THE COHEN-MACAULAY PROPERTY OF MULTIPLICATIVE INVARIANTS

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
Doctor Of Philosophy

---

by
Jawahar Pathak
May, 2003

ABSTRACT

THE COHEN-MACAULAY PROPERTY

OF MULTIPLICATIVE INVARIANTS

Jawahar Pathak

Doctor Of Philosophy

Temple University, May, 2003

Professor Martin Lorenz, Chair

In this dissertation we study the following general problem:

> *Suppose S is a Cohen-Macaulay ring and G is a finite group acting by automorphisms on S. When is the subring of G-invariants S^G again Cohen-Macaulay?*

The Cohen-Macaulay condition is one of the central notions in commutative algebra; it will be discussed in detail in Chapter 2 below. Our main focus in this thesis is on the case where $S = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial algebra and G acts by k-algebra automorphisms on S in such a way that each variable x_i is sent to a monomial $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ depending on i and the acting group element from G. Actions of this type are known as multiplicative actions. (Occasionally, they are also called "purely monomial" or "exponential" actions; see, e.g., [Bou$_2$].) Article [LP] initiated the investigation of the

Cohen-Macaulay property of multiplicative invariants. In Chapters 3 and 4, we give an account of [LP] with some simplifications. In Chapter 5, we use this material and direct computations to determine all multiplicative invariants in dimension 3 that are Cohen-Macaulay, with the exception of two cases that remain open. This classification is our main contribution to the Cohen-Macaulay problem for invariant rings.

ACKNOWLEDGEMENTS

I take this opportunity to convey my sincere thanks to my advisor Professor Martin Lorenz. He introduced me to the invariant theory, showed me the finer elements and walked me through the subject. His well organized way of teaching, presentation of high quality subject matter, and dedication to his students is a great inspiration for me. Without his help this dissertation would not have come to exist.

I thank all of the faculty members at the Temple University for teaching me graduate courses, especially Professor Boris Datskovsky for Algebraic Number Theory. I thank all the graduate students for their valuable discussions. Among many are Ibrahim Al-Rasasi, Marc Renault, Mathias Beck and Mohammed Tessema. I would also like to thank Jaiwant Mulik from the CIS department for help with LaTeX and XY-pic.

My research was supported in part by the National Science Foundation through a Research Assistantship under grant DMS-9988756 (PI M. Lorenz). I thank NSF for enabling me to concentrate most of my efforts on this research.

Finally, I thank my wife Pallavi and my daughter Noopur for supporting me throughout my life. Their love has made positive impact on my research.

To My Parents

TABLE OF CONTENTS

CHAPTER 1

Introduction

When a finite group G acts on a commutative Noetherian ring S, many properties of S are inherited by the ring of invariant elements

$$S^G = \{s \in S \mid g(s) = s \text{ for all } g \in G\} \ .$$

For example, if S is a normal domain, then so is S^G. In this dissertation we try to understand to what extent the Cohen-Macaulay property passes from S to S^G.

In the classical literature of invariant theory, S is a polynomial algebra $k[x_1, \ldots, x_n]$ over a field k and the group G acts by sending each variable x_i to a k-linear combination of all variables. This type of action is known as a *linear action*; these actions provide the setting of the ground-breaking works of D. Hilbert and E. Noether. A crucial feature of linear actions is that $S = k[x_1, \ldots, x_n]$ is equipped with a non-trivial grading (by "total degree") which is preserved under the G-action. This need not be the case for arbitrary group actions. One such instance is when G acts *multiplicatively*. For these actions, S is a Laurent polynomial algebra $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ (which is Cohen-Macaulay) and the elements of G act by sending each variable x_i to a monomial $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ depending on i and the acting group element from G; see, e.g., Lorenz [Lo$_1$, Lo$_2$, Lo$_3$] and Farkas [F$_1$, F$_2$, F$_3$].

