\) if \(\mathfrak{p} = \text{ann}_S(m)\) for some nonzero element \(m \in M\).

The set of associated primes of \(M\) is denoted by \(\text{Ass}(M)\). When we are dealing with more than one ring, we will employ \(\text{Ass}_S(M)\) instead. In the following proposition we record two important properties of \(\text{Ass}(M)\); the proof can be found in [E, Theorem 3.1]. We will denote the union of the associated primes of \(M\) by \(\bigcup \text{Ass}(M)\).

**Proposition 2.1.6.** Let \(M\) be a finitely generated module over a Noetherian ring, then \(\text{Ass}(M)\) is a finite set and \(\bigcup \text{Ass}(M) = \{x \in S \mid xM = 0 \text{ for some } m \in M, m \neq 0\}\).
The elements of $\bigcup \text{Ass}(M)$ are called the zero divisors on $M$. We end this section with a lemma from [E, Lemma 3.3, p. 90] known as the “prime avoidance lemma”.

**Lemma 2.1.7.** Suppose $a_1, a_2, \ldots, a_n, b$ are ideals of a commutative ring $S$ and suppose that $b \subseteq \bigcup_j a_j$. If at most two of the $a_j$ are not prime, then $b$ is contained in one of the $a_j$.

### 2.2 Cohen-Macaulay modules

The material in this section is partly from [BH]. We continue to assume that $S$ is a commutative Noetherian ring and $M$ is a finitely generated left $S$-module.

**Definition 2.2.1.** An element $x \in S$ is called $M$-regular if $x$ is not a zero divisor on $M$, that is, for any nonzero element $m$ of $M$, $xm \neq 0$. A sequence $\{x\} = \{x_1, x_2, \ldots, x_r\}$ is called $M$-regular sequence of length $r$, if

(i) $\sum_{i=1}^{r} x_i M \neq M$.

(ii) $x_i$ is $M/(\sum_{j=1}^{i-1} x_j M)$-regular for $1 \leq i \leq r$, where the empty sum denotes the 0-module.

A maximal $M$-regular sequence in an ideal $a$ is an $M$-regular sequence $\{x\} = \{x_1, x_2, ..., x_r\}$ that is a subset of $a$ and such that $\{x_1, x_2, \ldots, x_r, x_{r+1}\}$ is not
an $M$-regular sequence, for any $x_{r+1} \in a$. For convenience, we will denote the submodule $\sum_{i=1}^{n} x_i M$ by $(\overline{x})M$.

**Remark 1.** If $\{x\} = \{x_1, \ldots, x_r\}$ is $M$-regular, then so is $\{x^n\} = \{x_1^n, \ldots, x_r^n\}$ for any positive integer $n$; see [BH, Exercise 1.1.10(b)]. Many other interesting properties of regular sequences can be found in [BH]. We make one more useful remark about maximal $M$-regular sequences. Suppose $\{x\}$ is a maximal $M$-regular sequence in an ideal $a$. Then for any element $a$ of $a$, there exists a nonzero element $\bar{m} \in M/(\overline{x})M$ such that $a\bar{m} = 0$. In other words, $a$ consists of zero divisors of $M/(\overline{x})M$. Applying Proposition 2.1.6, we get $a \subset \cup \text{Ass}(M/(\overline{x})M)$. Hence, by the Lemma 2.1.7, we conclude that $a \subset \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(M/(\overline{x})M)$.

**Definition 2.2.2.** Let $a$ be any ideal of $S$. If $aM \neq M$, then we define the depth of $a$ on $M$ to be $\sup\{ r \mid a$ contains an $M$-regular sequence of length $r \}$. If $aM = M$ then we say that the depth of $a$ on $M$ is infinite. The depth of $a$ on $M$ is denoted by $\text{depth}(a, M)$.

In literature the depth is also called the grade of an ideal, e.g. see [BH]. Depth has a homological characterization in terms of “Ext”, which we state in part (b) of the following theorem. For the proof of (b), see [BH, Theorem 1.2.5]; for (a) and (c), see [BH, Exercise 1.2.22(a)] and [BH, p. 10], respectively.

**Theorem 2.2.3.** (a) For any finitely generated $S$-module $M$ and an ideal $a$
of $S$, $\text{depth}(a, M) \leq \text{height}(a, M)$.

(b) Suppose that $aM \neq M$. Then the length of any maximal $M$-regular sequence in $a$ depends on $a$ and $M$ only and is equal to the number $n$ such that $\text{Ext}^i_S(S/a, M) = 0$ for $i < n$ and $\text{Ext}^n_S(S/a, M) \neq 0$.

(c) If $aM = M$ then $\text{Ext}^i_S(S/a, M) = 0$ for all $i$.

The characterization in (b) shows that all maximal $M$-regular sequences in a fixed ideal have same length. Parts (b) and (c) also give the formula

$$\text{depth}(a, M) = \inf \{i \mid \text{Ext}^i_S(S/a, M) \neq 0\},$$

where $\inf \emptyset = \infty$. In view of the beginning of Remark 1, it is clear that $\text{depth}(a, M) = \text{depth}(\sqrt{a}, M)$. Even more is true:

**Lemma 2.2.4.** For any ideal $a$ of $S$, there exists a prime ideal $p$ containing $a$ such that $\text{depth}(a, M) = \text{depth}(p, M)$

**Proof.** If $aM = M$, then any prime ideal containing $a$ would serve. Assume therefore that $\text{depth}(a, M)$ is finite, say $\text{depth}(a, M) = n$. Let $\{x_1, \ldots, x_n\}$ be a maximal $M$-regular sequence in $a$. By Remark 1, there exists a prime $p \in \text{Ass}_S(\bar{M})$ such that $a \subseteq p$, where $\bar{M} = M/(x)M$. By definition of associated primes, $p = \text{ann}_S(\bar{m})$ for some nonzero element $\bar{m}$ of $\bar{M}$. But then $x_1, \ldots, x_n$ is a maximum $M$-regular sequence in $p$. Thus $\text{depth}(a, M) = \text{depth}(p, M)$. \qed
Recall that always \( \text{depth}(a, M) \leq \text{height}(a, M) \). The following definition stresses on the equality of these two constants.

**Definition 2.2.5.** The module \( M \) is called Cohen-Macaulay if \( \text{depth}(a, M) = \text{height}(a, M) \) holds for all the ideals \( a \) of \( S \). The ring \( S \) is Cohen-Macaulay if \( S \) is Cohen-Macaulay as a module over itself.

The following lemma states that the Cohen-Macaulay is a local property is local.

**Lemma 2.2.6.** The following are equivalent for the \( S \)-module \( M \):

(i) \( M \) is Cohen-Macaulay;

(ii) \( \text{depth}(m, M) = \text{height}(m, M) \) holds for all maximal ideals \( m \) of \( S \) containing \( \text{ann}_S(M) \);

(iii) all localizations \( M_m \) of \( M \) at the maximal ideals \( m \) of \( S \) are Cohen-Macaulay as \( S_m \)-modules.

**Proof.** The implication (i) \( \Rightarrow \) (iii) is contained in \([BH, \text{Theorem } 2.1.3(b)]\), and (iii) \( \Rightarrow \) (ii) follows from the equalities \( \text{height}(m, M) = \text{height}(mS_m, M_m) \) and \( \text{depth}(m, M) = \text{depth}(mS_m, M_m) \); see \([BH, \text{Proposition } 1.2.10(a)]\). Finally, for (ii) \( \Rightarrow \) (i), see \([E, \text{Ex. } 18.4]\). \( \square \)

We now turn to the important special case where \( S \) is an affine domain over a field \( k \). Then, by Noether normalization \([E, \text{Theorem A1 on p. } 221]\), there
exists a subset \( \{x_1, \ldots, x_n \} \) of \( S \) which is algebraically independent over \( k \) and so that \( k[x_1, \ldots, x_n] \hookrightarrow S \) is a finite extension. The following proposition gives what is probably the most accessible description of affine Cohen-Macaulay domains. The freeness property described therein has many important applications of Cohen-Macaulay domains, e.g. to generating functions in combinatorics.

**Proposition 2.2.7.** Let \( S \) be any affine domain over a field \( k \). Then \( S \) is Cohen-Macaulay if and only if \( S \) is a finite free module over some polynomial subalgebra \( k[x] = k[x_1, \ldots, x_n] \subseteq S \). In this case, \( S \) is free over any polynomial subalgebra \( k[x] \subseteq S \) such that \( S \) is finite over \( k[x] \).

*Proof.* Suppose \( S \) is any affine domain over a field \( k \). Let \( k[x] \subseteq S \) be any polynomial subalgebra such that \( S \) is finite over \( k[x] \). By [Lo\textsuperscript{3}, Lemma in Section 1.3], \( S \) is a projective module over \( k[x] \), and hence \( S \) is free over \( k[x] \), by the Quillen-Suslin theorem. Conversely, suppose that \( S \) is finite and free over some polynomial subalgebra \( k[x] \). Then \( S \) is faithfully flat over \( k[x] \), and \( k[x] \) is Cohen-Macaulay. Moreover, for all primes \( p \) of \( S \), the fibre \( S_p/(p \cap k[x])S_p \) has Krull dimension 0, by Lemma 2.1.2(b), and hence it is Cohen-Macaulay. Therefore, [BH, Exercise 2.1.23] implies that \( S \) is Cohen-Macaulay as well. \( \square \)
2.3 Group actions

The main background references for this section are [Bou1] and [Be1]. Let $S$ be any commutative ring. The set $\text{Aut}(S)$ of all the ring automorphisms of $S$ forms a group under composition.

**Definition 2.3.1.** We say that a finite group $G$ acts on $S$ if there is a group homomorphism $\varphi : G \to \text{Aut}(S)$. If this homomorphism is injective then the $G$-action on $S$ is called **faithful**. We will write $\varphi(g)$ for image of an element $s \in S$ under the action of a group element $g \in G$.

Classical invariant theory is primarily concerned with a specific form of group action on polynomial algebras $k[x] = k[x_1, \ldots, x_n]$, called “linear action”; see the Introduction. Reference [Sm1], for example, gives a detailed treatment of linear actions. There is another type of action which is our prime interest, called “multiplicative action”. These actions have been briefly described in the Introduction and will be formally introduced in Chapter 4. Here is a simple example comparing a linear action and a multiplicative action in rank $2$.

**Example 2.3.2.** Consider the linear action of

$$G = \{ Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}$$

on the polynomial algebra $k[x_1, x_2]$ over a field $k$ of characteristic $\neq 2$ defined
as follows: \(-Id(x_1) = -x_1\) and \(-Id(x_2) = -x_2\). Thus \(-Id(f(x_1, x_2)) = f(-x_1, -x_2)\). Note that the elements such \(x_1^2, x_1x_2\) and \(x_2^2\) are invariant (fixed) under the action of \(G\). In fact, it is easy to see that these elements generate the algebra of invariants \(k[x_1, x_2]^G\). Observe that, under this action, the degree of a polynomial is unchanged.

Now we consider a multiplicative action of the same group on the Laurent polynomial algebra \(k[x_1^\pm 1, x_2^\pm 1]\). This action is defined by \(-Id(x_1) = x_1^{-1}\) and \(-Id(x_2) = x_2^{-1}\). Note that this action does not preserve degree. It is not hard to show that the invariant algebra \(k[x_1^\pm 1, x_2^\pm 1]^G\) is generated by the elements \(x_1 + x_1^{-1}, x_2 + x_2^{-1}\) and \(x_1x_2^{-1} + x_1^{-1}x_2\) (in any characteristic).

Now assume that \(G\) is a finite group acting on the commutative ring \(S\). The skew group ring \(S \ast G\) that is associated with this action is defined as follows: \(S \ast G\) is the free left \(S\)-module with basis \(G\). Thus, every element of \(S \ast G\) can be uniquely written in the form

\[
\sum_{i=1}^{m} a_i g_i \quad \text{with} \quad a_i \in S, \ g_i \in G.
\]

The product in \(S \ast G\) is defined by distributive extension of the rule \((ag)(bh) = ag(b)gh\). The following computation shows that \(S \ast G\) is an associative ring:

\[
((ag)(bh))ck = (ag(b)gh)(ck) = ag(b)gh(c)ghk = ag(bh)(ch) = ag((bh)(ck)),
\]

where \(g, h, k \in G\) and \(a, b, c \in S\). If \(G\) acts trivially on \(S\), then \(S \ast G\) is the ordinary group ring and is denoted by \(S[G]\). Now consider any left \(S \ast G\)-module
$M$ and define the module of invariant elements by

$$M^G = \{ m \in M \mid g(m) = m, \forall g \in G \}.$$

The ring of invariant elements $S^G$ is defined likewise; indeed, $S$ is a left $S \ast G$-module via $(ag)s = ag(s)$ for $a, s \in S$, $g \in G$. Elements of $S$ can be viewed as elements of $S \ast G$ by the identification $s = s1$, where $1$ is the identity element of $G$. With this in mind, let $a \in S^G$. Then for a generator $sg \in S \ast G$, $(sg)(a) = sg(a)(g1) = sag = a(sg)$. This shows that $S^G$ is in the center of $S \ast G$, or $S \ast G$ is an $S^G$-algebra.

Clearly we have $S^G \hookrightarrow S$. In the following well-known lemma we show that this extension is integral. Once we have established this, we can apply lemma 2.1.2 to obtain properties such as lying over, going up etc. In fact, more is true in the present setting.

**Lemma 2.3.3.** Let $G$ be a finite group acting on $S$.

(a) $S^G \hookrightarrow S$ is an integral extension. Further, if $S$ is normal domain, then so is $S^G$.

(b) If $p$ and $q$ are two prime ideals of $S$ such that $p \cap S^G = q \cap S^G$, then there exist $g$ in $G$ such that $p = g(q)$. In other words, $G$ acts transitively on the set of prime ideals lying over a fixed prime ideal of $S^G$.

(c) (Going down) Suppose $p \subseteq q$ are prime ideals of $S^G$ and suppose $\Omega$ is
a prime ideal in $S$ lying over $\mathfrak{q}$. Then there exists a prime ideal $\mathfrak{P}$ in $S$
lying over $\mathfrak{p}$ such that $\mathfrak{P} \subset \mathfrak{Q}$.

Proof. (a) For any $s \in S$, $f(x) = \prod_{g \in G}(x - g(s))$ is a monic polynomial with
coefficients in $S^G$ that is satisfied by $s$, which shows that $S^G \hookrightarrow S$ is integral.
For the second assertion, let $\alpha$ be an element of the quotient field of $S^G$ which
is integral over $S^G$. Then there exist $a, b \in S^G$ such that $\alpha = a/b$. Since
$S^G \subset S$, $\alpha$ is integral over $S$. But $S$ is a normal domain, therefore, $\alpha \in S$.
Now $a$ and $b$ are invariant under $G$. Therefore $\alpha$ is invariant and hence belongs
to $S^G$.

(b) Let $a \in \mathfrak{p}$. Then $\prod_{g \in G} g(a)$ is invariant under $G$ and also contained
in $\mathfrak{p}$. Therefore $\prod_{g \in G} g(a) \in \mathfrak{p} \cap S^G = \mathfrak{q} \cap S^G \subset \mathfrak{q}$. Now $\mathfrak{q}$ is a prime ideal,
and so there exists $g \in G$ such that $g(a) \in \mathfrak{q}$. Thus $a \in g^{-1}(\mathfrak{q})$. Therefore,
$\mathfrak{p} \subset \cup_{g \in G} g(\mathfrak{q})$. By Lemma 2.1.7, we conclude that $\mathfrak{p} \subseteq g(\mathfrak{q})$ for some $g \in G$.
Since $\mathfrak{p} \cap S^G = \mathfrak{q} \cap S^G = g(\mathfrak{q}) \cap S^G$, Lemma 2.1.2(b) yields $\mathfrak{p} = g(\mathfrak{q})$.

(c) To prove this part, let $\mathfrak{P}'$ be any lift of $\mathfrak{p}$ in $S$. By going up part of
Lemma 2.1.2, there exists a prime ideal $\mathfrak{Q}'$ in $S$ containing $\mathfrak{P}'$ which lies over
$\mathfrak{q}$. Then $\mathfrak{Q}$ and $\mathfrak{Q}'$ are both lifts of $\mathfrak{q}$. Therefore, by part (b), there exists
$g \in G$ such that $g(\mathfrak{Q}') = \mathfrak{Q}$. Put $g(\mathfrak{P}') = \mathfrak{P}$. Then $\mathfrak{P} \cap S^G = g(\mathfrak{P}') \cap S^G = \mathfrak{p}$
and $\mathfrak{P} = g(\mathfrak{P}') \subset g(\mathfrak{Q}') = \mathfrak{Q}$. 

For the remaining part of this chapter we will assume that $S$ is Noetherian
as $S^G$-module. This hypothesis is always satisfied when $S$ is an affine commu-
tative algebra over some commutative Noetherian ring $k$ and $G$ acts on $S$ by $k$-algebra automorphisms; see [Bou$_1$, Theorem 2 in Chap. 5 §1]. Moreover, $M$ will always denote a finitely generated left $S \ast G$-module. Hence, $M$ is also finitely generated over $S$ and over $S^G$. The following lemma relates the depth and heights of an ideal on $M$ and $M^G$. This result is presumably known, but we could not find a proof in the literature. We present here an elegant proof which the author learned from his advisor.

**Lemma 2.3.4.** With the above hypothesis we have the following:

(a) For any prime ideal $\mathfrak{p}$ of $S$, $\text{ann}_S(M) \subset \mathfrak{p}$ iff $\text{ann}_{S^G}(M) \subset \mathfrak{p} \cap S^G$.

(b) For any ideal $\mathfrak{b}$ of $S$, $\text{depth}(\mathfrak{b}, M) = \text{depth}(\mathfrak{b} \cap S^G, M)$ and $\text{height}(\mathfrak{b}, M) = \text{height}(\mathfrak{b} \cap S^G, M)$.

**Proof.** (a) Set $\mathfrak{p} = \mathfrak{p} \cap S^G$. If $\text{ann}_S(M) \subset \mathfrak{p}$, then clearly $\text{ann}_{S^G}(M) \subset \mathfrak{p}$. For the converse, assume that $\text{ann}_{S^G}(M) \subset \mathfrak{p}$. Let $\mathfrak{p}'$ be a lift of $\mathfrak{p}$ in $S$ containing $\text{ann}_S(M)$ (by Lemma 2.1.2(c), such a lift exists). Then there exists $g \in G$ such that $g(\mathfrak{p}') = \mathfrak{p}$. Since $g(\text{ann}_S(M)) = \text{ann}_S(M)$, $\mathfrak{p}$ contains $\text{ann}_S(M)$.