The definition of Cohen-Macaulay rings is rather technical. It involves two constants, the *height* and the *depth*, each defined for an arbitrary ideal $\mathfrak{a}$ of a commutative Noetherian ring S. The notion of height is essentially topological; it is related to the dimension of the set

$$V(I) = \{\mathfrak{p} \mid \ \mathfrak{p} \text{ is a prime ideal of } S \text{ with } \mathfrak{p} \supseteq \mathfrak{a}\} \ ,$$

endowed with the so-called Zariski topology. On the other hand, depth is purely algebraic and measures the length of a maximal regular sequence contained in $\mathfrak{a}$; for detailed definitions, see Chapter 2 below. Depth is always bounded above by height. If equality holds for all ideals $\mathfrak{a}$, then the ring S is called Cohen-Macaulay. In 1916 Macaulay [M] showed that polynomial algebras over a field are Cohen-Macaulay. Later in 1945, Cohen [Co] generalized this result and proved that regular local rings are Cohen-Macaulay. Subsequently, it turned out that Cohen-Macaulay rings form a rather wide class of commutative Noetherian rings which nevertheless enjoys some very desirable properties. The equality of an algebraic and a topological constant, as postulated in the definition of Cohen-Macaulay rings, allows in particular for a powerful dimension theory.

Returning to rings of invariants, if the order of G is invertible in S and S is Cohen-Macaulay then S^G is easily seen to be Cohen-Macaulay as well. Also, $\text{height}(\mathfrak{a}) = \text{height}(\mathfrak{a} \cap S^G)$ holds in this case, for any ideal $\mathfrak{a}$ in S. Thus, the Cohen-Macaulay property of invariant rings S^G is problematic only when the order of G is *not* invertible in S; this is called the *modular case*. Some depth estimates for modular actions were found by Ellingsrud and Skjebred [ES] (when G is a p-group) and by G. Kemper [Ke_1].

The main problem addressed in this thesis, stated specifically, is as follows:

> *Suppose $S = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial algebra over a field k of characteristic p and G is a finite group acting multiplicatively on S such that p divides the order $|G|$. When is the invariant algebra S^G Cohen-Macaulay?*

The dissertation is divided into 4 chapters. In Chapter 2, we give a brief survey of background material from commutative algebra and homological algebra. In Chapter 3, we define trace modules and obtain a height formula. The main result here are certain depth estimates; see in particular Lemma 3.4.2. Using this result, we give an alternative proof of some of Kemper's results in Section 3.5. Chapter 4 is devoted specifically to multiplicative actions. Lemma 4.1.3 reduces the Cohen-Macaulay problem for multiplicative invariants to the special case of effective actions. This Lemma is as yet unpublished and is due to Lorenz. The material in Chapters 3 and 4 is a simplified version of [LP]. In Chapter 5 we have classified the multiplicative invariant rings of dimension 3 in Cohen-Macaulay and non-Cohen-Macaulay classes. We use the classification list from [T] and use the notation $W_i(n)$ for the group W_i on the page n in the following results:

> If $\operatorname{char} k \neq 2$ then all multiplicative invariants $k[A]^G$ in dimension 3 are Cohen-Macaulay. If $\operatorname{char} k = 2$, then, with exception of the groups conjugate to one of the following
>
> - $W_5(173)$ (order 2),
> - $W_2(174)$ and $W_4(174)$ (both cyclic of order 4),
> - $W_3(184)$ and $W_4(184)$ (both cyclic of order 6),
>
> and the possible exception of the conjugates of
>
> $$W_{14}(174)\ (\simeq C_2 \times C_2) \quad \text{and} \quad W_{10}(191)\ (\simeq A_4)$$
>
> all finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$ have Cohen-Macaulay multiplicative invariant algebras.

CHAPTER 2

Basic Concepts

In this chapter, we will develop the basic concepts and set up notations. Our main focus will be on the Cohen-Macaulay property of a module of invariant elements under the action of a finite group. Section 2.1 contains the necessary background from commutative algebra. In Section 2.2, we recall the definition of Cohen-Macaulay modules over a commutative Noetherian ring using the notion of depth and height. In Section 2.3, we consider actions of a finite group G on an arbitrary commutative Noetherian ring S and study the height and depth of modules over the associated skew group ring $S*G$. In Section 2.4, we also recall the fundamentals of spectral sequences, an important tool to be used in later chapters.

2.1 Background from commutative algebra

Throughout we assume that S is a commutative Noetherian ring with identity and M is a finitely generated unitary left S-module. Most of the material of this section can be found in [Bou$_1$] and in [E]. We also follow terminology and notations used in these references, and make an explicit mention if any change is made.

Definition 2.1.1. Let R be a subring of S. An element $s \in S$ is called *integral* over R if s satisfies a monic polynomial with coefficients in R. The set of all elements of S that are integral over R forms a subring $\bar{R}$ of S which is called the *integral closure* of R in S. If $S = \bar{R}$ then we say that $R \hookrightarrow S$ is an *integral extension*. On the other hand, if $R = \bar{R}$ then we say that R is *integrally closed* in S. A *normal domain* is an integral domain which is integrally closed in its own quotient field.

If S is finitely generated R-algebra which is integral over R, then it is easy to show that S is a finitely generated module over R. The converse is also true, i.e. if S is a finite R-module then the extension is integral. The following lemma is a collection of standard results on integral extensions.

Lemma 2.1.2. *Suppose $R \hookrightarrow S$ is integral extension. Then the following statements hold:*

(a) (Lying over) *Given a prime $\mathfrak{p} \subset R$, there exist a prime ideal $\mathfrak{P} \subset S$ such* *that* $\mathfrak{P} \cap R = \mathfrak{p}$. [Bou$_1$, ch V.2.2, Th.1, p. 328].

(b) (Incomparability) *If* $\mathfrak{P} \subset \mathfrak{Q}$ *are two prime ideals of* S *lying over a prime* $\mathfrak{p}$ *of* R*, then* $\mathfrak{P} = \mathfrak{Q}$ [E, cor. 4.18 on p. 131].

(c) (Going up) *Suppose* $\mathfrak{b}$ *is an ideal of* S *and* $\mathfrak{a} = \mathfrak{b} \cap R$*. Then for any prime ideal* $\mathfrak{p} \subset R$ *containing* $\mathfrak{a}$*, there exists a prime ideal* $\mathfrak{P} \subset S$ *lying over* $\mathfrak{p}$ *which contains* $\mathfrak{b}$. [Bou$_1$, ch. V.2.1, cor. 2, p. 328].

(d) (Going down) *If, in addition,* S *is an integral domain and* R *is normal, then given prime ideals* $\mathfrak{p} \subset \mathfrak{q}$ *in* R *and a prime* $\mathfrak{Q}$ *in* S *lying over* $\mathfrak{q}$*, there exists a prime* $\mathfrak{P} \subset \mathfrak{Q}$ *such that* $\mathfrak{P} \cap R = \mathfrak{p}$. [E, Theorem 13.9, p. 204] .

The *annihilator* of M is an ideal in S which is denoted by $\mathrm{ann}_S(M)$. So $\mathrm{ann}_S(M) = \{a \in S \mid aM = 0\}$. If R is a subring of S, then any S-module is also an R-module. It is clear that $\mathrm{ann}_R(M) = \mathrm{ann}_S(M) \cap R$. The following lemma is an easy consequence of the Cayley-Hamilton theorem [E, Theorem 4.3.7]. We give a quick self-contained proof.

Lemma 2.1.3. *For any ideal* $\mathfrak{a}$ *of* S*,* $\mathfrak{a} + \mathrm{ann}_S(M) = S$ *iff* $\mathfrak{a}M = M$.