(b) Set $\mathfrak{a} = \mathfrak{b} \cap S^G$. If $\mathfrak{a}M = M$, then so is $\mathfrak{b}M = M$. Therefore we can assume that $\text{depth}(\mathfrak{a}, M)$ is finite. Clearly $\text{depth}(\mathfrak{a}, M) \leq \text{depth}(\mathfrak{b}, M)$. To prove the reverse inequality, suppose $\text{depth}(\mathfrak{a}, M) = n$. Let $\{x_1, \ldots, x_n\}$ be a maximal $M$-regular sequence in $\mathfrak{a}$. Then $\mathfrak{a}$ consists of zero divisors of $\bar{M} = M/(x_1 \cdots x_n)M$. By Proposition 2.1.6, $\mathfrak{a} \subset \cup_{\mathfrak{Q} \in \text{Ass}(\bar{M})} \mathfrak{Q}$. Thus $\mathfrak{a} \subset \cup_{\mathfrak{Q} \in \text{Ass}(\bar{M})} \mathfrak{Q} \cap$
$S^G$. Now we apply Lemma 2.1.7 to find $\mathfrak{Q} \in \text{Ass}_S(\bar{M})$ such that $a \subset \mathfrak{Q} \cap S^G$. By definition, $\mathfrak{Q} = \text{ann}_S(\bar{m})$ for some nonzero element $\bar{m} \in \bar{M}$. But then $x_1, \ldots, x_n$ is a maximal $M$-regular sequence in $\mathfrak{Q}$; so $n = \text{depth}(\mathfrak{Q}, M)$, by Theorem 2.2.3(b). By going up, there exist a prime $\mathfrak{P}$ in $S$ containing $b$ such that $\mathfrak{Q} \cap S^G = \mathfrak{P} \cap S^G$. As before, there exists $g \in G$ such that $g(\mathfrak{P}) = \mathfrak{Q}$. Hence, $\text{depth}(a, M) = \text{depth}(\mathfrak{Q}, M) = \text{depth}(\mathfrak{P}, M) \geq \text{depth}(b, M)$.

Now for height. We first assume that $a$ and $b$ both are prime ideals in $S^G$ and $S$ respectively and $\text{ann}_S(M) \subset b$ (and hence, $\text{ann}_{S^G}(M) \subset a$). Our goal is to show that $\text{height}(a, M) = \text{height}(b, M)$, that is, $\text{height}(a/\text{ann}_{S^G}(M)) = \text{height}(b/\text{ann}_S(M))$. But Lemma 2.1.2(b) (Incomparability), applied to the integral extension $S^G/\text{ann}_{S^G}(M) \hookrightarrow S/\text{ann}_S(M)$, gives $\text{height}(b/\text{ann}_S(M)) \leq \text{height}(a/\text{ann}_{S^G}(M))$. The reverse inequality follows from Lemma 2.1.2(c) (Going Up) together with part(b) of Lemma 2.3.3.

For the general case, let $\mathfrak{P}$ be a prime ideal of $S$ containing $b + \text{ann}_S(M)$ such that $\text{height}(b, M) = \text{height}(\mathfrak{P}, M)$. Putting $p = \mathfrak{P} \cap S^G$, we note that $p$ contains $a + \text{ann}_{S^G}(M)$. Thus by above discussion,

$$\text{height}(a, M) \leq \text{height}(p, M) = \text{height}(\mathfrak{P}, M) = \text{height}(b, M).$$

Hence $\text{height}(a, M) \leq \text{height}(b, M)$. Conversely, let $p$ be a prime ideal containing $a + \text{ann}_{S^G}(M)$ such that $\text{height}(a, M) = \text{height}(p, M)$. Then $p \supset (b + \text{ann}_S(M)) \cap S^G$. Indeed, let $a + b \in S^G$ with $a \in b$ and $b \in \text{ann}_S(M)$.
Then
\[(a + b)^{|G|} = \prod_{g \in G} g(a + b) = \prod_{g \in G} g(a) + c\]
with \(c \in \text{ann}_S(M)\). Clearly, \(\prod_{g \in G} g(a) \in \mathfrak{a}\) and \(c \in \text{ann}_{S^G}(M)\). Therefore, \((a + b)^{|G|} \in \mathfrak{a} + \text{ann}_{S^G}(M) \subseteq \mathfrak{p}\), and so \(a + b \in \mathfrak{p}\), as desired. By Lemma 2.1.2(c), we may choose a prime \(\mathfrak{p}\) of \(S\) with \(\mathfrak{p} \cap S^G = \mathfrak{p}\) and \(\mathfrak{p} \supset (b + \text{ann}_S(M))\). Since \(\text{height}(\mathfrak{p}, M) = \text{height}(\mathfrak{p}, M)\), we finally obtain \(\text{height}(\mathfrak{a}, M) = \text{height}(\mathfrak{p}, M) = \text{height}(\mathfrak{p}, M) \geq \text{height}(\mathfrak{b}, M)\).

As a consequence, we obtain the following result.

**Theorem 2.3.5.** Let \(M\) be a finitely generated left \(S \ast G\)-module. Then \(M\) is Cohen-Macaulay as \(S\)-module if and only if \(M\) is Cohen-Macaulay as an \(S^G\)-module.

**Proof.** We use the Cohen-Macaulay criterion in Lemma 2.2.6(ii). Note that, by Lemmas 2.1.2 and 2.3.4(a), the maximal ideals of \(S^G\) containing \(\text{ann}_{S^G}(M)\) are precisely the ideals of the form \(\mathfrak{m} \cap S^G\), where \(\mathfrak{m}\) is a maximal ideal of \(S\) containing \(\text{ann}_S(M)\). Moreover, by Lemma 2.3.4(b), neither depth nor height on \(M\) changes when passing from \(\mathfrak{m}\) to \(\mathfrak{m} \cap S^G\).

**Remark 2.** Suppose \(M\) is a finitely generated \(S \ast G\)-module. Our main concern is to determine when \(M^G\) is Cohen-Macaulay as an \(S^G\)-module, given that \(M\) is a Cohen-Macaulay \(S\)-module. The above theorem enables us to replace \(S \ast G\) by the group ring \(S^G[G]\) and rephrase the question as follows:
Let $M$ be an $S^G[G]$-module which is Cohen-Macaulay as an $S^G$ module. When can we say that $M^G$ is Cohen-Macaulay over $S^G$?

### 2.4 Background from homological algebra

In the Section 3.5, we make use of certain spectral sequences associated with “derived functors”. We do not define categories and functors here; a good background reference is [Rot]. Since we are interested in categories of modules over a ring, all our categories are assumed to be abelian (in the sense of [Be$_1$, ch.2, sec. 1]). In the following examples, we introduce certain module categories that are important for our purposes, along with some notation. In this section, $R$ denotes an arbitrary ring (associative with identity) and $S$ will denote a commutative ring. Moreover, $G$ will be a finite group acting on $S$, as in the previous sections.

**Definition 2.4.1.** For any ring $R$, $R$-$\text{mod}$ is the category of all left $R$-modules. For two left $R$-modules $M$ and $N$, the group of morphisms $\text{Hom}_R(M, N)$ in $R$-$\text{mod}$ is the group of all $R$-linear transformations $M \to N$. Similarly one defines category $\text{mod-}R$ of right $R$-modules.

We are primarily interested in the categories $S \ast G$-$\text{mod}$ and $S^G[G]$-$\text{mod}$. Note that any $S \ast G$-module is also a $S^G[G]$-module and any $S \ast G$-linear map is also $S^G[G]$-linear. In this sense, $S \ast G$-$\text{mod}$ is a subcategory of $S^G[G]$-$\text{mod}$. Two more categories of which $S \ast G$-$\text{mod}$ is a subcategory are $S$-$\text{mod}$ and
$S^G$-mod. This situation is summarized in the following diagram, where all arrows are the so-called “restriction functors”:

$$
\begin{array}{c}
S \ast G \text{-mod} \longrightarrow S \text{-mod} \\
\downarrow \quad \downarrow
\end{array}
$$

$$
\begin{array}{c}
S^G[G] \text{-mod} \longrightarrow S^G \text{-mod} \\
\end{array}
$$

We give two further examples of functors which are very important for us.

**Example 2.4.2.** Our first example is the fixed-point functor $(\_)^G$:

$S^G[G]$-mod → $S^G$-mod. For any $S^G[G]$-module $M$, one defines $(M)^G$ as the collection of all $G$-invariant elements in $M$; this is easily seen to be an $S^G$-module. For simplicity, we will write $M^G$ in place of $(M)^G$. Now, for any $S^G[G]$-linear homomorphism $f : M \to N$, $f(M^G) \subset N^G$. Therefore the restriction of $f$ to $M^G$ gives a morphism $f^G \in \text{Hom}_{S^G}(M^G, N^G)$. By composing $(\_)^G$ with the restriction functor $S \ast G \text{-mod} \to S^G[G] \text{-mod}$, we also obtain a fixed point functor $(\_)^G : S \ast G \text{-mod} \to S^G \text{-mod}$.

**Example 2.4.3.** Our second example is the a-torsion functor $\Gamma_a(\_): S \text{-mod} \to S \text{-mod}$ that is associated an ideal $a$ in a commutative ring $S$. Under this functor, the image of an $S$-module $M$ is

$$
\Gamma_a(M) = \{m \in M \mid a^nm = 0, \text{ for some } n \in \mathbb{N}\}.
$$

For any morphism $f : M \to N$ in $S$-mod, $\Gamma_a(f)$ is the restriction of $f$ to $\Gamma_a(M)$.

We will often consider the situation where $a$ is an ideal of $S^G$ and $M$ is an $S^G[G]$-module (or an $S \ast G$-module). In this case, $\Gamma_a(M)$ is defined by first restricting $M$ to $S^G$, as in the above diagram.
We recall the (standard) notion of an exact functor.

**Definition 2.4.4.** A functor \( \mathcal{F} \) defined on \( R\text{-mod} \) is called *left exact* if, given an exact sequence \( 0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \) in \( R\text{-mod} \) (i.e., \( f \) is injective and \( \ker(g) = \text{Im}(f) \)), the resulting sequence

\[
0 \to \mathcal{F}(M_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M_2)
\]

is also exact. Right exactness can be defined in the same manner. A functor that is both right exact and left exact is called *exact*.

We will use \( \circ \) to denote the composite of functors.

**Lemma 2.4.5.** The fixed-point functor \( (\ . \ )^G \) and the a-torsion functors \( \Gamma_a \) defined above are left exact. Moreover, for any ideal \( a \) of \( S^G \), \( (\ . \ )^G \circ \Gamma_a = \Gamma_a \circ (\ . \ )^G \) as functors \( S^G[G]\text{-mod} \to S^G\text{-mod} \).

**Proof.** The commuting part can be verified directly from the expression

\[
\Gamma_a(M^G) = \{ m \in M \mid m \text{ is } G\text{-invariant and for some } n, a^n m = 0 \} = \Gamma_a(M)^G
\]

for any \( S^G[G]\text{-module} \) \( M \).

To show left exactness, let \( 0 \to L \xrightarrow{f} M \xrightarrow{f'} N \) be an exact sequence. Being restrictions of \( f \), both \( \Gamma_a(f) \) and \( f^G \) are injective. Also since \( f'f = 0 \), \( \text{Im}(\Gamma_a(f)) \subset \ker(\Gamma_a(f')) \) and \( \text{Im}(f^G) \subset \ker(f'^G) \). To complete the proof, let \( m \in \ker(\Gamma_a(f')) \) and \( m' \in \ker((f')^G) \); so \( m \in \Gamma_a(M) \cap \ker f' \) and \( m' \in M^G \cap \ker f' \). Then there exists \( l, l' \in L \) such that \( f(l) = m \) and \( f(l') = m' \).
Since $f$ is injective and $f(a^n l) = f(l' - g(l')) = 0$ for some value of $n$ and for any $g \in G$, we conclude that $l \in \Gamma_a(L)$ and $l' \in L^G$. This completes the proof.

The above functors $(\cdot)^G$ and $\Gamma_a$ are however usually not exact. In general, if a left exact functor $\mathcal{F}$ on some module category $R$-mod fails to be exact, the "exactness defect" can be measured by the right derived functors $R^n(\mathcal{F}) (n \geq 0)$. We briefly discuss the construction of these functors, referring to [Rot] for complete details. For this, we will need the concept of injective modules.

**Definition 2.4.6.** A left $R$-module $I$ is called injective if given an injective $R$-module map $f: M \hookrightarrow N$ and any $R$-module map $g: M \rightarrow I$, there exists an $R$-module map $\phi: N \rightarrow I$ such that $\phi \circ f = g$. In other words, $\text{Hom}_R(\cdot, I)$ is an exact functor on $R$-mod.

Let $M$ be any left $R$-module. It is known that $M$ can be embedded into some injective module, say $I_0$. The cokernel of this embedding, $I_0 / M$, can be embedded in to another injective module, $I_1$. Continuing in this way we construct a long exact sequence,

$$I_\ast: 0 \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots \xrightarrow{d_n} I_n \xrightarrow{d_{n+1}} I_{n+1} \rightarrow \ldots$$

$I_\ast$ is called an injective resolution of $M$. Note that $\text{Ker} d_0 \simeq M$; so the above sequence is exact everywhere except at 0th level. Now apply $\mathcal{F}$ to this sequence
and put \( \delta^i = \mathfrak{F}(d^i) \) to get

\[
0 \to \mathfrak{F}(I_0) \xrightarrow{\delta^0} \mathfrak{F}(I_1) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} \mathfrak{F}(I_{n+1}) \to .
\]

One defines

\[
R^n \mathfrak{F}(M) = \text{Ker}(\delta^n)/\text{Im}(\delta_{n-1}) .
\]

The following two properties will be important for us.

- \( R^0 \mathfrak{F}(M) = \mathfrak{F}(M) . \)

- Any short exact sequence \( 0 \to M_1 \to M \to M_2 \to 0 \) gives rise to the
  a “natural” long exact sequence, \( 0 \to \mathfrak{F}(M_1) \to \mathfrak{F}(M) \to \mathfrak{F}(M_2) \xrightarrow{\delta^0}
  R^1 \mathfrak{F}(M_1) \to \cdots \xrightarrow{\delta^n} R^n \mathfrak{F}(M_2) \xrightarrow{\delta^n} R^{n+1} \mathfrak{F}(M_1) \to \cdots \)

We will now turn specifically to the fixed-point and torsion functors for

\( (.)^G \) and \( \Gamma_a . \)

**Example 2.4.7.** (a) Let \( M \) be a left module over \( S^G[G] \). The \( n^{th} \) right
derived functor of the \( G \)-fixed point functor \( (.)^G \), evaluated on \( M \), is called

\( n^{th} \) cohomology group of \( M \). The usual notation is

\[
R^n (.)^G(M) = H^n(G, M) .
\]

Note that for \( s \in S^G \), the map \( \sigma: M \to M, m \mapsto sm \) is a morphism in
\( S^G[G]-\text{mod} \), and hence it induces a map \( \sigma^n_s: H^n(G, M) \to H^n(G, M) \) for each
\( n \). Letting \( s \) act on \( H^n(G, M) \) via \( \sigma^n_s \), we make \( H^n(G, M) \) into a left \( S^G \)-
module.
(b) The $n^{th}$ right derived functor of the $\alpha$-torsion functor, $R^n\Gamma_\alpha$, is commonly denoted by $H^*_\alpha(\cdot)$; it is called the $n^{th}$ local cohomology functor. If $M$ is a left module over $S^G[G]$ then, for any $\alpha \in S^G[G]$, the left multiplication map $\alpha : M \to M$, $m \mapsto \alpha m$ is an $S^G$-module morphism. Hence, as in (a) above, we obtain maps $\alpha_\alpha^* : H^n_\alpha(M) \to H_\alpha^n(M)$ which make each $H^n_\alpha(M)$ into a left $S^G[G]$-module.

(c) In view of the above considerations, for any left $S^G[G]$-module $M$ and any ideal $\alpha$ of $S^G$, we may consider the groups

$$H^p_\alpha(H^q(G, M)) \quad \text{and} \quad H^p(G, H^q_\alpha(M))$$

for $p, q \geq 0$. These groups will be important later on.

For our next result we need some preparations. Let $\alpha : R \to R'$ be any homomorphism of (not necessary commutative) rings. Then $R'$ can be viewed as a $(R, R')$-bimodule and as a as a $(R', R)$-bimodule through $\alpha$. Furthermore, any left $R'$-module $M'$ can be viewed as left $R$-module via $\alpha$; the resulting $R$-module is often called the “restricted module”. Let $M$ be any left $R$-module. Then the tensor product $R' \otimes_R M$ is a left $R'$-module, sometimes called the “induced module”, and $\text{Hom}_R(\cdot, M)$ is another left $R'$-module, usually called the “co-induced module”. For the detailed construction of these modules, see [Rot] or [Br, Chap. III.3]. These references also contain proofs for the following
natural isomorphisms, the so-called “adjoint isomorphisms”:

\[ \text{Hom}_R(M', M) \simeq \text{Hom}_R(M', \text{Hom}_R(R', M)) \]  \hspace{1cm} (2.4.1)

\[ \text{Hom}_R(M, M') \simeq \text{Hom}_R(R' \otimes_R M, M') . \]  \hspace{1cm} (2.4.2)

One consequence of these isomorphisms is the following proposition.

**Proposition 2.4.8.** Let \( I \) be an injective left \( S^G[G] \)-module and let \( a \) be an ideal of \( S^G \). Then:

(a) \( I \) and \( I^a \) are injective \( S^G \)-modules;

(b) \( \Gamma_a(I) \) is injective \( S^G \)-module.

**Proof.** (a) Since \( S^G[G] \) is a free right \( S^G \) module, \( S^G[G] \otimes_{S^G} (.) : S^G \text{-mod} \to S^G[G] \text{-mod} \) is an exact functor. Also since \( I \) is injective, \( \text{Hom}_{S^G[G]}(., I) \) is an exact functor on \( S^G[G] \text{-mod} \). Thus, the composite \( \text{Hom}_{S^G[G]}(S^G[G] \otimes_{S^G} (.), I) \) is an exact functor on \( S^G \text{-mod} \). Now we apply the isomorphism (2.4.2), with \( \alpha : R = S^G \hookrightarrow R' = S^G[G] \) the inclusion map, to see that \( \text{Hom}_{S^G}(., I) \) is exact on \( S^G \text{-mod} \). Thus, \( I \) is an injective \( S^G \)-module.

For the second assertion of (a), consider the augmentation map \( \omega : S^G[G] \to S^G \), \( \omega(\sum_{g \in G} s_g g) = \sum_{g \in G} s_g \). If \( M \) is any left \( S^G[G] \)-module then the co-induced \( S^G \)-module \( \text{Hom}_{S^G[G]}(S^G, M) \) is naturally isomorphic with the module of \( G \)-fixed points \( M^G \), via \( f \mapsto f(1) \). Applying the isomorphism (2.4.1) with \( \alpha = \omega \) and \( M = I \), we get isomorphisms of functors on \( S^G \text{-mod} \),

\[ \text{Hom}_{S^G[G]}(., I) \simeq \text{Hom}_{S^G}(., \text{Hom}_{S^G[G]}(S^G, I)) \simeq \text{Hom}_{S^G}(., I^G) \, . \]
Since $\Hom_{S^G[G]}(\cdot, I)$ is exact, this shows that $\Hom_{S^G}(\cdot, I^G)$ is an exact functor on $S^G\text{-mod}$, $I^G$ is an injective $S^G$-module.

(b) Let $b$ be an ideal of $S^G$ and let $f : b \to \Gamma_a(I)$ be an $S^G$-linear map. By [Rot, Theorem 3.20], it is enough to show that there exists an element $m \in \Gamma_a(I)$ such that $f(s) = sm$ holds for all $s \in b$. Since $I$ is injective as $S^G$-module, by part (a), there certainly exists an element $m' \in I$ with this property. Now, since $S^G$ is Noetherian, $f(b)$ is a finitely generated submodule of $\Gamma_a(I)$. Hence there exist a positive integer $r$ such that $a^r f(b) = 0$. Further, $f(b)$ is a submodule of the finitely generated $S^G$-module $S^G m'$. By the Artin-Rees Lemma [Bou1, Cor. 1, Chap. III, §3.1], there exists a positive integer $t$ such that for all $n \geq t$,

$$a^n(S^G m') \cap f(b) = a^{n-t}(a^t(S^G m') \cap f(b)).$$

Therefore $a^{r+t}(S^G m') \cap f(b) = a^r(a^t(S^G m') \cap f(b)) \subset a^r f(b) = 0$. This implies that we can extend $f$ to $h : a^{r+t} + b \to \Gamma_a(I)$ by defining

$$h(s + s') = s'm'$$

for all $s \in a^{r+t}$ and $s' \in b$. Indeed, if $s + s' = u + u'$ with $s, u \in a^{r+t}$ and $s', u' \in b$ then $(u' - s')m' = (s - u)m' \in a^{r+t}(S^G m') \cap f(b) = 0$. Once again we use injectivity of $I$ over $S^G$ to find $m \in I$ such that $h(s) = sm$ holds for all $s \in S^G$. We claim that $m \in \Gamma_a(I)$. Indeed, if $s \in a^{r+t}$ then $sm = h(s) = h(s + 0) = 0$. This completes the proof. \qed
Figure 2.1: $E_2$-page of a spectral sequence.

We conclude this section with a brief introduction to spectral sequences. The spectral sequences considered here are sometimes called “third quadrant (homology)” or “first quadrant (cohomology)” spectral sequences. Again, we refer to [Rot] for details and more general versions.

Definition 2.4.9. A spectral sequence is a sequence of bigraded $S$-modules $E_r = \{E_r^{p,q}\}_{p,q \geq 0}$ ($r \geq 2$) equipped with an $S$-linear maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ such that $d_r^{p,q} \circ d_r^{p-r,q-1+r} = 0$ and $E_{r+1}^{p,q} = \ker d_r^{p,q} / \text{Im}(d_r^{p-r,q-1+r}) = H^{p,q}(E_r)$. We say that for a fixed pair $p, q$, $\{E_r^{p,q}\}$ converges to $E_\infty^{p,q}$ and write $E_r^{p,q} \Rightarrow E_\infty^{p,q}$ if there exists a positive integer $r$ dependent on $p$ and $q$ such that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty$.

We are particularly interested in $r = 2$. In this case the differential $d_2$ has bi-degree $(2, -1)$. Figure 1.1 shows the $E_2$-page of the spectral sequence.

We end this chapter with a remarkable theorem of Grothendieck which gives a convenient way to construct certain spectral sequences. The categories in the theorem can be any abelian categories with enough injective objects
(i.e., every object embeds into an injective object), but for our purposes, it will be enough to think quite specifically of module categories. We refer to [Rot, Theorem 11.38] for a complete proof.

**Theorem 2.4.10.** Let \( \mathcal{G} : \mathcal{A} \rightarrow \mathcal{B} \) and \( \mathcal{F} : \mathcal{B} \rightarrow \mathcal{C} \) be functors with \( \mathcal{F} \) left exact such that, for each injective object \( I \) in \( \mathcal{A} \), the image \( \mathcal{G}(I) \) in \( \mathcal{B} \) satisfies \( R^q \mathcal{F}(\mathcal{G}(I)) = 0 \) for \( q > 0 \). Then, for each module \( A \) in \( \mathcal{A} \), there exists a spectral sequence

\[
E_2^{p,q} = R^p \mathcal{F}(R^q \mathcal{G}(A)) \Rightarrow R^{p+q}(\mathcal{F} \circ \mathcal{G})(A)
\]
CHAPTER 3

General Group Actions

In this chapter, we consider the following question: Suppose $G$ is a finite group acting on the commutative ring $S$ and $M$ is an $S \times G$-module which is Cohen-Macaulay as an $S$-module, when is $M^G$ Cohen-Macaulay as an $S^G$-module? As we saw in Theorem 2.3.5, if $M$ is a Cohen-Macaulay $S$-module then $M$ is also Cohen-Macaulay $S^G$-module. Therefore, later in Section 3.4 of this chapter, we will replace $S$ by $S^G$ and assume that $S$ is pointwise fixed under $G$. Most of the literature on invariant theory is concerned with the special case of linear actions, as explained in the Introduction; see, e.g., [H$_1$, H$_2$, N$_1$, N$_2$, ST, C]. In this chapter, we will consider arbitrary group actions.
We assume throughout this chapter that

\begin{align*}
S & \text{ is a commutative ring,} \\
G & \text{ is a finite group acting on } S, \\
S^G & \text{ is the subring of } G\text{-invariants in } S, \\
S \ast G & \text{ is the skew group ring that is associated with the} \\
& \text{given } G\text{-action on } S, \\
M & \text{ is a left } S \ast G\text{-module, and} \\
M^G & \text{ denotes the submodule of } G\text{-invariant elements of } M.
\end{align*}

Further hypotheses will be explicitly mentioned whenever they are needed.

## 3.1 Trace map

The relative trace map is defined for any subgroup \( H \) of \( G \). Let \( \tau \) be a set of representatives for the cosets \( gH \) of \( H \) in \( G \).

**Definition 3.1.1.** The relative trace map \( \text{tr}_{G/H} : M^H \to M^G \) is defined by

\[ \text{tr}_{G/H}(m) = \sum_{g \in \tau} g(m) \quad (m \in M^H). \]

The image of \( \text{tr}_{G/H} \) is denoted by \( M^G_H \).

Thus,

\[ M^G_H = \{ \text{tr}_{G/H}(m) \mid m \in M^H \}. \]

It is easy to check that \( \text{tr}_{G/H} \) does not depend on the choice of \( \tau \). In the following lemma we will explore the elementary properties of the trace map. In particular, we will show that \( M^G_H \) is an \( S^G \)-submodule of \( M^G \). We use \( gH \) to denote the conjugate subgroup \( gHg^{-1} \) for \( g \in G \).
Lemma 3.1.2. Let $M, S, G$ be as above. Then:

(a) $\text{tr}_{G/H}$ is a homomorphism of $S^G$-modules. Thus $M^G_H$ is an $S^G$-submodule of $M^G$.

(b) If $K$ is any subgroup of $H$, then $\text{tr}_{G/K} = \text{tr}_{G/H} \text{tr}_{H/K}$ Consequently, $M^G_K \subset M^G_H$.

(c) For any $m \in M^G$, $\text{tr}_{G/H}(m) = [G:H]m$

(d) For any $g$ in $G$, $M^G_H = M^G_{gH}$.

Proof. (a) Since the trace map is defined by summation, it is additive. To show $S^G$-linearity, let $r \in S^G$ and let $m \in M$. Choose a set $\tau$ of cosets representatives of $H$ in $G$ Then,

$$\text{tr}_{G/H}(rm) = \sum_{g \in \tau} g(rm) = \sum_{g \in \tau} rg(m) = r \text{tr}_{G/H}(m).$$

Thus $\text{tr}_{G/H}$ is $S^G$-linear. Since an image of any module homomorphism is a module, the second statement follows.

(b) Let $\tau$ be a set of cosets representatives of $H$ in $G$ and $\mu$ be a set of cosets representatives of $K$ in $H$. It is clear that we can choose $\gamma = \{gh | g \in \tau, h \in \mu\}$ as a set of cosets representatives of $K$ in $G$. Therefore,

$$\text{tr}_{G/K}(m) = \sum_{gh \in \gamma} gh(m) = \sum_{g \in \tau} g(\sum_{h \in \mu} h(m)) = \text{tr}_{G/H}(\text{tr}_{H/K}(m)).$$

The second assertion follows from this identity.
(c) If $m \in M^G$, then $gm = m$ for all $g \in G$. Therefore, $\text{tr}_{G/H}(m) = |G : H|m$.

(d) Let $m \in M^G_H$. Then there exists $m' \in M^H$ such that $\text{tr}_{G/H}(m') = m$. Now for any $ghg^{-1} \in gHg^{-1}$, $ghg^{-1}(gm') = gm'$ shows that $gm' \in M^{gH}$. Also, $\{gtg^{-1} \mid t \in \tau\}$ is a set of coset representatives for $gH$ in $G$. Thus, $\text{tr}_{G/\tau H}(gm') = \sum_{t \in \tau} gtg^{-1} gm' = g \text{tr}_{G/H}(m') = m$, we get $M^G_H \subseteq M^G_{\tau H}$. By the symmetry of the situation, we get $M^G_H = M^G_{\tau H}$. \hfill \Box

Applying property (a) to the special case $M = S$, we deduce that $S^G_H = \text{tr}_{G/H}(S^H)$ is an ideal of $S^G$.

**Definition 3.1.3.** For any additive subgroup $N$ of $M$, we define the *inertia group* $I_G(N)$ as follows,

$$I_G(N) = \{g \in G \mid (g - 1)(M) \subseteq N\}.$$ 

In the following lemma we justify the word subgroup in the definition. We also list elementary properties.

**Lemma 3.1.4.** With the above notations we have the following:

(a) $I_G(N)$ is a subgroup of $G$.

(b) If $N_1 \subseteq N_2$ then $I_G(N_1) \subseteq I_G(N_2)$.

(c) For $H \subseteq G$, $I_G(N) \cap H = I_H(N)$.
Proof. (a) Since \( G \) is finite, it suffices to show that if \( g, h \in I_G(N) \) then \( gh \in I_G(N) \). But this follows from the calculation

\[
(gh - 1)(m) = (g - 1)(h - 1)(m) + (g - 1)(m) + (h - 1)(m),
\]
because all three terms on the right belong to \( N \).

(b) and (c) are clear from the definition. \( \Box \)

In the following lemma we derive an identity which will be used later with a special case \( M = S \). We will use the notation \( K \backslash G/H \) to denote a set of double coset representatives \( \{KgH\} \) of \( (K, H) \) in \( G \).

**Lemma 3.1.5.** For any \( m \in M^H \),

\[
\text{tr}_{G/H}(m) \equiv \sum_{g \in I_G(N) \setminus G/H} [I_G(N) : I_{gH}(N)] g(m) \pmod{N}
\]

Proof. Write \( G \) as a disjoint union \( G = \bigcup_g I_G(N)gH \) with \( g \) running over \( I_G(N) \setminus G/H \), and for each \( g \), let \( \tau_g \) be a set of the coset representatives of \( H \) in the double coset \( I_G(N)gH \). Then,

\[
\text{tr}_{G/H}(m) = \sum_{g \in I_G(N) \setminus G/H} \sum_{g' \in \tau_g} g'(m).
\]

Now each \( g' \) has the form \( g' = agh \) with \( a \in I_G(N) \) and \( h \in H \). Thus, we have for any \( m \in M^H \), \( g'(m) = ag(m) = ag(m) \equiv g(m) \pmod{N} \). Thus the above formula yields,

\[
\text{tr}_{G/H}(m) \equiv \sum_{g \in I_G(N) \setminus G/H} |\tau_g| g(m) \pmod{N}.
\]
We claim that \( |\tau_3| = [I_G(N) : I_{sH}(N)] \). To show this, let \( g_1, g_2 \in I_G(N)gH \). Then \( g_1 = a_1gh_1, g_2 = a_2gh_2 \) for some \( a_1, a_2 \in I_G(N) \) and \( h_1, h_2 \in H \). Then \( g_1H = g_2H \Leftrightarrow a_1gH = a_2gH \Leftrightarrow a_1 I_{sH}(N) = a_2 I_{sH}(N) \), where the last equivalence uses Lemma 3.1.4(c). This proves the claimed formula for \( |\tau_3| \), and hence the lemma. \( \square \)

The proof of the following lemma is based on an earlier proof that was communicated to M. Lorenz by Don Passman (e-mail of Oct. 18, 2000). The special case where \( S \) is an affine algebra over a field is covered by [Ke2, Satz 4.7]. Let \( I_G(\Omega) = \{g \in G \mid (g - 1)(S) \subseteq \Omega\} \) be the inertia group of an ideal \( \Omega \) of \( S \).

**Lemma 3.1.6.** For any prime ideal \( \Omega \) of \( S \),

\[
\Omega \supseteq S^G_H \iff [I_G(\Omega) : I_{sH}(\Omega)] \in \Omega \quad \text{for all } g \in G
\]

**Proof.** By Lemma 3.1.5 the implication \( \Leftarrow \) is clear. For \( \Rightarrow \), assume that \( \Omega \supseteq S^G_H \). Since \( S^G_H = S^G_{sH} \) by Lemma 3.1.2(d), it suffices to show that

\[
[I_G(\Omega) : I_{sH}(\Omega)] \in \Omega
\]

To simplify notation, put \( I = I_G(\Omega) \) and let \( P \) denote a Sylow \( p \)-subgroup of \( I \cap H = I_H(\Omega) \), where \( p \geq 0 \) is the characteristic of the commutative domain \( S/\Omega \). (Here \( P = \{1\} \) if \( p = 0 \).) Then our desired conclusion, \( [I : I \cap H] \in \Omega \),
is equivalent with

\[ [I : P] \in \Omega . \]

Furthermore, our assumption \( \Omega \supseteq S^G_{\Omega} \) entails that \( \Omega \supseteq S^G_P \), by Lemma 3.1.2(b).

Thus, leaving \( H \) for \( P \), we may assume that \( H = P \) is a \( p \)-subgroup of \( I \). Let \( D = \{ g \in G \mid g(\Omega) = \Omega \} \) denote the decomposition group of \( \Omega \); so \( I \leq D \).

We claim that

\[ \Omega \supseteq S^D_P . \]

To see this, choose \( r \in S \). Then that \( r \in g(\Omega) \) for all \( g \in G \setminus D \) but \( r \notin \Omega \).

Then \( s = \prod_{g \in D} g(r) \) also belongs to \( \bigcap_{g \in G \setminus D} g(\Omega) \) but not to \( \Omega \) and, in addition, \( s \in S^D \). Now assume that, contrary to our claim, there exists an element \( f \in S^P \) so that \( \text{tr}_{D/P}(f) \notin \Omega \). Then \( \text{tr}_{D/P}(sf) = s \text{tr}_{D/P}(f) \in \bigcap_{g \in G \setminus D} g(\Omega) \setminus \Omega \). Hence, \( \text{tr}_{G/P}(sf) = \text{tr}_{G/D}(\text{tr}_{D/P}(sf)) = \text{tr}_{D/P}(sf) + \sum_{1 \neq g \in G / D} g \text{tr}_{D/P}(sf) \notin \Omega \), a contradiction.

By the claim, we may replace \( G \) by \( D \), thereby reducing to the case where \( \Omega \) is \( G \)-stable. (Note that \( I \) is unaffected by this replacement.) So \( G \) acts on \( S/\Omega \) with kernel \( I \), \( P \) is a \( p \)-subgroup of \( I \), and \( S^G_P \subseteq \Omega \). Thus, for all \( r \in S^P \), \( 0 \equiv \text{tr}_{G/P}(r) \equiv [I : P] \cdot \sum_{g \in G/I} g(r) \) (mod \( \Omega \)). Our desired conclusion, \( [I : P] \in \Omega \), will follow if we can show that \( \sum_{g \in G/I} g(r) \notin \Omega \) holds for some \( r \in S^P \). But \( \sum_{g \in G/I} g \) induces a nonzero endomorphism on \( S/\Omega \), by linear independence of automorphisms of \( K = \text{Fract}(S/\Omega) \); so \( \sum_{g \in G/I} g(s) \notin \Omega \) holds for some \( s \in S \). Putting \( r = \prod_{h \in P} h(s) \), we have \( r \in S^P \) and \( r \equiv s^{[P]} \) (mod \( \Omega \)). Since
$|P|$ is 1 or a power of $p = \text{char } K$, we obtain $\sum_{g \in G/H} g(r) \equiv \sum_{g \in G/H} g(s^{|P|}) \equiv \left( \sum_{g \in G/H} g(s) \right)^{|P|} \equiv 0 \pmod{\wp}$, as required.

\[ \square \]

### 3.2 Height formula

For any collection $\mathcal{X}$ of subgroups of $G$, we define the ideal $S^G_{\mathcal{X}}$ of $S^G$ by

$$S^G_{\mathcal{X}} = \sum_{H \in \mathcal{X}} S^G_H.$$  

By Lemma 3.1.2(b),(d), we can assume that $\mathcal{X}$ is closed under $G$-conjugation and under taking subgroups without changing $S^G_{\mathcal{X}}$.

Moreover, for any subgroup $H \leq G$, we define

$$I_S(H) = \sum_{h \in H} (h - 1)(S)S.$$  

Thus, $I_S(H)$ is an ideal of $S$, and $\wp \supseteq I_S(H)$ is equivalent with $H \leq I_G(\wp)$.

**Lemma 3.2.1.** Assume that $S$ has characteristic $p$, a positive prime, and let $\mathcal{X}$ be a collection of subgroups of $G$ that is closed under $G$-conjugation and under taking subgroups. Then

$$\text{height } S^G_{\mathcal{X}} = \inf \{ \text{height } I_S(P) \mid P \text{ is a } p\text{-subgroup of } G, P \notin \mathcal{X} \}$$

**Proof.** One has height $S^G_{\mathcal{X}} = \inf q \text{ height } q = \inf \wp \text{ height } \wp$, where $q$ runs over the prime ideals of $S^G$ containing $S^G_{\mathcal{X}}$ and $\wp$ runs over the primes of $S$ containing $S^G_{\mathcal{X}}$. Here, the first equality is just the definition of height, while the second equality is a consequence of Lemma 2.3.3.
By Lemma 3.1.6,

$$\Omega \supseteq S^G_X \iff p \mid \left| I_G(\Omega) : I_H(\Omega) \right| \text{ for all } H \in X.$$  

Since $I_H(\Omega) = I_G(\Omega) \cap H$ belongs to $X$ for $H \in X$, the latter condition just says that the Sylow $p$-subgroups of $I_G(\Omega)$ do not belong to $X$ or, equivalently, some $p$-subgroup $P \leq I_G(\Omega)$ does not belong to $X$. Therefore,

$$\Omega \supseteq S^G_X \iff \Omega \supseteq \bigcap_{P \leq G \text{ a } p\text{-subgroup, } P \not\in X} I_S(P),$$

which implies the asserted height formula. \qed

3.3 Annihilators of cohomology classes

Recall that $M$ denotes a module over the skew group ring $S \ast G$. Recall further that the $G$-cohomology $H^*(G, M)$ is a left module over $S^G$; see Example 2.4.7(a).

The following lemma generalizes [Ke₃, Corollary 2.4].