Proof. If $\mathfrak{a} + \mathrm{ann}_S(M) = S$, then $M = SM = \mathfrak{a}M + \mathrm{ann}_S(M)M = \mathfrak{a}M$. To prove the converse, suppose $\mathfrak{a}M = M$. Choose a set of generators $m_1, \dots, m_n$ of M. Now for each j, we have $m_j = \sum_{i=1}^{n} \lambda_{ij} m_i$, $\lambda_{ij} \in \mathfrak{a}$. Therefore

$(1-\lambda_{nn})m_n = \sum_{i=1}^{n-1} \lambda_{in} m_i$. Thus $(1-\lambda_{nn})M$ has less than n generators. Inductively we can find $a \in \mathfrak{a}$ such that $(1-a)M = 0$. Hence $1-a \in \operatorname{ann}_S(M)$. Therefore $\mathfrak{a} + \operatorname{ann}_S(M) = S$. $\square$

Definition 2.1.4. The *height* of a prime ideal $\mathfrak{p}$ of S is the maximal integer r such that there exist a strictly ascending chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$. For an arbitrary ideal $\mathfrak{a}$, $\operatorname{height}(\mathfrak{a}) = \inf\{\operatorname{height}(\mathfrak{p}) \mid \mathfrak{p}$ is a prime containing $\mathfrak{a}\}$. (If $\mathfrak{a} = S$ then $\operatorname{height}(\mathfrak{a}) = \infty$.) The height of an ideal $\mathfrak{a}$ in a module M is denoted by $\operatorname{height}(\mathfrak{a}, M)$; it is defined to be the height of the ideal $(\mathfrak{a} + \operatorname{ann}_S(M))/\operatorname{ann}_S(M)$ in the ring $S/\operatorname{ann}_S(M)$. Thus, by Lemma 2.1.3, if $\mathfrak{a}M = M$ then $\operatorname{height}(\mathfrak{a}, M) = \infty$.

Definition 2.1.5. A prime ideal $\mathfrak{p}$ of S is called an *associated prime* of the module M if $\mathfrak{p} = \operatorname{ann}_S(m)$ for some nonzero element $m \in M$.

The set of associated primes of M is denoted by $\operatorname{Ass}(M)$. When we are dealing with more than one ring, we will employ $\operatorname{Ass}_S(M)$ instead. In the following proposition we record two important properties of $\operatorname{Ass}(M)$; the proof can be found in [E, Theorem 3.1]. We will denote the union of the associated primes of M by $\cup \operatorname{Ass}(M)$.

Proposition 2.1.6. *Let M be a finitely generated module over a Noetherian ring, then $\operatorname{Ass}(M)$ is a finite set and $\cup \operatorname{Ass}(M) = \{x \in S \mid xm = 0$ for some $m \in M$, $m \neq 0\}$.*

The elements of $\cup \operatorname{Ass}(M)$ are called the *zero divisors* on M. We end this section with a lemma from [E, Lemma 3.3, p. 90] known as the "prime avoidance lemma".

Lemma 2.1.7. *Suppose $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \mathfrak{b}$ are ideals of a commutative ring S and suppose that $\mathfrak{b} \subset \cup_j \mathfrak{a}_j$. If at most two of the $\mathfrak{a}_j$ are not prime, then $\mathfrak{b}$ is contained in one of the $\mathfrak{a}_j$.*

2.2 Cohen-Macaulay modules

The material in this section is partly from [BH]. We continue to assume that S is a commutative Noetherian ring and M is a finitely generated left S-module.

Definition 2.2.1. An element $x \in S$ is called *M-regular* if x is not a zero divisor on M, that is, for any nonzero element m of M, $xm \neq 0$. A sequence $\{\underline{x}\} = \{x_1, x_2, \ldots, x_r\}$ is called *M-regular sequence* of length r, if

(i) $\sum_{i=1}^{r} x_i M \neq M$.

(ii) x_i is $M/(\sum_{j=1}^{i-1} x_j M)$-regular for $1 \leq i \leq r$, where the empty sum denotes the 0-module.