**Lemma 3.3.1.** The ideal $S^G_H$ of $S^G$ annihilates the kernel of the restriction map $\text{res}^G_H : H^*(G, M) \rightarrow H^*(H, M)$.

**Proof.** The action of $S^G = H^0(G, S)$ on $H^*(G, M)$ described in Example 2.4.7(a) can also be interpreted as coming from the cup product

$$H^0(G, S) \times H^*(G, M) \xrightarrow{\cup} H^*(G, S \otimes \mathbb{Z} M) \xrightarrow{\text{red}} H^*(G, M),$$
where the map denoted by \( \cdot \) comes from the \( G \)-equivariant map \( S \otimes_{\mathbb{Z}} M \rightarrow M \), \( r \otimes m \mapsto rm \); see, e.g., [Br, Exerc. 1 on p. 114]. Furthermore, the relative trace map \( \mathrm{tr}_{G/H} : S^H \rightarrow S^G \) is identical with the corestriction map \( \mathrm{cor}^G_H : H^0(H, S) \rightarrow H^0(G, S) \); cf. [Br, p. 81]. Thus, the transfer formula for cup products ([Br, (3.8) on p. 112]) gives, for \( s \in S^H \) and \( x \in H^*(G, M) \),

\[
\mathrm{tr}_{G/H}(s)x = - (\mathrm{tr}_{G/H}(s) \cup x) = - (\mathrm{cor}^G_H(s \cup \mathrm{res}^G_H(x))) .
\]

Therefore, if \( \mathrm{res}^G_H(x) = 0 \) then \( \mathrm{tr}_{G/H}(s)x = 0 \). \( \square \)

We summarize the material of this section in the following proposition. For convenience, we write \( \mathrm{res}^G_H(.) = . \mid_p \).

**Proposition 3.3.2.** Assume that \( S \) has characteristic \( p \), and let \( M \) be an \( S \ast G \)-module. Then, for any \( x \in H^*(G, M) \),

\[
\text{height } \mathrm{ann}_{S^G}(x) \geq \inf \{ \text{height } I_S(P) \mid P \text{ a } p\text{-subgroup of } G, x \mid_p \neq 0 \} .
\]

**Proof.** Let \( \mathcal{X} \) denote the splitting data of \( x \), that is, \( \mathcal{X} = \{ H \leq G \mid x \mid_H \neq 0 \} \); cf. [CoR]. By Lemma 3.3.1, \( \mathrm{ann}_{S^G}(x) \supseteq S^G_{\mathcal{X}} \), and by Lemma 3.2.1, \( \text{height } S^G_{\mathcal{X}} = \inf \{ \text{height } I_S(P) \mid P \text{ is a } p\text{-subgroup of } G, x \mid_p \neq 0 \} \). The proposition follows. \( \square \)
3.4 Depth estimates

In this section, we will assume that $S$ is Noetherian and that $G$ acts trivially on $S$. Thus, $S = S^G$ and the skew group ring $S \ast G$ is the ordinary group ring $S[G]$. Throughout, $\mathfrak{a}$ will denote an ideal of $S$. The $S[G]$-module $M$ will be assumed to be finitely generated. Hence, $M$ is Noetherian as an $S$-module, and likewise for the submodule $M^G$ of $G$-invariant elements of $M$.

Our goal is to find estimates on the $\operatorname{depth}(\mathfrak{a}, M^G)$ in terms of $\operatorname{depth}(\mathfrak{a}, M)$. We then apply this results to get a sufficient condition for $M^G$ to be Cohen-Macaulay, in Corollary 3.4.4.

We recall from Theorem 2.2.3 that

$$\operatorname{depth}(\mathfrak{a}, M) = \inf_i \{ i \mid \operatorname{Ext}^i_S(S/\mathfrak{a}, M) \neq 0 \}.$$

In the following lemma we replace ‘Ext’ by local cohomology.

**Lemma 3.4.1.** Let $\mathfrak{a}$ be an ideal in the ring $S$ and let $H^i_\mathfrak{a}$ denote the $i$-th local cohomology functor with respect to $\mathfrak{a}$. Then, for any finitely generated $S$-module $M$,

$$\operatorname{depth}(\mathfrak{a}, M) = \inf \{ i \mid H^i_\mathfrak{a}(M) \neq 0 \}.$$

**Proof.** This is [BS, Theorem 6.2.7].

Since $M$ is a Noetherian $S$-module, all $H^q(G, M)$ are Noetherian $S$-modules as well. Indeed, $H^q(G, M)$ is the $q$th cohomology group of the cochain complex
\[ \text{Hom}_{S[G]}(F_*, M) = \text{Hom}_S(F_*, M)^G, \] where \( F_* \) is any free resolution of the trivial \( S[G]\)-module \( S \). Since \( S \) is Noetherian, all \( F_q \) may be chosen to be finitely generated. Thus, each \( \text{Hom}_S(F_q, M)^G \) is finitely generated over \( S \), and hence so is the subquotient \( H^q(G, M) \). Therefore, the above lemma applies to the \( S \)-module \( H^q(G, M) \) and gives

\[
\text{depth}(a, H^q(G, M)) = \inf \{ i \mid H^i_a(H^q(G, M)) \neq 0 \} .
\]

These \( S \)-modules \( H^p_a(H^q(G, M)) \) feature as the \( E_2^{pq} \)-terms of a certain spectral sequence due to Ellingsrud and Skjelbred [ES]. In fact, two related spectral sequences are constructed in [ES] which we describe now. We take the functors \( \Gamma_a \) and \(( . )^G \) from Examples 2.4.2 and 2.4.3 and apply Theorem 2.4.10, to obtain two Grothendieck spectral sequences as follows.

- By restriction, we may view \( \Gamma_a \) as a functor \( \Gamma_a : S[G]\text{-mod} \to S[G]\text{-mod} \).

  By Lemma 2.4.5, this functor is left exact, and Proposition 2.4.8(b) says that \( \Gamma_a(I) \) is injective whenever \( I \) is an injective (left) \( S[G]\)-module.

  Therefore, \( H^q(G, \Gamma_a(I)) = 0 \) holds for all \( q > 0 \); see the remarks on the construction of right derived functors by means of injective resolutions in Chapter 1. Theorem 2.4.10 now yields a spectral sequence

\[
E_2^{p,q} = H^p(G, H^q_a(M)) \Longrightarrow R^{p+q}(( . )^G \circ \Gamma_a)(M) .
\]

- Similarly, by Lemma 2.4.5, the fixed point functor \(( . )^G : S[G]\text{-mod} \to S\text{-mod} \) is left exact and from Proposition 2.4.8(a), we know that \( I^G \) is
injective over $S$ for any injective $S[G]$-module $I$. Hence, $H^q_a(I^G)$ for all $q > 0$. Again, by Theorem 2.4.10, we obtain a third quadrant (cohomology) spectral sequence

$$E_2^{p,q} = H^p_a(H^q(G,M)) \Longrightarrow R^{p+q}(\Gamma_a \circ (\cdot)^G)(M).$$

- Finally, by Lemma 2.4.5, we have a canonical isomorphism $\Gamma_a(M^G) \simeq \Gamma_a(M)^G$; so the right hand sides of both spectral sequences are isomorphic. For simplicity, we write

$$H^p_a(G,M) = R^p(\Gamma_a \circ (\cdot)^G)(M) \simeq R^p((\cdot)^G \circ \Gamma_a)(M).$$

To summarize, we have the following spectral sequences:

$$E_2^{p,q} = H^p_a(H^q(G,M)) \quad (3.4.1)$$

$$E_2^{p,q} = H^p(G,H^q_a(M))$$

We use this to obtain estimates for $\text{depth}(a, M^G)$ in the following lemma.

**Lemma 3.4.2.** Let $a$ be an ideal of $S$ and let $M$ be a finitely generated $S[G]$-module. Put $m = \text{depth}(a, M)$ and $h_a = \inf_{q>0} \{q + \text{depth}(a, H^q(G,M))\}$. Then:

(a) **lower bound:** $\text{depth}(a, M^G) \geq \min\{m, h_a + 1\}$,

(b) **upper bound:** Assume that $H^{p_0}_a(H^q(G,M)) \neq 0$ for some $p_0 \geq 0$, $q_0 > 0$ with $s = p_0 + q_0 < m$. Assume further that $H^{s+1-t}_a(H^t(G,M)) = 0$.
holds for \( \ell = 1, \ldots, q_0 - 1 \) and \( H^{n-1-\ell}_a(H^\ell(G, M)) = 0 \) holds for \( \ell > q_0 \).

Then \( \text{depth}(a, M^G) \leq s + 1 \).

Proof. Since \( m = \text{depth}(a, M) \), \( H^q_a(M) = 0 \) for \( q < m \), and so the \( E \)-sequence in (3.4.1) implies that \( H^n_a(G, M) = 0 \) for \( n < m \). Therefore, the \( E \)-sequence satisfies

\[
E^{p,q}_\infty = 0 \quad \text{if} \quad p + q < m. \quad \tag{3.4.2}
\]

Furthermore, \( E^{p,0}_2 = H^p_a(M^G) \); so

\[
\text{depth}(a, M^G) = \inf\{p \mid E^{p,0}_2 \neq 0\}.
\]

Finally,

\[
h_a = \inf\{p + q \mid q > 0, E^{p,q}_2 \neq 0\}.
\]

To prove (a), assume that \( p < \min\{m, h_a + 1\} \). Then \( E^{p,0}_\infty = 0 \), by (3.4.2), and \( E^{i,j}_r = 0 \) for \( j > 0 \), \( i + j < p \), \( r \geq 2 \). Recall that the differential \( d_r \) of \( E_r \) has bidegree \((r, 1 - r)\). Thus, \( E^{p,0}_r \) has no nontrivial boundaries and consists entirely of cycles. This shows that \( E^{p,0}_2 = E^{p,0}_3 = \cdots = E^{p,0}_\infty \), and hence \( E^{p,0}_2 = 0 \). Thus, (a) is proved.

For (b), we check that \( E^{p+1,0}_2 \neq 0 \). Our hypotheses imply that, at position \((p_0, q_0)\), all incoming differentials \( d_r \) (\( r \geq 2 \)) are 0 as well as all outgoing \( d_r \) (\( r \geq 2, r \neq q_0 + 1 \)). Therefore, \( E^{p_0,0}_{q_0+1} = E^{p_0,0}_{q_0} = E^{p_0,0}_\infty = E^{p_0,0}_{q_0+2} = \text{Ker}(a^{p_0,0}_{q_0+1}) \). The former implies that \( E^{p_0,0}_{q_0+1} \neq 0 \), by hypothesis on \((p_0, q_0)\), and the latter shows that \( a^{p_0,0}_{q_0+1} \) is injective, because \( E^{p_0,0}_{\infty} = 0 \) by (3.4.2). Thus, \( a^{p_0,0}_{q_0+1} \) embeds
\[ E_{q_0+1}^{0,0} \] into \( E_{q_0+1}^{s+1,0} \), forcing the latter to be nonzero. Hence, \( E_{2}^{s+1,0} \) is nonzero as well, as desired. \( \square \)

For future reference, we note the following simple lemma.

**Lemma 3.4.3.** Assume that \( sM \) is Cohen-Macaulay and that \( \sqrt{\mathfrak{a}} \supseteq \text{ann}_S M^G \). Then \( \text{depth}(\mathfrak{a}, M) = \text{height}(\mathfrak{a}, M) \geq \text{height}(\mathfrak{a}, M^G) \).

**Proof.** Note that \( \sqrt{\mathfrak{a}} \supseteq \text{ann}_S M^G \supseteq \text{ann}_S M \). This implies that \( \text{height}(\mathfrak{a}, M) \geq \text{height}(\mathfrak{a}, M^G) \). Indeed, for every prime \( \mathfrak{p} \) of \( S \) containing \( \mathfrak{a} \), we clearly have \( \text{height}(\mathfrak{p}/\text{ann}_S M) \geq \text{height}(\mathfrak{p}/\text{ann}_S M^G) \), and hence

\[
\text{height}(\mathfrak{a}, M) = \inf\{\text{height}(\mathfrak{p}/\text{ann}_S M) \mid \mathfrak{p} \supseteq \mathfrak{a}\} \\
\geq \inf\{\text{height}(\mathfrak{p}/\text{ann}_S M^G) \mid \mathfrak{p} \supseteq \mathfrak{a}\} = \text{height}(\mathfrak{a}, M^G).
\]

Further, \( \text{height}(\mathfrak{a}, M) = \text{depth}(\mathfrak{a}, M) \), because \( sM \) is Cohen-Macaulay. The lemma follows. \( \square \)

We now give a sufficient condition for \( M^G \) to be Cohen-Macaulay. Here \( \dim_S M \) stands for \( \dim(S/\text{ann}_S M) \).

**Corollary 3.4.4.** Assume that \( sM \) is Cohen-Macaulay. If \( H^q(G, M) = 0 \) holds for \( 0 < q < \dim_S M - 1 \) then \( M^G \) is a Cohen-Macaulay \( S \)-module as well. In particular, this holds whenever the group order is a unit in \( S \).

**Proof.** Let \( \mathfrak{a} \) be an ideal of \( S \) with \( \mathfrak{a} \supseteq \text{ann}_S M^G \). Since \( H^q(G, M) = 0 \) for \( 0 < q < \dim_S M - 1 \), the value of \( h_\mathfrak{a} \) in Lemma 3.4.2 satisfies \( h_\mathfrak{a} \geq \)}
\( \dim_S M - 1 \). Also, \( \dim M \geq \text{height}(\mathfrak{a}, M) \geq \text{height}(\mathfrak{a}, M^G) \), by Lemma 3.4.3.

Thus, Lemma 3.4.2(a) gives \( \text{depth}(\mathfrak{a}, M^G) \geq \text{height}(\mathfrak{a}, M^G) \), which proves that \( M^G \) is Cohen-Macaulay.

For the last assertion, just note that the group order \( |G| \) annihilates \( H^q(G, M) \) for all \( q > 0 \). Thus if the group order is invertible in \( S \) then \( H^q(G, M) = 0 \) holds for all \( q > 0 \).

\[ \square \]

Note that the condition \( H^q(G, M) = 0 \) for \( 0 < q < \dim_S M - 1 \) is vacuous if \( \dim_S M \leq 2 \). For \( \dim_S M = 3 \), the condition becomes \( H^1(G, M) = 0 \). The latter holds, for example, whenever \( M \) is a \( G \)-permutation module without \( |G| \)-torsion; explicitly, as \( G \)-module, \( M \cong \bigoplus_H (\mathbb{Z}[G] \otimes \mathbb{Z}^{[H]} M(H)) \), where \( H \) runs over certain subgroups of \( G \) and each \( M(H) \) is an \( H \)-submodule of \( M^H \) such that \( |H| m = 0, m \in M(H) \) implies \( m = 0 \).

### 3.5 Depth formula

We continue to work under the hypotheses of Section 3.4 and use the same notation.

In view of Corollary 3.4.4, we may concentrate on the case where \( M \) has non-vanishing positive \( G \)-cohomology. The following proposition is a version of results of Kemper, see [Ke1, Corollary 1.6] and [Ke2, Kor. 1.18].

**Proposition 3.5.1.** Assume that \( sM \) is Cohen-Macaulay and that \( \sqrt{\mathfrak{a}} \supseteq \)
ann_S M^G. Furthermore, assume that, for some \( r \geq 0 \), \( H^q(G, M) = 0 \) holds for \( 0 < q \leq r \) and \( ax = 0 \) for some \( 0 \neq x \in H^r(G, M) \). Then \( \text{depth}(a, M^G) = \min\{r + 1, \text{depth}(a, M)\} \).

Remark 3. By Lemma 3.4.3, \( \text{height}(a, M) = \text{depth}(a, M) \) holds in the above formula.

Proof of Proposition 3.5.1. Our hypothesis \( ax = 0 \) for some \( 0 \neq x \in H^r(G, M) \) is equivalent with \( H^q_a(H^r(G, M)) \neq 0 \); so \( \text{depth}(a, H^r(G, M)) = 0 \). The asserted equality is trivial for \( r = 0 \), since \( \text{depth}(a, M^G) = \text{depth}(a, M) = 0 \) holds in this case. Thus we assume that \( r > 0 \). Then, in the notation of Lemma 3.4.2, we have \( r = h_a \). Therefore, by part (a) of the lemma, we have

\[
\text{depth}(a, M^G) \geq \min\{r + 1, \text{depth}(a, M)\}.
\]

To prove the reverse inequality, note that Lemma 3.4.3 and Theorem 2.2.3(a) give \( \text{depth}(a, M) \geq \text{depth}(a, M^G) \). Thus, it suffices to show that \( \text{depth}(a, M^G) \leq r + 1 \) if \( \text{depth}(a, M) > r + 1 \). For this, we quote Lemma 3.4.2(b) with \( p_0 = 0 \) and \( q_0 = r \) (so \( s = r \)). \( \square \)

3.6 The Sylow subgroup of \( G \)

In this section, we focus on rings of invariants \( S^G \) rather than modules. Throughout, \( S \) is assumed to be Noetherian as an \( S^G \)-module. As we remarked
earlier, this hypothesis is satisfied whenever $S$ is an affine $k$-algebra for some commutative Noetherian ring $k$ and $G$ acts on $S$ by $k$-algebra automorphisms. Furthermore, we assume that $S$ has characteristic $p$, a positive prime; that is, $pS = \{0\}$. We let $P$ denote a fixed Sylow $p$-subgroup of $G$.

Put
\[ \mu = \mu(G, S) = \inf \{ r > 0 \mid H^r(G, S) \neq 0 \} . \]

**Proposition 3.6.1.** Put $\mathcal{P} = \{ P' \leq P \mid \text{height } I_S(P') \leq \mu + 1 \}$. If $S$ and $S^G$ are both Cohen-Macaulay and $\mu < \infty$ then the restriction map
\[ \text{res}^G_{P'} : H^p(G, R) \to \prod_{P' \in \mathcal{P}} H^p(P', S) \]
is injective.

**Proof.** Let $0 \neq x \in H^p(G, S)$ be given and put $a = \text{ann}_{S^G}(x)$. Then, by Proposition 3.3.2,

\[ \text{height } a \geq \inf \{ \text{height } I_S(P') \mid P' \text{ a } p\text{-subgroup of } G, x|_{P'} \neq 0 \} . \]

Since $S^G$ is Cohen-Macaulay, $\text{height } a = \text{depth } a$. Finally, Proposition 3.5.1 with $M =_{S^G} S$ (note that $M$ is Cohen-Macaulay, by Theorem 2.3.5) gives $\text{depth } a \leq \mu + 1$. Thus, there exists a $p$-subgroup $P'$ of $G$ with $x|_{P'} \neq 0$ and $\text{height } I_S(P') \leq \mu + 1$. Note that both the condition $x|_{P'} \neq 0$ and the value of $\text{height } I_S(P')$ are preserved upon replacing $P'$ by a conjugate $gP'$ with $g \in G$. Therefore, we may assume that $P' \in \mathcal{P}$, which proves the proposition. \qed
3.7 Bireflections

We continue to use the notations and hypotheses of Section 3.6.

Following [Ke2], we will call an element $g \in G$ a bireflection on $S$ if height $I_S(\langle g \rangle) \leq 2$.