A *maximal* M-regular sequence in an ideal $\mathfrak{a}$ is an M-regular sequence $\{\underline{x}\} = \{x_1, x_2, ..., x_r\}$ that is a subset of $\mathfrak{a}$ and such that $\{x_1, x_2, \ldots, x_r, x_{r+1}\}$ is not an M-regular sequence, for any $x_{r+1} \in \mathfrak{a}$. For convenience, we will denote the submodule $\sum_{i=1}^{n} x_i M$ by $(\underline{x})M$.

Remark 1. If $\{\underline{x}\} = \{x_1, \ldots, x_r\}$ is M-regular, then so is $\{\underline{x^n}\} = \{x_1^n, \ldots, x_r^n\}$ for any positive integer n; see [BH, Exercise 1.1.10(b)]. Many other interesting properties of regular sequences can be found in [BH]. We make one more useful remark about maximal M-regular sequences. Suppose $\{\underline{x}\}$ is a maximal M-regular sequence in an ideal $\mathfrak{a}$. Then for any element a of $\mathfrak{a}$, there exists a nonzero element $\bar{m} \in M/(\underline{x})M$ such that $a\bar{m} = 0$. In other words, $\mathfrak{a}$ consists of zero divisors of $M/(\underline{x})M$. Applying Proposition 2.1.6, we get $\mathfrak{a} \subset \cup \operatorname{Ass}(M/(\underline{x})M)$. Hence, by the Lemma 2.1.7, we conclude that $\mathfrak{a} \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M/(\underline{x})M)$.

Definition 2.2.2. Let $\mathfrak{a}$ be any ideal of S. If $\mathfrak{a}M \neq M$, then we define the *depth* of $\mathfrak{a}$ on M to be $\sup\{r \mid \mathfrak{a}\,\text{contains an } M\text{-regular sequence of length } r\}$. If $\mathfrak{a}M = M$ then we say that the depth of $\mathfrak{a}$ on M is infinite. The depth of $\mathfrak{a}$ on M is denoted by $\operatorname{depth}(\mathfrak{a}, M)$.

In literature the depth is also called the grade of an ideal, e.g. see [BH]. Depth has a homological characterization in terms of “Ext”, which we state in part (b) of the following theorem. For the proof of (b), see [BH, Theorem 1.2.5]; for (a) and (c), see [BH, Exercise 1.2.22(a)] and [BH, p. 10], respectively.

Theorem 2.2.3. (a) *For any finitely generated S-module M and an ideal $\mathfrak{a}$*

of S *,* $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{height}(\mathfrak{a}, M)$.

(b) *Suppose that* $\mathfrak{a}M \neq M$*. Then the length of any maximal* M*-regular sequence in* $\mathfrak{a}$ *depends on* $\mathfrak{a}$ *and* M *only and is equal to the number* n *such that* $\operatorname{Ext}^i_S(S/\mathfrak{a}, M) = 0$ *for* $i < n$ *and* $\operatorname{Ext}^n_S(S/\mathfrak{a}, M) \neq 0$.

(c) *If* $\mathfrak{a}M = M$ *then* $\operatorname{Ext}^i_S(S/\mathfrak{a}, M) = 0$ *for all* i.

The characterization in (b) shows that all maximal M-regular sequences in a fixed ideal have same length. Parts (b) and (c) also give the formula

$$\operatorname{depth}(\mathfrak{a}, M) = \inf_i \{ i \mid \operatorname{Ext}^i_S(S/\mathfrak{a}, M) \neq 0 \} ,$$

where $\inf \varnothing = \infty$. In view of the beginning of Remark 1, it is clear that $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{depth}(\sqrt{\mathfrak{a}}, M)$. Even more is true:

Lemma 2.2.4. *For any ideal* $\mathfrak{a}$ *of* S*, there exists a prime ideal* $\mathfrak{p}$ *containing* $\mathfrak{a}$ *such that* $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{p}, M)$