**Corollary 3.7.1.** Assume that $S$ and $S^G$ are both Cohen-Macaulay. Let $H$ denote the subgroup of $G$ that is generated by all elements of $G$ whose order is coprime with $p$ and all bireflections in $P$. Then $S^G = S^G_H$.

**Proof.** First note that $H$ is a normal subgroup of $G$ and $G/H$ is a $p$-group. Thus, if $S^G \neq S^G_H$ or, equivalently, $\hat{H}^0(G/H, S^H) \neq 0$ then also $H^1(G/H, S^H) \neq 0$; see [Br, Theorem VI.8.5]. In view of the exact sequence

$$0 \to H^1(G/H, S^H) \longrightarrow H^1(G, S) \xrightarrow{\text{res}^G_H} H^1(H, S)$$

(see [Ba, 35.3]) we further obtain $H^1(G, S) \neq 0$. Thus, $\mu = 1$ holds in Proposition 3.6.1 and every $P' \in \mathcal{P}$ consists of bireflections. Therefore, $P' \subseteq H$ and Proposition 3.6.1 implies that $\text{res}^G_H : H^1(G, S) \to H^1(H, S)$ is injective, contradicting the above exact sequence. Therefore, we must have $S^G = S^G_H$. \qed

**Corollary 3.7.2.** Assume that $S$ and $S^G$ are both Cohen-Macaulay. If $\mathbb{F}_p = \mathbb{Z} \cdot 1_S$ is a $G$-module direct summand of $S$ then $G = H$. In particular, if $G$ is a $p$-group then $G$ is generated by bireflections.

**Proof.** If $\text{tr}_{G/H} : S^H \to S^G$ is onto then we can write $1 = \text{tr}(a) = \sum G/H g(a)$. Since $\mathbb{F}_p$ is a direct summand of $S$, $a = u + r$ with $u \in \mathbb{F}_p$ and $r \in M$. Applying
\( \text{tr}_{G/H} \) on both sides, we get
\[ 1 = [G : H]u + \text{tr}_{G/H}(r). \]
Since \( \mathbb{F}_p \) is a \( G \)-summand,
\[ 1 = [G : H]u. \]
Now by definition of \( H \), \( [G : H] \) is a power of \( p \). This shows that
\[ [G : H] = 1 \] and \( G = H \). The second assertion is immediate from this. \( \square \)
CHAPTER 4

Multiplicative Actions

Let $G$ be a finite group acting by automorphisms on a $\mathbb{Z}$-lattice (free abelian group) $A$ of rank $n$. In this case, $A$ is often called a $G$-lattice. We extend the $G$-action on $A$ action linearly to the group algebra $S = k[A]$ over a commutative ring $k$; so $G$ acts on $S = k[A]$ by $k$-algebra automorphisms. Actions of this type are called multiplicative; the terms exponential or purely monomial are also found in the literature. The term “purely monomial” refers to the fact that, by choosing a $\mathbb{Z}$-basis for $A$, one can identify $k[A]$ with the Laurent polynomial algebra $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the elements of $A$ correspond to the monomials in the variables $x_i$ and their inverses. Traditionally, the group operation in $A$ is written as addition, $+$. On the other hand, when viewing $A \subset k[A]$, the lattice $A$ becomes a multiplicative group.

The algebra $S^G$ of $G$-invariant elements in $S$ is called the algebra of mul-
tiplicative invariants. It is a standard fact that the group algebra $S = k[A]$ is Cohen-Macaulay precisely if the base ring $k$ is Cohen-Macaulay; see [BH, Theorems 2.1.3 and 2.1.9]. In this chapter, we assume throughout that $k$ is a field, although much of the material generalizes directly to the case where $k$ is a Cohen-Macaulay ring. The above notations remain valid throughout the chapter. We will investigate the following question:

When is $S^G$ Cohen-Macaulay?

4.1 Passage to an effective lattice

Let

$$A^G = \{ a \in A \mid g(a) = a \ \forall g \in G \}$$

denote the sublattice of $G$-invariants in $A$ and let $\overline{\cdot} : A \to A/A^G$ denote the canonical map. Extend this map to a map

$$\overline{\cdot} : S = k[A] \to \overline{S} = k[A/A^G].$$

**Definition 4.1.1.** We say that $A$ is an effective $G$-lattice if $A^G = 0$.

**Lemma 4.1.2.** $\overline{A} = A/A^G$ is an effective $G$-lattice.

**Proof.** Let $\overline{a} \in \overline{A}$, with $a \in A$. Fix $g \in G$ and set $b = g(a) - a \in A^G$. Then $|G|b = trb = tr(a) - tr(a) = 0$. Since $A^G$ is torsion free, $b = 0$. Therefore $g(a) = a$ for all $g \in G$ or $a \in A^G$. This shows that $\overline{A^G} = 0$. $\square$
Let $G(a)$ denote the $G$-orbit $\{g(a) \mid g \in G\}$ of $a \in A$. The orbit sum of $a$ is the element of $S = k[A]$ that is defined by

$$\sigma(a) = \sum_{b \in G(a)} b .$$

Note that $\sigma(a)$ belong to $S^G = k[A]^G$.

The following lemma reduces the general case of the Cohen-Macaulay problem for $S^G$ to the case of the effective $G$-lattice $\overline{A} = A/A^G$. The lemma and its proof have been given to me by my advisor Martin Lorenz.

**Lemma 4.1.3.** With the above notations, $S^G$ is Cohen-Macaulay if and only if $\overline{S^G} = k[\overline{A}]^G$ is Cohen-Macaulay.

**Proof.** We begin by constructing an auxiliary $G$-lattice $\overline{A}$ as follows. View $G$ as acting on the $\mathbb{Q}$-vector space

$$A_\mathbb{Q} = A \otimes \mathbb{Q}$$

and let $\rho: A_\mathbb{Q} \to A_\mathbb{Q}^G$ denote the Reynolds operator, $\rho(v) = |G|^{-1} \sum_{g \in G} g(v)$. Define

$$\overline{A} = A + \rho(A) \subseteq A_\mathbb{Q} .$$

We make the following claims:

(i) $\overline{A} \simeq A^G \oplus \overline{A}$ as $G$-lattices;

(ii) $A \subseteq \overline{A}$ and $\overline{A} = \bigcup_{t \in \mathcal{T}} (t + A)$ (disjoint union) for some finite subset $\mathcal{T} \subseteq \overline{A}^G$. 
To prove this, note that \( \rho \) and \( \pi = 1 - \rho \) are orthogonal idempotents in \( \text{End}_{\mathbb{Q}[G]}(A_{\mathbb{Q}}) \). Therefore, \( A_{\mathbb{Q}} = \rho(A_{\mathbb{Q}}) \oplus \pi(A_{\mathbb{Q}}) = A_{\mathbb{Q}}^G \oplus \pi(A_{\mathbb{Q}}) \) and \( A \subset \rho(A) \oplus \pi(A) \subset |G|^{-1}A \subset A_{\mathbb{Q}} \). Consequently,

\[
\overline{A} = A + \rho(A) = \rho(A) \oplus \pi(A)
\]

and \( \rho(A) = \overline{A}^G \). Moreover, \( \pi(A) \cong A/\text{Ker}_A(\pi) = A/A^G \). This proves Claim (i). Since \( \overline{A}/A \subset (|G|^{-1}A)/A \), the index of \( A \) in \( \overline{A} \) is finite, and since \( \overline{A} = A + \overline{A}^G \), a transversal, \( T \), for \( A \) in \( \overline{A} \) can be chosen from \( \overline{A}^G \). Claim (ii) follows.

Now, turning to group rings, Claims (i) and (ii) above immediately imply that

\[
k[\overline{A}] = \bigoplus_{t \in T} tk[A] \cong k[A] \otimes_k k[\overline{A}^G]
\]

and \( t \) can chosen to be in \( \rho(A) \subset A^G \). By taking \( G \)-invariants, we obtain

\[
k[\overline{A}]^G = \bigoplus_{t \in T} tk[A]^G \cong k[\overline{A}]^G \otimes_k k[\overline{A}^G] ;
\]

so \( k[\overline{A}]^G \) is finite and free over \( k[A]^G \) and a Laurent polynomial algebra over \( k[\overline{A}]^G \). In view of [BH, Theorems 2.1.3 and 2.1.9 and Exercise 2.1.23], we conclude that

\[
k[A]^G \text{ is Cohen-Macaulay } \Leftrightarrow k[\overline{A}]^G \text{ is Cohen-Macaulay}
\]

\[
\Leftrightarrow k[\overline{A}]^G \text{ is Cohen-Macaulay} .
\]

This completes the proof of the lemma. \( \square \)
4.2 Cohomology $H^*(G, S)$

**Definition 4.2.1.** Let $H$ be a subgroup of $G$. A left $k[G]$-module $M$ is called *induced* from $H$ to $G$ if there exist a $k[H]$-module $N$ such that $M \simeq k[G] \otimes_{k[H]} N$. In this case, we will write $M = N \uparrow^G_H$.

Since $G$ is assumed finite, there is a natural isomorphism $\text{Hom}_{k[H]}(k[G], N) \simeq N \uparrow^G_H$; see [Br, Proposition III(5.9)].

Fix a set of representatives $\tau$ for the $G$-orbits in $A$; so

$$A = \bigcup_{a \in \tau} G(a),$$

where the union is a disjoint union. For each $a \in A$, let $G_a = \{g \in G \mid g(a) = a\}$ denote the stabilizer of $a$ in $G$. Then

$$S = k[A] \cong \bigoplus_{a \in \tau} k \uparrow^G_{G_a}$$

as $k[G]$-modules, where $k$ is the “trivial” $k[G_a]$-module, that is, every element of $G_a$ acts as the identity on $k$. The Eckman-Shapiro Lemma [Br, Proposition III(6.2)] gives

$$H^*(G, k[A]) \cong \bigoplus_{a \in \tau} H^*(G_a, k).$$

Here, we use finiteness of $G$ for the fact that $H^*(G, \_)$ commutes with direct sums; see [Br, Proposition VIII(4.6)].

Now assume that $G$ acts *fixed-point freely* on $A$, that is, $G_a = \{1\}$ holds
for all $0 \neq a \in A$. Then the above equation reduces to

$$H^*(G, k[A]) \cong H^*(G, k) \quad (* > 0)$$

Therefore, if $G$ acts fixed-point freely on $A$ then the constant $\mu = \mu(G, k[A])$ introduced in Section 3.6 is given by

$$r_k(G) = \inf \{ r > 0 \mid H^r(G, k) \neq 0 \}.$$ 

It is known that $r_k(G) < \infty$ if and only if char $k$ divides the order $|G|$; see [Be$_2$, Theorem 4.1.3].

Before stating and proving the main result of this section, we record the following simple lemma determining the height of the ideals $I_S(H)$ for subgroups $H \leq G$ in the present setting of multiplicative actions.

**Lemma 4.2.2.** For any subgroup $H \leq G$, height $I_S(H) = \text{rank } A - \text{rank } A^H$.

**Proof.** By definition, the ideal $I_S(H)$ of $S = k[A]$ is generated by the elements $h(a) - a$, or else by the elements $h(a)a^{-1} - 1$ for $h \in H, a \in A$. Thus, $S/I_S(H) \cong k[A/[H, A]]$, where we have put $[H, A] = \langle h(a)a^{-1} \mid h \in H, a \in A \rangle \leq A$. Consequently, height $I_S(H) = \dim S - \dim S/I_S(H) = \text{rank } A - \text{rank } A/[H, A]$. Finally, since the group algebra $\mathbb{Q}[H]$ is semisimple, $A \otimes \mathbb{Q} = (A^H \otimes \mathbb{Q}) \oplus ([H, A] \otimes \mathbb{Q})$; so $\text{rank } A/[H, A] = \text{rank } A^H$. \square

The following proposition is the main result of this section.
Proposition 4.2.3. Assume that the action of $G$ on $\overline{A}$ is fixed-point free and that $p = \text{char } k$ divides $|G|$. If $\text{rank}(\overline{A}) \geq r_k(G) + 2$ then $S^G$ is not Cohen-Macaulay.

Proof. By Lemma 4.1.3, we can assume without loss of generality that $\overline{A} = A$; so $G$ acts fixed-point freely on $A$. In particular, by Lemma 4.2.2 above, we have height $I_S(H) = \text{rank } A$ for every subgroup $1 \neq H \leq G$. Now assume that $\text{rank } A \geq r_k(G) + 2$ but $S = k[A]$ is Cohen-Macaulay. To derive a contradiction, we will apply Proposition 3.6.1. As we pointed out above, $\mu = r_k(G) < \infty$. Let $P$ be a fixed Sylow $p$-subgroup of $G$ and define the set $\mathcal{P}$ as in Proposition 3.6.1. By the foregoing, $\mathcal{P}$ consists of the identity subgroup alone. Therefore, the product $\prod_{P' \in \mathcal{P}} H^\mu(P', S)$ in Proposition 3.6.1 is the trivial group, and hence $H^\mu(G, S)$ must be trivial as well, contradicting the definition of $\mu$. 

The following corollary settles the case of groups of order $p = \text{char } k$.

Corollary 4.2.4. Suppose $|G| = p = \text{char } k$. Then $S^G$ is Cohen-Macaulay if and only if $\text{rank}(\overline{A}) \leq 2$. In particular, if $p > 3$ and $G$ acts non-trivially on $A$ then $S^G$ is never Cohen-Macaulay.

Proof. Since $|G| = p$, the $G$-action on $\overline{A}$ is actually fixed-point free. Also, $r_k(G) = 1$. Thus, by Proposition 4.2.3, if $\text{rank}(\overline{A}) \geq 3$ then $S^G$ is not Cohen-Macaulay. On the other hand, if $\text{rank}(\overline{A}) \leq 2$, then $k[\overline{A}]^G$ is certainly Cohen-
Macaulay; see [BH, Exercise 2.2.30(a)]. Thus, in view of Lemma 4.1.3, we obtain that $S^G$ is Cohen-Macaulay iff $\text{rank}(\overline{A}) \leq 2$, proving the first assertion of the corollary.

Since $\text{GL}_n(\mathbb{Z})$ for $n \leq 2$ does not contain a subgroup of order $p > 3$, the second assertion also follows. \qed

Here is a simple explicit example.

**Example 4.2.5.** Suppose that $\text{char } k = 2$ and $|G| = \{ \pm I_{n \times n} \}$ acts canonically on the lattice $A = \mathbb{Z}^n$. Then $k[A]^G$ is Cohen-Macaulay if and only if $n \leq 2$. Since $A = \overline{A}$ for the given action, this is immediate from Corollary 4.2.4.

### 4.3 Cyclic Sylow subgroups

The notation employed in the previous section remains in effect. In particular, we assume that $p = \text{char } k$ divides the order $|G|$ of $G$, and we let $P$ denote a fixed Sylow $p$-subgroup of $G$. Our main interest in this section is on the case where $P$ is cyclic. The following lemma, however, is valid generally, even for arbitrary $G$-actions on a commutative ring $S$.

**Lemma 4.3.1.** Suppose that $G$ has a subgroup $H$ such that $S^G_H = S^G$ and $S^H$ is Cohen-Macaulay. Then $S^G$ is Cohen-Macaulay as well.

**Proof.** By virtue of our hypothesis $S^G_H = S^G$, the relative trace map $\text{tr}_{G/H} : S^H \to S^G$ is surjective; so we may fix an element $c \in S^H$ with $\text{tr}_{G/H}(c) = 1$. Define
the map $\rho: S^H \to S^G$ by $\rho(s) = \text{tr}_{G/H}(sc)$. Then $\rho$ is a “Reynolds operator”, that is, $\rho$ is $S^G$-linear and the restriction of $\rho$ to $S^G$ is the identity map on $S^G$. Since $S^H$ is integral over the subring $S^H$, we may apply a result of Hochster and Eagon [HE] (see also [BH, Theorem 6.4.5]) to conclude that the Cohen-Macaulay property descends from $S^H$ to $S^G$, as we have claimed. □

Note that the hypothesis $S^G_H = S^G$ in the lemma is certainly satisfied whenever the index $[G : H]$ is invertible in $S$. For, in this case, the element $c = [G : H]^{-1} \in S^H$ satisfies $\text{tr}_{G/H}(c) = 1$. Our main application of these observations in Chapter 4 will via the following

**Corollary 4.3.2.** If $S^P$ is Cohen-Macaulay then so is $S^G$.

The converse of this corollary is false in general:

**Example 4.3.3.** Let $G = S_p$ denote the symmetric group on $p$ symbols and let $A = \mathbb{Z}^p$. Then $G$ acts on $A$ by permuting the canonical basis $\{x_1, \ldots, x_p\}$ of $A$. By the classical theorem on elementary symmetric functions, the corresponding multiplicative invariant algebra is given by

$$k[A]^G = k[x_1^\pm 1, \ldots, x_p^\pm 1]^{S_p} = k[s_1, \ldots, s_{p-1}, s_p^\pm 1],$$

where the $s_i$ denote the elementary symmetric functions. Thus, $k[A]^G$ is certainly Cohen-Macaulay. On the other hand, the Sylow $p$-subgroup $P$ is cyclic of order $p$, generated by the $p$-cycle $(1, 2, \ldots, p)$. If $p > 3$ then $P$ is not gen-
erated by a bireflection on $A$. Therefore, Corollary 3.7.2 shows that $k[A]^P$ is not Cohen-Macaulay.

For any finite group $G$, $O^p(G)$ denotes the intersection of all normal subgroups $N$ of $G$ so that $G/N$ is a $p$-group. In other words, $O^p(G)$ is the subgroup of $G$ that is generated by all $p'$-elements.

**Theorem 4.3.4.** Assume that $O^p(G) \neq G$ and that $P$ is cyclic. Then $S^G$ is Cohen-Macaulay if and only if $P$ is generated by a bireflection on $A$.

**Proof.** First assume that $S^G$ is Cohen-Macaulay. Let $H$ denote the subgroup of $G$ that is generated by $O^p(G)$ and all bireflections in $P$. Then Corollary 3.7.2 implies that $G = H$. It follows that $G/O^p(G) = P/P \cap O^p(G)$ is generated by the images of the bireflections in $P$. Since $P$ is cyclic, it follows that $P$ is generated by a bireflection.

Conversely, if $P$ is generated by a bireflection on $A$ then $A/A^P$ has rank at most 2. Therefore, the invariant algebra $k[A/A^P]^P$ is Cohen-Macaulay; see [BH, Exercise 2.2.30(a)]. Lemma 4.1.3 gives that $k[A]^P$ is Cohen-Macaulay, and Corollary 4.3.2 further implies that $k[A]^G$ is Cohen-Macaulay.
CHAPTER 5

Multiplicative Invariants of

Rank 3 Groups

If $A$ is any $G$-lattice of rank at most 2 then the multiplicative invariant algebra $k[A]^G$ is a normal domain of dimension at most 2, and hence $k[A]^G$ is Cohen-Macaulay; see [BH, Exercise 2.2.30(a)]. As we have seen in Example 4.2.5, this does not hold in rank 3. Our aim in this chapter is to classify all multiplicative invariant algebras $k[A]^G$ with rank $A = 3$ into two classes, Cohen-Macaulay and non-Cohen-Macaulay. In view of Lemma 4.1.3, this will entail an answer to the Cohen-Macaulay problem for all $G$-lattices $A$ so that $\overline{A} = A/A^G$ has rank at most 3.