Proof. If $\mathfrak{a}M = M$, then any prime ideal containing $\mathfrak{a}$ would serve. Assume therefore that $\operatorname{depth}(\mathfrak{a}, M)$ is finite, say $\operatorname{depth}(\mathfrak{a}, M) = n$. Let $\{x_1, \ldots, x_n\}$ be a maximal M-regular sequence in $\mathfrak{a}$. By Remark 1, there exists a prime $\mathfrak{p} \in \operatorname{Ass}_S(\bar{M})$ such that $\mathfrak{a} \subset \mathfrak{p}$, where $\bar{M} = M/(\underline{x})M$. By definition of associated primes, $\mathfrak{p} = \operatorname{ann}_S(\bar{m})$ for some nonzero element $\bar{m}$ of $\bar{M}$. But then $x_1, \ldots, x_n$ is a maximum M-regular sequence in $\mathfrak{p}$. Thus $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{p}, M)$. $\square$

Recall that always $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{height}(\mathfrak{a}, M)$. The following definition stresses on the equality of these two constants.

Definition 2.2.5. The module M is called *Cohen-Macaulay* if $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{height}(\mathfrak{a}, M)$ holds for all the ideals $\mathfrak{a}$ of S. The ring S is Cohen-Macaulay if S is Cohen-Macaulay as a module over itself.

The following lemma states that the Cohen-Macaulay is a local property is local.

Lemma 2.2.6. *The following are equivalent for the S-module M:*

(i) *M is Cohen-Macaulay;*

(ii) $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{height}(\mathfrak{m}, M)$ *holds for all maximal ideals $\mathfrak{m}$ of S containing* $\operatorname{ann}_S(M)$*;*

(iii) *all localizations $M_{\mathfrak{m}}$ of M at the maximal ideals $\mathfrak{m}$ of S are Cohen-Macaulay as $S_{\mathfrak{m}}$-modules.*

Proof. The implication (i) $\Rightarrow$ (iii) is contained in [BH, Theorem 2.1.3(b)], and (iii) $\Rightarrow$ (ii) follows from the equalities $\operatorname{height}(\mathfrak{m}, M) = \operatorname{height}(\mathfrak{m}S_{\mathfrak{m}}, M_{\mathfrak{m}})$ and $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}(\mathfrak{m}S_{\mathfrak{m}}, M_{\mathfrak{m}})$; see [BH, Proposition 1.2.10(a)]. Finally, for (ii) $\Rightarrow$ (i), see [E, Ex. 18.4]. $\square$

We now turn to the important special case where S is an affine domain over a field k. Then, by Noether normalization [E, Theorem A1 on p. 221], there exists a subset $\{x_1, \ldots x_n\}$ of S which is algebraically independent over k and so that $k[x_1, \ldots, x_n] \hookrightarrow S$ is a finite extension. The following proposition gives what is probably the most accessible description of affine Cohen-Macaulay domains. The freeness property described therein has many important applications of Cohen-Macaulay domains, e.g. to generating functions in combinatorics.

Proposition 2.2.7. *Let S be any affine domain over a field k. Then S is Cohen-Macaulay if and only if S is a finite free module over some polynomial subalgebra $k[\underline{x}] = k[x_1, \ldots, x_n] \subseteq S$. In this case, S is free over any polynomial subalgebra $k[\underline{x}] \subseteq S$ such that S is finite over $k[\underline{x}]$.*

Proof. Suppose S is any affine domain over a field k. Let $k[\underline{x}] \subseteq S$ be any polynomial subalgebra such that S is finite over $k[\underline{x}]$. By [Lo$_3$, Lemma in Section 1.3], S is a projective module over $k[\underline{x}]$, and hence S is free over $k[\underline{x}]$, by the Quillen-Suslin theorem. Conversely, suppose that S is finite and free over some polynomial subalgebra $k[\underline{x}]$. Then S is faithfully flat over $k[\underline{x}]$, and $k[\underline{x}]$ is Cohen-Macaulay. Moreover, for all primes $\mathfrak{p}$ of S, the fibre $S_{\mathfrak{p}}/(\mathfrak{p} \cap k[\underline{x}])S_{\mathfrak{p}}$ has Krull dimension 0, by Lemma 2.1.2(b), and hence it is Cohen-Macaulay. Therefore, [BH, Exercise 2.1.23] implies that S is Cohen-Macaulay as well. $\square$

2.3 Group actions

The main background references for this section are [Bou_1] and [Be_1]. Let S be any commutative ring. The set $\mathrm{Aut}(S)$ of all the ring automorphisms of S forms a group under composition.