Thus, for the rest of this chapter, we assume that

$$A = \mathbb{Z}^3$$
and

$$G \text{ is a finite subgroup of } \text{GL}_3(\mathbb{Z})$$

acting canonically on $A$. As is customary in group theory, we shall use the exponential notation $a^g$ ($a \in A, g \in G$) for the $G$-action on $A$, and likewise for the extended action on the group algebra $S = k[A]$. In other words, we think of the elements of $A$ as integer row vectors on which the matrices in $G$ act by right multiplication. Writing $a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1)$ for the canonical basis of $A$, the group algebra $S = k[A]$ can be thought of as the Laurent polynomial algebra

$$S = k[a_{\pm 1}, b_{\pm 1}, c_{\pm 1}].$$

For example, the matrix $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \text{GL}_3(\mathbb{Z})$ acts on these generators via $a^g = a, b^g = c^{-1}$ and $c^g = b$.

Throughout, $k$ will denote a field of characteristic $p$; it will turn out that only the case $p = 2$ needs to be considered in detail.

Our approach to the classification is to proceed case-by-case, using the machinery explained in the previous chapters along with explicit computations. To this end, we use the list of all finite subgroups of $\text{GL}_3(\mathbb{Z})$, up to conjugation, that is given in reference [T]. There are 73 such groups; one group listed in [T] is superfluous, as has been pointed out in [AGr]. Borrowing a convention from [HK], we will denote the group $W_i$ on the page $j$ in [T] by

$$W_i(j).$$
(Occasionally, if there is no ambiguity, we will drop the page number $j$ from the notation.) For example, $W_5(198)$ is the group listed as $W_5$ on the page 198 in [T]. It is claimed there that $W_5(198)$ has order 24, while the actual order is 12, as was pointed out in [AGr]; so the group $W_5(198)$ should be disregarded.

With this notation, our findings can be summarized in the following

**Theorem 5.0.1.** (a) If $\text{char } k \neq 2$ then all multiplicative invariants $k[A]^G$ in dimension 3 are Cohen-Macaulay.

(b) Assume that $\text{char } k = 2$. Then, with exception of the groups conjugate to one of the following

- $W_3(173)$ (order 2),
- $W_2(174)$ and $W_4(174)$ (both cyclic of order 4),
- $W_3(184)$ and $W_4(184)$ (both cyclic of order 6),

and the possible exception of the conjugates of

$$W_{14}(174) \simeq C_2 \times C_2 \quad \text{and} \quad W_{10}(191) \simeq A_4$$

all finite subgroups of $\text{GL}_3(\mathbb{Z})$ have Cohen-Macaulay multiplicative invariant algebras.

Thus far, we have not been able to settle the cases $W_{14}(174)$ and $W_{10}(191)$. Here, $W_{14}(174)$ is (up to conjugation) the Sylow 2-subgroup of $W_{10}(191)$. Thus,
once $W_{14}(174)$ will be shown to be Cohen-Macaulay, which we expect, then it will follow that $W_{10}(191)$ is Cohen-Macaulay as well.

5.1 Some reductions

We first show that the five exceptions listed part (b) of Theorem 5.0.1 do indeed have non-Cohen-Macaulay multiplicative invariant algebras in characteristic 2.

**Proposition 5.1.1.** Assume that $p = 2$. If $G$ is conjugate to one of the following

- $W_5(173)$ (order 2),
- $W_2(174)$ and $W_4(174)$ (both cyclic of order 4),
- $W_3(184)$ and $W_4(184)$ (both cyclic of order 6),

then the invariant algebra $k[A]^G$ is not Cohen-Macaulay.

**Proof.** Let $P$ denote a Sylow 2-subgroup of $G$. By Theorem 4.3.4, it suffices to show that $P$ is not generated by bireflection. Recall that bireflections are matrices $g \in \text{GL}_3(\mathbb{Z})$ so that $\text{rank}(g - \text{Id}) \leq 2$. (It follows from Lemma 4.2.2 that this condition is equivalent with the earlier definition of bireflections given in Section 3.7.) Since powers and conjugates of bireflections are again bire-
flections, it suffices to exhibit, for each of the groups \(W_5(173), \ldots, W_4(184),\) a generator of \(P\) that is not a bireflection.

In each of the cases \(W_5(173), W_5(184), W_4(184),\) the Sylow 2-subgroup \(P\) is generated by \(-Id_{3\times3},\) clearly not a bireflection. For \(W_2(174)\) and \(W_4(174),\) \([T]\) gives the generators \(\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\) and \(\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\), respectively. Both are not bireflections.

We remark that the groups in the above proposition are indeed the only finite subgroups of \(\text{GL}_3(\mathbb{Z})\) that cannot be generated by bireflections.

Our next proposition gives a number of sufficient conditions for Cohen-Macaulayness of the invariant algebra. In particular, part (a) of Theorem 5.0.1 is covered by part (a) below.

**Proposition 5.1.2.** Each of the following conditions entails that the invariant algebra \(k[A]^G\) is Cohen-Macaulay:

(a) \(p = \text{char } k \neq 2;\)

(b) the order \(|G|\) is odd;

(c) \(G\) has non-trivial fixed points in \(A,\) that is, \(A^G \neq 0;\)

(d) \(G\) can be generated by reflections, that is, by matrices \(g \in \text{GL}_3(\mathbb{Z})\) so that \(\text{rank}(g - \text{Id}) = 1.\)

**Proof.** (a) The order of \(|G|\) is a divisor of 48; see \([T].\) Thus, if \(p \neq 2, 3\) then \(k[A]^G\) is Cohen-Macaulay by Corollary 3.4.4. Assume now that \(p = 3.\) Then
$P$ has order 3. Hence, by [T, Proposition 3], $P$ is conjugate to one of the following two groups:

$$W_1(173) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \quad \text{or} \quad W_2(173) = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$$

Both these matrices are bireflections; so $k[A]^P$ is Cohen-Macaulay by Corollary 4.2.4, and Corollary 4.3.2 further implies that $k[A]^G$ is Cohen-Macaulay. This proves (a).

(b) Assume that $|G|$ is odd. If $p \neq 2$ then $k[A]^G$ is Cohen-Macaulay by part (a), and if $p = 2$ then the same conclusion follows from Corollary 3.4.4 or Corollary 4.3.2.

(c) If $A^G \neq 0$ then $\overline{A} = A/A^G$ has rank at most 2. Therefore, $k[\overline{A}]^G$ is Cohen-Macaulay, by [BH, Exercise 2.2.30(a)]. Lemma 4.1.3 now gives that $k[A]^G$ is Cohen-Macaulay.

(d) This is a consequence of the main result of [Lo1] which asserts that multiplicative invariants $k[A]^G$ of reflection groups $G$ (in any rank) are affine normal semigroup algebras. Cohen-Macaulayness of $k[A]^G$ now follows from a result of Hochster; see [BH, Theorem 6.3.5(a)].

\[ \square \]

In view of part (a) above, we will henceforth assume that

$$p = \text{char } k = 2 .$$

The remaining sections each are devoted to finite subgroups $G \subseteq \text{GL}_3(\mathbb{Z})$ of a given even order.
We conclude this section with a simple lemma that will be useful in our explicit computations. The lemma is independent of the particular setting of multiplicative actions and holds more generally.

**Lemma 5.1.3.** Let $G$ be a finite group acting faithfully on a commutative integral domain $S$.

(a) Let $S_0 \subseteq S^G$ be a Noetherian subring such that $S$ is free of rank $r$ as $S_0$-module. Then $|G|$ divides $r$ and $S^G$ is finitely generated as $S_0$-module, with minimal number of generators $\geq r/|G|$. Moreover, $S^G$ is free as $S_0$-module if and only if it can be generated by $r/|G|$ elements.

(b) Let $H \subseteq G$ be a subgroup of $G$ and suppose that $S^H = \sum_{i=1}^{m} \eta_i S^G$ where $m = [G : H]$ is the index of $H$ in $G$. Then \{\eta_i\}_1^m$ is a free $S^G$-module basis of $S^H$.

**Proof.** Denote by $K$ the quotient field of $S$. The action of $G$ can be extended to $K$ by putting $(a/b)\vartheta = a\vartheta/b\vartheta$.

(a) The fact that $S^G$ is finitely generated as $S_0$-module follows from Noetherianness of $S_0$. Let $m$ denote the minimal number of module generators and write $S^G = \sum_{i=1}^{m} x_i S_0$. Let $F$ denote the quotient field of $S_0$; so $F \subseteq K$. The subring $S^G F = \sum_{i=1}^{m} x_i F \subseteq K$ has dimension at most $m$ over $F$ and is contained in the fixed field $K^G$. Therefore, $K^G = S^G F$ and $[K^G : F] \leq m$. Similarly, $K = SF$ and our freeness hypothesis on $S$ over $S_0$ entails that
\[ K : F \] = r. Since \([K : K^G] = |G|\), by Galois theory, we conclude that \(r = |G|[K^G : F] \leq |G|m\). This proves the estimate \(m \geq r/|G|\). Moreover, equality holds here if and only if \([K^G : F] = m\), which in turn is equivalent to \(\{x_i\}_1^m\) being \(F\)-independent or, equivalently, \(S_0\)-independent. This proves (a).

(b) The fixed field \(K^H\) contains the subring \(S^HK^G\) and \([K^H : K^G] = [G : H]\), by Galois theory. This entails that \(K^H = S^HK^G\). Now assume that \(S^H = \sum_{i=1}^m \eta_i S^G\) with \(m = [G : H]\). Then \(K^H = S^HK^G = \sum_{i=1}^m \eta_i K^G\); so \(\{\eta_i\}_1^m\) is a \(K^G\)-basis of \(K^H\). Therefore, \(\{\eta_i\}_1^m\) is surely \(S^G\)-independent, proving the lemma.

\[ \square \]

### 5.2 Groups of order 2

There are 5 conjugacy classes of subgroups \(G \subseteq \text{GL}_3(\mathbb{Z})\) of order 2.

**Proposition 5.2.1.** If \(G \subseteq \text{GL}_3(\mathbb{Z})\) is a group of order 2, then \(S^G\) is Cohen-Macaulay if and only if \(G \neq \langle -\text{Id}_{3 \times 3} \rangle = W_5(173)\).

**Proof.** We already know that the multiplicative invariants of \(W_5(173)\) are not Cohen-Macaulay; see Proposition 5.1.1. The remaining 4 conjugacy classes of subgroups \(G \subseteq \text{GL}_3(\mathbb{Z})\) of order 2 are represented by the groups with the following generators:

\[ \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]
These matrices all have fixed points. Thus for those four groups, $S^G$ is Cohen-Macaulay by Proposition 5.1.2(c).

5.3 Order 4

There are 14 conjugacy classes of subgroups $G \subseteq \text{GL}_3(\mathbb{Z})$ of order 4; these are represented by the groups $W_i(174)$ for $i = 1, \ldots, 14$.

**Proposition 5.3.1.** If $G = W_i(174)$ ($i \neq 14$), then $S^G$ is Cohen-Macaulay if and only if $i \neq 2$ or 4.

**Proof.** By Proposition 5.1.1, we know that the multiplicative invariants of $W_2(174)$ and of $W_4(174)$ are not Cohen-Macaulay. We consider the cases $G = W_i(174)$ ($i \neq 2, 4, 14$) in turn.

- The following 7 groups have fixed points; so the ring of invariants of these groups are Cohen-Macaulay by Proposition 5.1.2(c):

  (i) $W_1(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

  (ii) $W_3(174)$ has fixed points $(2, 1, 1)\mathbb{Z}$.

  (iii) $W_7(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

  (iv) $W_9(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

  (v) $W_{10}(174)$ has fixed points $(0, 1, 1)\mathbb{Z}$. 


(vi) $W_{13}(174)$ has fixed points $(0, 1, 1)\mathbb{Z}$.

It remains to show that if $G = W_i(174)$ where $i = 5, 6, 8, 11, 12$ then $S^G$ is Cohen-Macaulay. We will do this case by case.

- Let $G = W_5(174) = \left\{ g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$. Clearly,

  \[
  W_5(174) \subseteq H = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\}
  \]

Observe that

\[ S^H = S_0 = k[u, v, w] , \]

where we have put $u = a + a^{-1}$, $v = b + b^{-1}$ and $w = c + c^{-1}$. Thus,

\[ S_0 \subseteq S^G \subseteq k[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] = (S_0 \oplus aS_0) \otimes (S_0 \oplus bS_0) \otimes (S_0 \oplus cS_0) \]

\[ = S_0 \oplus aS_0 \oplus bS_0 \oplus cS_0 \oplus abS_0 \oplus acS_0 \oplus bcS_0 \oplus abcS_0 . \hspace{1cm} (5.3.1) \]

Let $f \in S^G$. Then there exists $f_i \in S_0$ ($0 \leq i \leq 7$), such that $f = f_0 + af_1 + bf_2 + cf_3 + abf_4 + acf_5 + bcf_6 + abc f_7$. Comparing $f^g = f$ and substituting $b^g = b^{-1} = v + b$, $c^g = c^{-1} = w + c$, we get the following equation:

\[ v f_2 + w f_3 + a(v f_4 + w f_5 + vw f_7) + b(w f_6) + c(v f_8 + ab w f_7) + abc(v f_7) = 0 . \]

This implies that $f_6 = f_7 = 0$ and $v f_2 = w f_3 = vw f'_2$, $v f_4 = w f_5 = vw f'_3$ for suitable $f'_1, f'_2 \in S_0$. Thus,

\[ f = f_0 + af_1 + (bw + cv)f'_1 + (abw + acv)f'_2 \hspace{1cm} (5.3.2) \]

Applying $h$ and using $f^h = f$, we get $f_1 = f'_2 = 0$ This shows that $S_0 \subseteq S^G \subseteq S_0 + (bw+cv)S_0 \subseteq S^G$. Thus $S^G = S_0 + (bw+cv)S_0$. Since the index of $G$ in $H$ is
2, Lemma 5.1.3(b) implies that this sum is direct. Thus $S^G$ is Cohen-Macaulay. Alternatively, this follows from the observation that $G = G_1 \times G_2$, where $G_1$ acts by inversion on $a$ and trivially on $\langle b, c \rangle$, while $G_2$ acts by inversion on $\langle b, c \rangle$ and is trivial on $a$. Therefore, $S^G = k[a^{\pm 1}]^G_1 \otimes_k k[b^{\pm 1}, c^{\pm 1}]^G_2$. Since both factors are Cohen-Macaulay, being normal domains of rank 1 and 2, $S^G$ is Cohen-Macaulay as well.

- When $i = 6$, $W_6(174) = \langle g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$. Note that $g$ is identical with the matrix denoted by $g$ in the case of $W_6(174)$. Let $f \in S^{W_6}$. Since $g \in W_6 \subset H$, we can still deduce equation (5.3.2) from the condition $f^g = f$. Applying $f^h = f$, we further derive $f = f_0 + (aw + bw + cw)f_1$. Thus $S^{W_6} = S_0 \oplus (aw + bw + cw)S_0$. Thus, as above, Lemma 5.1.3 implies that $S^{W_6}$ is Cohen-Macaulay.

- Suppose $G = W_8$:

$$W_8(174) = \langle g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle.$$  

Take $G' = \langle \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rangle$. Then $G'$ is a group of order 16 containing $W_8$ and $S^{G'} = S_0 = k[u, s, t]$, where $s = v + w$ and $t = vw$ with $u, v, w$ as above. As $S_0 \subset S^{W_8} \subset S$ and $S$ is a free module over $S_0$ with basis $1, a, av, b, bv, c, cv, ab, abv, bc, bcv, ac, acv, abc, abcv$, any $f \in S^{W_8}$ is an $S_0$-linear combination of this basis. Using $f^g = f, f^h = f$ we get

$$f = f_0 + (uv + as)f_1 + (bs + bv + cv)f_2 + (btu + cvs + abs^2 + abvs + acvs)f_3$$
Thus $S^{W_8} = S_0 + (uv + as)S_0 + (bs + bv + cv)S_0 + (btu + cvs + ab + cv + abv)S_0$.

Since the index of $W_8$ in $G$ is 4, Lemma 5.1.3(b) shows that this is a direct sum, and so $S^{W_8}$ is Cohen-Macaulay.

- Suppose $G = W_{11}$:

$$W_{11}(174) = \left\langle g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle .$$

$W_{11}$ is also a subgroup of $G^i$ defined for the case $i = 8$ above; so we can use the methods and notations developed there. If $f \in S^{W_{11}}$ then $f^g = f$, $f^h = f$ implies that $f = f_a + (u + as + bu + cu)f_1 + (bs + bv + cv)f_2 + (cvs + ab + abv + acv + bcu)f_3$. Thus $S^{W_{11}} = S_0 \oplus (u + as + bu + cu)S_0 \oplus (bs + bv + cv)S_0 \oplus (cuv + abs + abv + acv + bcu)S_0$. As above, this shows that $S^{W_{11}}$ is Cohen-Macaulay.

- Now we consider the case $G = W_{12}$:

$$W_{12}(174) = \left\langle g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle .$$

Denote by $H$ the subgroup of $W_{12}$ generated by $h$. It is easy to see that $S^H = k[\alpha, \beta, \gamma, \delta]$, where, $\alpha = a + a^{-1}$, $\beta = b^{-1} + c^{-1}$, $\gamma = bc$, $\delta = ab + a^{-1}c$. One can easily check that $\alpha$ satisfies the monic quadratic polynomial

$$t^2 - (\beta \delta + \beta^2)t + \gamma^{-1} \delta^2$$

over $A = k[\beta, \gamma, \delta]$. Therefore, $S^H = A \oplus \alpha A$. Since $H \subset W_{12}$, $S^{W_{12}} \subset S^H$.

Now $g$ acts on $S^H$ as follows: $\beta^g = \delta$, $\delta^g = \beta$, $\gamma^g = \gamma^{-1}$ and $\alpha^g = \alpha$. Put
\[ B = k[\beta + \delta, \gamma + \gamma^{-1}, \beta \delta]. \] Then \( B \subset S^{W_{12}} \subset S^H \) and \( A = B \oplus \delta B \oplus \gamma B \oplus \delta \gamma B \).