Definition 2.3.1. We say that a finite group G acts on S if there is a group homomorphism $G \to \mathrm{Aut}(S)$. If this homomorphism is injective then the G-action on S is called *faithful*. We will write $g(s)$ for image of an element $s \in S$ under the action of a group element $g \in G$.

Classical invariant theory is primarily concerned with a specific form of group action on polynomial algebras $k[\underline{x}] = k[x_1, \ldots, x_n]$, called "linear action"; see the Introduction. Reference [Sm_1], for example, gives a detailed treatment of linear actions. There is another type of action which is our prime interest, called "multiplicative action". These actions have been briefly described in the Introduction and will be formally introduced in Chapter 4. Here is a simple example comparing a linear action and a multiplicative action in rank 2.

Example 2.3.2. Consider the linear action of

$$G = \left\{ Id = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ -Id = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}$$

on the polynomial algebra $k[x_1, x_2]$ over a field k of characteristic $\neq 2$ defined as follows: $-Id(x_1) = -x_1$ and $-Id(x_2) = -x_2$. Thus $-Id(f(x_1, x_2)) = f(-x_1, -x_2)$. Note that the elements such x_1^2,,x_1x_2 and x_2^2 are invariant (fixed) under the action of G. In fact, it is easy to see that these elements generate the algebra of invariants $k[x_1, x_2]^G$. Observe that, under this action, the degree of a polynomial is unchanged.

Now we consider a multiplicative action of the same group on the Laurent polynomial algebra $k[x_1^{\pm 1}, x_2^{\pm 1}]$. This action is defined by $-Id(x_1) = x_1^{-1}$ and $-Id(x_2) = x_2^{-1}$. Note that this action does not preserve degree. It is not hard to show that the invariant algebra $k[x_1^{\pm 1}, x_2^{\pm 1}]^G$ is generated by the elements $x_1 + x_1^{-1}$, $x_2 + x_2^{-1}$ and $x_1x_2^{-1} + x_1^{-1}x_2$ (in any characteristic).

Now assume that G is a finite group acting on the commutative ring S. The *skew group ring* $S * G$ that is associated with this action is defined as follows: $S * G$ is the free left S-module with basis G. Thus, every element of $S * G$ can be uniquely written in the form

$$\sum_{i=1}^{m} a_i g_i \quad \text{with } a_i \in S,\ g_i \in G.$$

The product in $S*G$ is defined by distributive extension of the rule $(ag)(bh) = ag(b)gh$. The following computation shows that $S * G$ is an associative ring: $((ag)(bh))ck = (ag(b)gh)(ck) = ag(b)gh(c)ghk) = ag(bh(c)hk) = ag((bh)(ck))$, where $g, h, k \in G$ and $a, b, c \in S$. If G acts trivially on S, then $S * G$ is the ordinary *group ring* and is denoted by $S[G]$. Now consider any left $S*G$-module

M and define the module of invariant elements by

$$M^G = \{m \in M \mid g(m) = m, \forall\, g \in G\}.$$

The ring of invariant elements S^G is defined likewise; indeed, S is a left $S*G$-module via $(ag)s = ag(s)$ for $a, s \in S$, $g \in G$. Elements of S can be viewed as elements of $S*G$ by the identification $s = s1$, where 1 is the identity element of G. With this in mind, let $a \in S^G$. Then for a generator $sg \in S*G$, $(sg)(a) = sg(a)(g1) = sag = a(sg)$. This shows that S^G is in the center of $S*G$, or $S*G$ is an S^G-algebra.

Clearly we have $S^G \hookrightarrow S$. In the following well-known lemma we show that this extension is integral. Once we have established this, we can apply lemma 2.1.2 to obtain properties such as lying over, going up etc. In fact, more is true in the present setting.