Thus,

\[ S^H = B \oplus \delta B \oplus \gamma B \oplus \alpha B \oplus \delta \gamma B \oplus \delta \alpha B \oplus \gamma \alpha B \oplus \alpha \delta B. \]

Now let \( f \in S^{W_{12}} \subset S^H \). Then there exist \( f_0, \ldots, f_7 \in B \) such that

\[ f = f_0 + \delta f_1 + \gamma f_2 + \alpha f_3 + \delta \gamma f_4 + \delta \alpha f_5 + \gamma \alpha f_6 + \alpha \delta f_7. \]

Now using \( f^g = f \), we obtain relations among \( f_i \). Solving these relation gives,

\[ f = f_0 + [\delta (\gamma + \gamma^{-1}) + \gamma (\delta + \beta)]f_1' + \alpha f_2 + [\delta \alpha (\alpha + \alpha^{-1}) + \gamma \alpha (\beta + \delta)]f_2', \]

for some \( f_1', f_2' \in B \). Thus \( S^G = B + [\delta (\gamma + \gamma^{-1}) + \gamma (\delta + \beta)]B + \alpha B + [\delta \alpha (\alpha + \alpha^{-1}) + \gamma \alpha (\beta + \delta)]B \). Since the index of \( H \) in \( W_{12} \) is 2, this is a direct sum decomposition, by a Galois theoretic argument analogous to the one used in the proof of Lemma 5.1.3. Thus, \( S^{W_{12}} \) is a free module over a Cohen-Macaulay algebra, and hence it is Cohen-Macaulay.

\[ \square \]

## 5.4 Order 6

There are 10 conjugacy classes of subgroups \( G \subseteq \text{GL}_3(\mathbb{Z}) \) of order 6, with representative groups \( W_i(184) \) for \( i = 1, \ldots, 10 \).

**Proposition 5.4.1.** If \( G = W_i(184) \), then \( S^G \) is Cohen-Macaulay if and only if \( i \neq 3 \) or 4.
Proof. As we have pointed out in Proposition 5.1.1, the invariant algebras of $G = W_i(184)$ for $i = 3$ or $4$ are not Cohen-Macaulay.

For the remaining groups $G = W_i(184)$, $i \neq 3, 4$, let $P$ denote the Sylow 2-subgroup of $G$ as usual. In each case, one easily checks that $P \neq \langle -Id_{3 \times 3} \rangle$. Thus, Proposition 5.2.1 implies that $k[A]^P$ is Cohen-Macaulay. Hence, by Corollary 4.3.2, $k[A]^G$ is Cohen-Macaulay for these groups. \qed

5.5 Groups of order 12

Proposition 5.5.1. If the order of $G$ is 12, then $S^G$ is Cohen-Macaulay, with the possible exception of groups conjugate to $W_{10}(191) \simeq A_4$.

Proof. Representative groups for the 11 conjugacy classes of subgroups of order 12 in GL$_3(\mathbb{Z})$ are the groups $W_i(193)$ with $i = 1, \ldots, 11$. These groups are isomorphic to $C_6 \times C_2$ ($i = 1$), the dihedral group $D_{12}$ ($2 \leq i \leq 8$) and the alternating group $A_4$ ($9 \leq i \leq 11$). In each case, the Sylow 2-subgroup $P$ of $G$ is isomorphic to the Klein four-group $C_2 \times C_2$. Therefore, $P$ is not conjugate to the two non-Cohen-Macaulay cases of order 4, $W_2(174)$ and $W_2(174)$, both of which are cyclic; see Proposition 5.3.1. To conclude that $S^P$ is Cohen-Macaulay (and hence $S^G$ as well), it suffices to rule out the possibility that $P$ is conjugate to the as yet unsettled potential non-Cohen-Macaulay case $W_{14}(174)$. 

Note that $W_{14}(174) \subseteq \text{SL}_3(\mathbb{Z})$. Therefore, if $W_i(193)$ contains an element of determinant $-1$ then the 2-part of this element will then also have determinant $-1$, and hence $P$ cannot be conjugate to $W_{14}(174)$. This observation takes care of the groups $W_i(193)$ with $i \neq 2, 9, 10, 11$; see [T, Corollary on p. 191].

For the remaining groups $W_i(193)$ with $i = 2, 9, 10, 11$, we note that, up to conjugation, the Sylow 2-subgroups are given by:

- $W_8(174) \in \text{Syl}_2(W_2(191))$;

- $W_6(174) \in \text{Syl}_2(W_9(191))$;

- $W_{12}(174) \in \text{Syl}_2(W_{11}(191))$;

- $W_{14}(174) \in \text{Syl}_2(W_{10}(191))$;

Thus, except for the case of $W_{10}(191)$, the Sylow 2-subgroup has a Cohen-Macaulay invariant algebra, as desired.

\[\square\]

5.6 Orders 8 and 24

**Proposition 5.6.1.** The invariants $S^G$ for subgroups $G \subseteq \text{GL}_3(\mathbb{Z})$ of order 8 are all Cohen-Macaulay. Consequently, the same conclusion holds for all subgroups $G \subseteq \text{GL}_3(\mathbb{Z})$ of order 24.

**Proof.** By Corollary 4.3.2, we may concentrate on groups of order 8, since groups of order 24 have a Sylow 2-subgroup of order 8. There are 14 conjugacy
classes of subgroups of order 8 in $\text{GL}_3(\mathbb{Z})$, represented by the groups $W_i(187)$ ($i = 1, \ldots, 14$). We will consider these groups in turn.

- The groups $W_8(187) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ and $W_{12}(187) = \left\langle \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ have non-trivial fixed points; so Proposition 5.1.2(c) applies.

- It turns out that all subgroups $G \subseteq \text{GL}_3(\mathbb{Z})$ isomorphic to $C_2 \times C_2 \times C_2$ are generated by reflections, and hence their rings of invariants $S^G$ are Cohen-Macaulay, by Proposition 5.1.2(d). These are the groups $W_3(187), W_4(187), W_5(187)$ and $W_6(187)$.

- Next, we treat the groups $W_1 = W_1(187), W_7 = W_7(187), W_9 = W_9(187)$ and $W_{10} = W_{10}(187)$. To this end, let

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

Then $W_1 = \langle -\text{Id}_{3 \times 3}, h \rangle$, $W_7 = \langle g, h \rangle$, $W_9 = \langle g, -h \rangle$, and $W_{10} = \langle -g, -h \rangle$.

Note that all these groups are contained in the group

$$G = \left\langle \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, h \right\rangle.$$  

This is the group $W_1(194)$ of order 16. The invariants of $G$ are easy to calculate directly. Alternatively, since $G$ is a reflection group, we may use the methods of [Lo1]. The result is that $S^G$ is a polynomial algebra:

$$S^G = S_0 = k[u, s, t],$$  

where \( u = a + a^{-1} \), \( s = b + c + b^{-1} + c^{-1} \) and \( t = (b + b^{-1})(c + c^{-1}) \). In addition, we put \( v = b + b^{-1} \). Then one has the following decomposition of \( S \) as \( S_0 \)-module:

\[
S = S_0 \oplus vS_0 \oplus aS_0 \oplus avS_0 \oplus bS_0 \oplus bvS_0 \oplus cS_0 \oplus cvS_0 \oplus abS_0 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\]

Thus, when considering an invariant \( f \in S^H \) with \( H \in \{W_1, W_7, W_9, W_{10}\} \), we will assume that \( f \) is written in the form

\[
f = f_0 + v f_1 + a f_2 + av f_3 + b f_4 + bv f_5 + c f_6 + cv f_7 + ab f_8 + abv f_9 + ac f_{10} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\]

with \( f_i \in S_0 \subseteq S^H \). Our goal will be to show that \( S^H \), as module over \( S_0 \), can be generated by 2 elements. Lemma 5.1.3(b) then implies that \( S^H \) is indeed free over \( S_0 \), and hence Cohen-Macaulay. We now consider the four groups \( H \in \{W_1, W_7, W_9, W_{10}\} \) separately.

\underline{\( W_1 = \langle -Id_{3 \times 3}, h \rangle \):} Let \( f \in S^{W_1} \), written in the form (5.6.2). Then \( f^h = f \) gives the following relations:

\begin{enumerate}
\item \( f_5 s^2 + f_4 s + f_1 s + f_3 t = 0, f_9 t + f_8 s + f_3 s + f_9 s^2 = 0, \)
\item \( f_{12} s + f_7 s + f_6 + f_{13} s^2 + f_{13} t + f_4 = 0, f_9 + f_5 s + f_4 = 0, \)
\item \( f_{13} s + f_5 + f_{12} + f_7 = 0, f_4 + f_5 s = 0, f_5 + f_7 = 0, \)
\end{enumerate}
(iv) \( f_8 + f_{15}t + f_{10} + f_{15}s^2 + f_{11}s + f_{14}s = 0, \)

(v) \( f_{10} + f_{9}s + f_{8} = 0, f_{14} + f_{9} + f_{11} + f_{15}s = 0, f_{9} + f_{11} = 0, \)

(vi) \( f_{9}s + f_{8} = 0, f_{13}s = 0, f_{15}s = 0. \)

Similarly, \( f^{-td} = f \) gives the following relations:

(i) \( f_{9}ut + f_{14}ut + f_{11}ut + f_{12}t + f_{6}s + f_{10}us + f_{12}t + f_{2}u + f_{5}t = 0, \)

(ii) \( f_{9}t + f_{14}t + f_{11}t + f_{10}s = 0, f_{8}u + f_{13}ut + f_{12}s + f_{14}us = 0, \)

(iii) \( f_{13}t + f_{10}u + f_{15}t = 0, f_{9}u + f_{14}u + f_{12} = 0, 6. f_{10}u + f_{5}s + f_{13}t + f_{6} + f_{3}u + f_{8}u + f_{4} + f_{15}ut + f_{9}us = 0, \)

(iv) \( f_{14}u + f_{11}u + f_{12} + f_{15}us + f_{13}s = 0, f_{15}t + f_{14}s = 0, \)

(v) \( f_{15}t = 0, f_{14} = 0, f_{14} + f_{15}s = 0, f_{8} + f_{10} + f_{9}s + f_{15}t = 0, \)

Solving these equations yields \( f_i = 0 (i = 3, 6, 8, 9, 10, 11, 12, 13, 14, 15), f_{1}s = f_{5}t = stf' \) for some \( f' \in S_0 = k[u, s, t] \) and \( f_{4} = f_{5}s = f_{7}s = 0. \) Therefore, 
\[ S^{W_1(187)} = S_0 + (vt + bs^2 + bs + cs)S_0, \] whence \( S^{W_1} \) is Cohen-Macaulay.

\[ W_7 = \langle g, h \rangle: \] For \( f \in S^{W_7}, \) the condition \( f^g = f \) is equivalent to the following relations: \( f_3 = f_{15} = 0, f_9 = f_{11}, f_8 + f_9s = f_{10}, f_4 + f_5 + f_7s + f_8u = 0, f_1s = f_2u. \) These relations and the relations derived above (case \( W_1 \)) from \( f^h = f \) give us \( f_i = 0 (i = 6, 8, 9, 10, 11, 12, 13, 14, 15), f_5 = f_7, f_4 = sf_7, \) and \( f_1s = f_7t = f_2u = stuf' \) for some \( f' \in S_0. \) Therefore, \( f = f_0 + (vut + ast + \ldots) \)
\[ bs^2u + bvsu + cvsu \] \( f' \), and so \( S^{W_7} = S_0 + (vut + ast + bs^2u + bvsu + cvsu)S_0 \), as desired.

\[ W_9 = \langle g, -h \rangle: \] Let \( f \in S^{W_9} \). As in the case of \( W_7 \), the condition \( f^g = f \) amounts to following relations: \( f_3 = f_{15} = 0, f_9 = f_{11} \), \( f_8 + f_9s = f_{10} \), \( f_4 + f_6 + f_7s + f_8u = 0, f_1s = f_2u \). Similar relations result from \( f^{-h} = f \).

When combined with the above relations, they give \( f_i = 0 \) \( (i = 4, \ldots, 15) \) and \( f_3u = f_2s = usf \) for some \( f \in S_0 \). Thus \( f = f_0 + (uv + as)f' \) for some \( f' \in S_0 \).

Thus, as before, we conclude that \( S^{W_9} \) is Cohen-Macaulay.

\[ W_{10} = \langle -g, -h \rangle: \] For an invariant \( f \in S^{W_{10}} \), the condition \( f^{-g} = f \) amounts to the following relations: \( f_{13} = f_{15} = 0, f_9 + f_{11} + f_{14} = 0, f_8 + f_{10} + f_9s = 0, f_{12} + f_7 + f_5 = 0, f_9s + f_8 + f_3 = 0, f_4 + f_6 + f_5s = 0, f_6 = f_1 \). These relations and similar relations derived from \( f^{-h} = f \) give:

\[ f_i = 0 \] for \( i = 1, 3, 6, 8, 10, 11, 12, 13, 14, 15 \), \( f_7 = f_5, f_4 = f_5s, f_3t = f_2u \).

Therefore \( f = f_0 + (at + bsu + bvu + cvu)f' \). Since the index of \( W_{10} \) in \( G \) is 2, \( S^{W_{10}} \) is Cohen-Macaulay.

- It remains to consider the groups \( W_2, W_{11}, W_{13} \) and \( W_{14} \), where \( W_i = W_i(187) \). For this, we put

\[
g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .\]

The groups \( W_2, W_{11}, W_{13} \) and \( W_{14} \) are generated by \( \{g, -Id\}, \{g, -h\} \),
\(-g, -h\) and \(-g, h\) respectively. All of them are subgroups of the reflection group \(W_2(195)\) of order 16, whose ring of invariants (strictly) contains the polynomial subalgebra

\[ S_0 = k[r, s, t], \]

where \( r = a^2bc + (a^2bc)^{-1}, s = b + b^{-1} + c + c^{-1} \) and \( t = (b + b^{-1})(c + c^{-1}) \). Put \( u = a^2bc, v = b + b^{-1} \) and \( w = c + c^{-1} \). Then the ring \( S = k[a^\pm 1, b^\pm 1, c^\pm 1] \) is a free \( S_0 \)-module:

\[
S = S_0 + vS_0 + aS_0 + avS_0 + bS_0 + bvS_0 + cS_0 + cvS_0 + abS_0 \\
+ abvS_0 + acS_0 + acvS_0 + bcs_0 + bcvS_0 + abcS_0 + abcvS_0 + a^2bcS_0 \\
+ a^2bcvS_0 + a^3bcS_0 + a^3bcvS_0 + a^2b^2cS_0 + a^2b^2cvS_0 + a^2bc^2S_0 \\
+ a^2bc^2vS_0 + a^3b^2cS_0 + a^3b^2cvS_0 + a^3bc^2S_0 + a^3bc^2vS_0 \\
+ a^2b^2c^2S_0 + a^2b^2c^2vS_0 + a^3b^2c^2S_0 + a^3b^2c^2vS_0
\]

Thus any invariant \( f \in S^G \), where \( G \) is one of the 4 subgroups \( W_2, W_{11}, W_{13} \) or \( W_{14} \), can be written as follows:

\[
f = f_0 + vf_1 + af_2 + avf_3 + bf_4 + bvf_5 + cf_6 + cvf_7 + abf_8 + abvf_9 \\
+ acf_{10} + acvf_{11} + bcf_{12} + bcvf_{13} + abcf_{14} + abcvf_{15} \\
+ uf_{16} + uvf_{17} + uaf_{18} + uavf_{19} + ubf_{20} \\
+ ubvf_{21} + ucf_{22} + ucvf_{23} + uabf_{24} + uabvf_{25} \\
+ uacf_{26} + uacvf_{27} + ubcf_{28} + ubcvf_{29} + uabcf_{30} + uabcvf_{31}
\]
with \( f_i \in S_0 \). Our goal will be to show, for each of the groups \( G \in \{ W_2, W_{11}, W_{13}, W_{14} \} \) that the invariant algebra \( S^G \) can be generated as \( S_0 \)-module by 4 elements. Then, applying Lemma 5.1.3(a) with \( r = 32 \) and \( |G| = 8 \), we deduce that \( S^G \) is free over \( S_0 \), and hence \( S^G \) is Cohen-Macaulay.

\[
G = W_2 = \langle g, -I_{d_{3 \times 3}} \rangle: \text{ Suppose } f \in S^G. \text{ Using } f^g = f^{-1d} = f \text{ gives us the following relations:}
\]

(i) \( f_4 s + f_5 s^2 + f_1 s + f_5 t = 0, f_9 s + f_2 + f_8 = 0, \)
\( f_{13} t + f_4 + f_7 s + f_{13} s^2 + f_{12} s + f_6 = 0, \)

(ii) \( f_5 s + f_4 + f_6 = 0, f_{17} s + f_{21} t + f_{21} s^2 + f_{20} s = 0, \)

(iii) \( f_{29} s + f_{23} + f_{28} + f_{21} = 0, f_{23} + f_{21} = 0, f_5 + f_{12} + f_{13} s + f_7 = 0, \)
\( f_{15} s + f_8 + f_{14} = 0, f_{30} + f_{26} + f_{27} s = 0, \)

(iv) \( f_5 s + f_4 = 0, f_{10} + f_2 + f_3 s = 0, f_5 + f_7 = 0, f_{11} s + f_{10} + f_{14} = 0, \)
\( f_{24} + f_{25} s + f_{18} = 0, f_{13} s = 0, \)

(v) \( f_{22} + f_{28} s + f_{29} t + f_{23} s + f_{20} + f_{29} s^2 = 0, f_{20} + f_{21} s = 0, f_{31} + f_{27} = 0, \)

(vi) \( f_{21} s + f_{20} + f_{22} = 0, f_{31} + f_{25} = 0, f_{27} + f_{19} = 0, f_{11} + f_{15} = 0, f_{25} + f_{19} = 0, \)
\( f_{31} s + f_{30} + f_{24} = 0, f_{19} s + f_{26} + f_{18} = 0, f_{15} + f_9 = 0, \)

(vii) \( f_3 + f_{11} = 0, f_5 + f_9 = 0, f_{12} t + f_5 t + f_{23} r t + f_{22} r s + f_6 s + f_{28} r t + f_{16} r + \)
\( f_r t + f_{21} r t = 0, f_2 + f_{30} r^2 + f_{30} + f_{14} r = 0, \)
(viii) \[f_{28}rs + f_{12}r + f_{20}r + f_{13}t + f_{29}rt = 0, \quad f_{13}t + f_{22}r + f_{29}rt = 0, \quad f_{28}t + f_{21}t + f_{23}t + f_{22}s = 0, \quad f_{28} = 0, \quad f_{28} + f_{29}s = 0, \quad f_{12} + f_{28}r + f_{21}r = 0,\]

(ix) \[f_{10}r + f_{26}r^2 + f_{26} + f_8 = 0, \quad f_2 + f_{30} + f_{18}r = 0, \quad f_{13}t + f_{17}r + f_{20}r + f_6 + f_4 + f_5s + f_{22}r + f_{29}rt + f_{21}rs = 0, \quad f_{24} + f_{10} + f_{24}r^2 + f_8r = 0,\]

(x) \[f_{23}r + f_{28}r + f_{13}s + f_{12} + f_{29}rs = 0, \quad f_2r + f_{18}r^2 + f_{14} + f_{18} = 0, \quad f_{14} + f_{18} + f_{30}r = 0, \quad f_{28}r = 0, \quad f_{29}t + f_{21}s + f_20 + f_{22} = 0,\]

(xi) \[f_3 + f_{19}r + f_{31} = 0, \quad f_{25} + f_{11} + f_{27}r = 0, \quad f_9 + f_{27} + f_{25}r = 0, \quad f_{15} + f_{3}r + f_{19} + f_{19}r^2 = 0, \quad f_{31}r + f_{15} + f_{19} = 0, \quad f_{24} + f_{10} + f_{26}r = 0,\]

(xii) \[f_{24}r + f_{26} + f_8 = 0, \quad f_{27}r^2 + f_9 + f_{11}r + f_{27} = 0, \quad f_{11} + f_{9}r + f_{25}r^2 + f_{25} = 0, \quad f_{15}r + f_{3}1r^2 + f_{31} + f_3 = 0,\]

These equations imply that \(f_i = 0\) for \(i = 6, 12, 13, 16, 17, 20, 21, 22, 23, 28, 29\)

\[f_4 = f_5s = f_7s, \quad f_1s = f_5t, \quad f_3 = f_9 = f_{11} = f_{15} = f_{19}(r + 1), \quad f_{19} = f_{25} = f_{27} = f_{31}, \quad f_2 = f_{14} = f_{30}(r + 1), \quad f_{18} = f_{30}, \quad f_{24} = f_{26}, \quad f_{24}(r + 1) = f_8, \quad f_8 = f_{10}, \quad f_{3}s + f_2 + f_8, \quad f_{24} + f_{25}r + f_{18},\]

Solving this gives us,

\[f = f_0 + (tv + s^2b + svb + svc)x + (sac + srab + uva + sab + uvac + va + rva + uvab + bau + srac + uvabc + caus)y + (rac + rab + a + ac + rvabc + ab + uabc + ua + vab + rvab + vac + rva + vabc + uab + ra + abc + rabc + uac)z.\]

Therefore,
\[ S^{W_2} = S_0 + (tv + s^2b + svb + svc)S_0 + (sac + srb + uva + sab + uvac + va +
rv + uvab + baus + srac + uvabc + caus)S_0 + (rac + rab + a + ac + rvabc + ab +
uabc + ua + vab + rvab + vac + rvac + vabc + uab + ra + abc + rabc + uac)S_0. \]

Hence \( S^{W_2} \) is Cohen-Macaulay.

\[ G = W_{11} = \langle g, -h \rangle: \] Again, let \( f \in S^G \). From \( f^g = f^{-h} = f \) we get the following relations:

(i) \( f_4s + f_1s + f_5t + f_5s^2 = 0, f_2 + f_8 + f_9s = 0, f_{12}s + f_{13}s^2 + f_{13}t + f_6 + f_7s + f_4 = 0, \)

(ii) \( f_6 + f_5s + f_4 = 0, f_{20}s + f_{17}s + f_{21}s^2 + f_{21}t = 0, f_{10} + f_3s + f_2 = 0, \)

(iii) \( f_{20} + f_{29}s^2 + f_{22} + f_{29}t + f_{28}s + f_{23}s = 0, f_{21}s + f_{20} = 0, f_{20} + f_{21}s + f_{22} = 0, \)
\[ f_{18} + f_{25}s + f_{24} = 0, \]

(iv) \( f_{14} + f_8 + f_{15}s = 0, f_4 + f_5s = 0, f_5 + f_{13}s + f_7 + f_{12} = 0, f_7 + f_5 = 0, \)

(v) \( f_9 + f_3 = 0, f_{30} + f_{26} + f_{27}s = 0, f_{14} + f_{11}s + f_{10} = 0, f_{26} + f_{19}s + f_{18} = 0, \)
\[ f_{15} + f_9 = 0, \]

(vi) \( f_3 + f_{11} = 0, f_{19} + f_{25} = 0, f_{21} + f_{29}s + f_{28} + f_{23} = 0, f_{23} + f_{21} = 0, \)

(vii) \( f_{24} + f_{30} + f_{31}s = 0, f_{11} + f_{15} = 0, f_{25} + f_{31} = 0, \)

(viii) \( f_{27} + f_{19} = 0, f_{31} + f_{27} = 0, f_{29}s = 0, f_{13}s = 0, \)
(ix) \[ f_{17}rs + f_{12}t + f_t + f_23rt + f_5t + f_{13}st + f_{21}rt + f_1s + f_{20}rs + f_4s + f_{21}r^2 + f_{16}r + f_5s^2 + f_{29}rst + f_{28}rt = 0, \]
\[ f_{30}r^2 + f_2 + f_{31}s + f_{30} + f_{14}r + f_{31}r^2s + f_{15}s = 0, \]

(x) \[ f_6 + f_4 + f_{28}rs + f_7s + f_{22}r + f_{12}s + f_{23}rs + f_{13}s^2 + f_{13}t + f_{29}rs^2 + f_{29}rt = 0, \]
\[ f_{21}rs + f_4 + f_{20}r + f_6 + f_{13}t + f_{29}rt + f_5s = 0, \]
\[ f_{21}s + f_{29}st + f_{21}t + f_{28}t + f_{23}t + f_{21}s^2 + f_{20}s = 0, \]

(xi) \[ f_{27}r^2s + f_{10}r + f_{10} + f_{26}r^2 + f_{11}rs + f_{26} + f_{27}s = 0, \]
\[ f_{29}s^2 + f_{20} + f_{29}t + f_{22} + f_{28}s + f_{23}s = 0, \]
\[ f_{29}t + f_{20} + f_{21}s + f_{22} = 0, \]

(xii) \[ f_{29}t + f_{20} + f_{21}s + f_{22} = 0, \]
\[ f_{18} + f_{14} + f_{15}s + f_{30}r + f_{31}rs = 0, \]
\[ f_{24}r^2 + f_{25}s + f_8r + f_8 + f_9rs + f_{24} + f_{25}r^2s = 0, \]

(xiii) \[ f_{21}rs + f_{20}r + f_{22}r + f_{13}t + f_{29}rt + f_{17}r + f_6 + f_4 + f_5s = 0, \]
\[ f_{23}r + f_5 + f_7 + f_{28}r + f_{12} + f_{13}s + f_{29}rs = 0, \]

(xiv) \[ f_{28}r + f_5 + f_{21}r + f_{12} + f_7 = 0, \]
\[ f_{31} + f_{15}r + f_5 + f_3r^2 = 0, \]

(xv) \[ f_{30} + f_2 + f_{3}s + f_{19}rs + f_{18}r = 0, \]
\[ f_{19}s + f_{14} + f_{3}rs + f_{19}r^2s + f_{18} + f_{18}r^2 + f_{2}r = 0, \]
\[ f_{11}s + f_{26}r + f_{27}rs + f_{26} + f_{10} = 0, \]

(xvi) \[ f_9r + f_9 + f_{25} + f_{25}r^2 = 0, \]
\[ f_{27} + f_{27}r^2 + f_{11} + f_{11}r = 0, \]
\[ f_{19} + f_{15} + f_{31}r = 0, \]

(xvii) \[ f_{21} + f_{28} + f_{23} + f_{29} = 0, \]
\[ f_{21} + f_{23} + f_{28} = 0, \]

(xviii) \[ f_{24}r + f_{24} + f_{25}rs + f_9s + f_8 = 0, \]
\[ f_{15} + f_{19}r^2 + f_3r + f_{19} = 0, \]
(xix) \( f_{25} + f_{25} r + f_{9} = 0, f_{27} r + f_{11} + f_{27} = 0, f_{31} + f_{19} r + f_{5} = 0, f_{29} r = 0,\)

(xx) \( f_{29} rs + f_{13} s + f_{28} r = 0\)

Solving these equations we get, \( f_i = 0 \) for \( i = 6, 12, 13, 17, 20, 21, 22, 23, \) 28, 29, \( f_7 = f_5, f_4 = f_5 s, f_{18} = f_5 t, f_{18} = f_{16} r, f_{19} = f_{27} = f_{25} = f_{31}, \)

\( f_{19}(1 + r) = f_3 = f_9 = f_{11} = f_{15}, f_{10} = f_8, f_{26} = f_{24}, f_{30} = f_{18}, f_{14} = f_2, \)

\( f_2 + f_8 = f_{19}(1+r)s, f_{18} + f_{24} = f_{19} s, f_2 + f_{18}(1+r) = f_{19} s, f_8 + f_{24}(1+r) = f_{19} s. \)

This shows that

\[
f = f_0 + (vrt + cvrs + brs^2 + ust + burs)x + (ua + uab + uac + ar + abc + abr +
acr + ay + ab + uabc + ac + abcr)y + (uav + uabs + uacv + uabv + uav + avr +
as + abcs + abvr + uacs + abrs + abv + abcvr + acv + acvr + abc + av + acrs)z,
\]

where \( rstx = f_1 s = f_5 t = f_{16} r, y = f_{18} \) and \( z = f_{19} \) are in \( S_0. \)

In other words

\[
S^{W_{11}} = S_0 + (vrt + cvrs + brs^2 + ust + burs)S_0 + (ua + uab + uac + ar + abc +
abr + acr + ay + ab + uabc + ac + abcr)S_0 + (uav + uabs + uacv + uabv + uav + avr +
as + abcs + abvr + uacs + abrs + abv + abcvr + acv + acvr + abc + av + acrs)S_0.
\]

Hence \( S^{W_{11}} \) Cohen-Macaulay.

\[
G = W_{13} = \langle -g, -h \rangle; \text{ The conditions } f^{-g} = f^{-h} = f \text{ lead to the following relations:}
\]

(i) \( f_7 t + f_{16} r + f_{23} rt + f_1 s + f_{17} rs = 0, f_{11} rs + f_{27} r^2 s + f_{27} st + f_{10} r + f_{26} r^2 +
f_{26} t + f_2 = 0, \)
(ii) \( f_4 + f_6 + f_7 s + f_{23} rs + f_{22} r = 0, f_{29} r t + f_5 s + f_{13} t + f_{20} r + f_6 + f_{21} rs + f_4 = 0, \)

(iii) \( f_{17} s + f_{23} t = 0, f_6 + f_{17} r + f_{22} r = 0, f_{23} + f_{21} = 0, f_{23} + f_{21} + f_{28} = 0, \)

\[ f_{20} + f_{22} + f_{23} s = 0. \]

(iv) \( f_{28} r + f_{13} s + f_{29} rs = 0, f_{29} r = 0, f_{28} r + f_6 + f_7 + f_{21} r + f_{12} = 0, \)

(v) \( f_{18} + f_{10} + f_{27} rs + f_{26} r + f_{11} s = 0, f_{27} t + f_{27} r^2 + f_3 + f_{11} r = 0, f_{27} r + f_{11} + f_{19} = 0, \)

(vi) \( f_{30} t + f_{14} r + f_{10} + f_{31} s t + f_{16} r s + f_{30} r^2 + f_{31} r^2 s = 0, f_9 + f_{25} r + f_{31} = 0, \)

\[ f_{30} + f_{24} r + f_9 s + f_{25} rs + f_8 = 0, \]

(vii) \( f_{25} r^2 + f_{9} r + f_{25} t + f_{15} = 0, f_{14} + f_{25} r^2 s + f_9 rs + f_{25} s t + f_{24} t + f_{24} r^2 + f_8 r = 0, \)

(viii) \( f_{19} r + f_3 + f_{25} = 0, f_3 s + f_{19} rs + f_{24} + f_{18} r + f_2 = 0, f_{19} t + f_{19} r^2 + f_3 r + f_9 = 0, \)

(ix) \( f_{19} s t + f_{18} r^2 + f_3 rs + f_2 r + f_{19} r^2 s + f_8 + f_{18} t = 0, \)

(x) \( f_{15} + f_{27} + f_{31} r = 0, \)

\[ f_{14} + f_{30} r + f_{31} rs + f_{26} + f_{15} s = 0, \]

(xi) \( f_{15} r + f_{11} + f_{31} r^2 + f_{31} t = 0, f_{20} + f_{29} t + f_{21} s + f_{22} = 0, f_{23} r + f_{7} + f_{5} = 0, \)

(xii) \( f_7 t + f_{16} r + f_{23} rt + f_1 s + f_{17} rs = 0, \)

(xiii) \( f_{11} rs + f_{27} r^2 s + f_{27} st + f_{10} r + f_{26} r^2 + f_{26} t + f_2 = 0, f_4 + f_6 + f_7 s + f_{23} rs + f_{22} r = 0, \)
(xiv) \( f_{29}rt + f_5s + f_{13}t + f_{20}r + f_6 + f_{21}rs + f_4 = 0, f_{17}s + f_{23}t = 0, \)

(xv) \( f_6 + f_{17}r + f_{22}r = 0, f_{23} + f_{21} = 0, \)

(xvi) \( f_{23} + f_{21} + f_{28} = 0, f_{20} + f_{22} + f_{23}s = 0, f_{29}s = 0, f_{28}r + f_{13}s + f_{29}rs = 0, \)

(xvii) \( f_{29}r = 0, f_{28}r + f_5 + f_7 + f_{21}r + f_{12} = 0, \)

(xviii) \( f_{18} + f_{10} + f_{27}rs + f_{26}r + f_{11}s = 0, f_{27}t + f_{27}r^2 + f_3 + f_{11}r = 0, \)

(xix) \( f_{27}r + f_{11} + f_{19} = 0, f_{30}t + f_{14}r + f_{10} + f_{31}st + f_{15}rs + f_{30}r^2 + f_{31}r^2s = 0, \)

(xx) \( f_9 + f_{25}r + f_{31} = 0, f_{30} + f_{24}r + f_9s + f_{25}rs + f_8 = 0, \)

(xxi) \( f_{25}r^2 + f_9r + f_{25}t + f_{15} = 0, f_{14} + f_{25}r^2s + f_9rs + f_{25}st + f_{24}t + f_{24}r^2 + f_8r = 0, \)

(xxii) \( f_{19}r + f_3 + f_{25} = 0, f_{3}s + f_{19}r + f_{24} + f_{18}r + f_2 = 0, \)

(xxiii) \( f_{19}t + f_{19}r^2 + f_3r + f_9 = 0, f_{19}st + f_{18}r^2 + f_3rs + f_2r + f_{19}r^2s + f_8 + f_{18}t = 0, \)

(xxiv) \( f_{15} + f_{27} + f_{31}r = 0, f_{14} + f_{30}r + f_{31}rs + f_{26} + f_{15}s = 0, f_{15}r + f_{11} + f_{31}r^2 + f_{31}t = 0, \)

(xxv) \( f_{20} + f_{29}t + f_{21}s + f_{22} = 0, f_{23}r + f_7 + f_5 = 0, f_{22} = 0, \)

Which shows that,

\[
 f_i = 0 \text{ for } i = 2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22, 24, 25, 26, 27 , 28, 29, 30, 31. \text{ and } f_7 + f_{16}r + f_{1}s = 0, f_{21} + f_{23} = 0, f_{20} = f_{21}s, f_{17}s + f_{21}t = 0, f_6 + f_{17}r = 0, f_5 + f_7 + f_{21}r = 0.
\]
Thus $S^{W_{13}}$ is generated by four generators, and so it is Cohen-Macaulay.

\[ G = W_{14} = \langle -g, h \rangle \]: Comparing $f^{-g} = f^h = f$ for any $f \in S^G$, we find the following relations:

(i) \[ f_7t + f_{23}rt + f_{16}r + f_1s + f_{17}rs = 0. \]

(ii) \[ f_{26}r^2 + f_{27}st + f_{27}r^2s + f_{10}r + f_{26}t + f_{11}rs - f_2 = 0, \] \[ f_7s + f_6 + f_{22}r - f_4 + f_{23}rs = 0. \]

(iii) \[ f_6 + f_{20}r + f_{21}rs + f_{29}rt + f_4 + f_5s + f_{13}t = 0. \]

(iv) \[ f_{17}s + f_{23}t = 0, f_{22}r + f_6 + f_{17}r = 0, f_{22} = 0. \]

(v) \[ f_5 + f_7 + f_{23}r = 0, f_{20} + f_{29}t + f_{21}s = 0. \]

(vi) \[ f_{23}s - f_{20} = 0, f_3r + f_9 + f_{19}t + f_{19}r^2 = 0. \]

(vii) \[ f_{31}r^2 + f_{15}r + f_{31}t - f_{11} = 0, f_{27} + f_{15} + f_{31}r = 0, f_7 + f_5 + f_{21}r + f_{12} + f_{28}r = 0. \]

(viii) \[ f_{2r} + f_{18}t - f_8 + f_3rs + f_{18}r^2 + f_{19}st + f_{19}r^2s = 0, f_{30}r^2 + f_{14}r - f_{10} + f_{30}t + f_{15}rs + f_{31}r^2s + f_{31}st = 0, f_{24}t - f_{14} + f_{24}t^2 + f_8r + f_9rs + f_{25}r^2s + f_{25}st = 0, \]

(ix) \[ f_{15}s + f_{31}rs + f_{30}r - f_{26} + f_{14} = 0, f_{19} + f_{11} + f_{27}r = 0. \]

(x) \[ f_{26}r - f_{18} + f_{27}rs + f_{10} + f_{11}s = 0, f_{11}r + f_{27}t - f_3 + f_{27}r^2 = 0, f_{29} = 0. \]

(xi) \[ f_9 + f_{25}r - f_{31} = 0, f_{30} + f_8 + f_{24}r + f_{25}rs + f_9s = 0, f_{23} + f_{21} + f_{28} = 0. \]
(xii) $f_{15} + f_9 r + f_{25} r^2 + f_{25} t = 0, f_{13} s + f_{28} r + f_{29} rs = 0.$

(xiii) $f_{25} + f_{19} r + f_5 = 0, f_{18} r + f_5 s + f_2 - f_{24} + f_{19} rs = 0.$ $f_{21} + f_{23} = 0,$

These combined with $f^h = f$ gives,

$f_i = 0$ for $i = 1, 3, 6, 9, 11, 12, 13, 15, 17, 19, 20, 21, 22, 23, 25, 27, 28, 29, 31.$

$f_4 = f_5 s = sf_r = 0, f_7 t = f_{10} r, f_8 = f_{30} + f_{24} r = 0, f_2 = f_{18} r + f_{24} = 0,$

$f_{18} t = f_{30}, f_{26} = f_{24} t,$

$f_{26} r = f_{10} + f_{30} t, f_{30} r = f_{26} + f_{14} = 0, f_{26} r = f_{18} + f_{10} = 0$

which shows that

$f = f_0 + (c v r + u t + b r s + b v r) x + (u a b + a + u a c t) y + (u a b c t + a r) z$

Thus $S^W_{14} = S_0 + (c v r + u t + b r s + b v r) S_0 + (u a b + a + u a c t) S_0 + (u a b c t + a r) S_0$

has 4 generators as $S_0$-module, and hence $S^W_{14}$ is Cohen-Macaulay.

\[ \square \]

5.7 Groups of order 16 and 48

There are 2 non-conjugate subgroups of order 16 and 3 non-conjugate subgroups of order 48.

**Proposition 5.7.1.** If $G \subset GL_3(\mathbb{Z})$ is a group of order 16 or 48, then $S^G$ is Cohen-Macaulay.

*Proof.* All groups of order 16, the conjugates of the groups $W_1(194)$ and
$W_2(195)$, are reflection groups. Hence their invariant algebras are Cohen-Macaulay by Proposition 5.1.2.

If $G$ is of order 48, then it has a 2-Sylow subgroup of order 16. In view of Corollary 4.3.2, this implies that $S^G$ is Cohen-Macaulay.
REFERENCES