Lemma 2.3.3. *Let G be a finite group acting on S.*

(a) *$S^G \hookrightarrow S$ is an integral extension. Further, if S is normal domain, then so is S^G.*

(b) *If $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of S such that $\mathfrak{p} \cap S^G = \mathfrak{q} \cap S^G$, then there exist g in G such that $\mathfrak{p} = g(\mathfrak{q})$. In other words, G acts transitively on the set of prime ideals lying over a fixed prime ideal of S^G.*

(c) (Going down) *Suppose $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals of S^G and suppose $\mathfrak{Q}$ is* *a prime ideal in S lying over $\mathfrak{q}$. Then there exists a prime ideal $\mathfrak{P}$ in S lying over $\mathfrak{p}$ such that $\mathfrak{P} \subset \mathfrak{Q}$.*

Proof. (a) For any $s \in S$, $f(x) = \prod_{g \in G}(x - g(s))$ is a monic polynomial with coefficients in S^G that is satisfied by s, which shows that $S^G \hookrightarrow S$ is integral. For the second assertion, let α be an element of the quotient field of S^G which is integral over S^G. Then there exist $a, b \in S^G$ such that $\alpha = a/b$. Since $S^G \subset S$, α is integral over S. But S is a normal domain, therefore, $\alpha \in S$. Now a and b are invariant under G. Therefore α is invariant and hence belongs to S^G.

(b) Let $a \in \mathfrak{p}$. Then $\prod_{g \in G} g(a)$ is invariant under G and also contained in $\mathfrak{p}$. Therefore $\prod_{g \in G} g(a) \in \mathfrak{p} \cap S^G = \mathfrak{q} \cap S^G \subset \mathfrak{q}$. Now $\mathfrak{q}$ is a prime ideal, and so there exists $g \in G$ such that $g(a) \in \mathfrak{q}$. Thus $a \in g^{-1}(\mathfrak{q})$. Therefore, $\mathfrak{p} \subset \cup_{g \in G} g(\mathfrak{q})$. By Lemma 2.1.7, we conclude that $\mathfrak{p} \subseteq g(\mathfrak{q})$ for some $g \in G$. Since $\mathfrak{p} \cap S^G = \mathfrak{q} \cap S^G = g(\mathfrak{q}) \cap S^G$, Lemma 2.1.2(b) yields $\mathfrak{p} = g(\mathfrak{q})$.

(c) To prove this part, let $\mathfrak{P}'$ be any lift of $\mathfrak{p}$ in S. By going up part of Lemma 2.1.2, there exists a prime ideal $\mathfrak{Q}'$ in S containing $\mathfrak{P}'$ which lies over $\mathfrak{q}$. Then $\mathfrak{Q}$ and $\mathfrak{Q}'$ are both lifts of $\mathfrak{q}$. Therefore, by part (b), there exists $g \in G$ such that $g(\mathfrak{Q}') = \mathfrak{Q}$. Put $g(\mathfrak{P}') = \mathfrak{P}$. Then $\mathfrak{P} \cap S^G = g(\mathfrak{P}') \cap S^G = \mathfrak{p}$ and $\mathfrak{P} = g(\mathfrak{P}') \subset g(\mathfrak{Q}') = \mathfrak{Q}$. $\square$

For the remaining part of this chapter we will assume that S is Noetherian as S^G-module. This hypothesis is always satisfied when S is an affine commutative algebra over some commutative Noetherian ring k and G acts on S by k-algebra automorphisms; see [Bou$_1$, Theorem 2 in Chap. 5 §1]. Moreover, M will always denote a finitely generated left $S * G$-module. Hence, M is also finitely generated over S and over S^G. The following lemma relates the depth and heights of an ideal on M and M^G. This result is presumably known, but we could not find a proof in the literature. We present here an elegant proof which the author learned from his advisor.

Lemma 2.3.4. *With the above hypothesis we have the following:*

(a) *For any prime ideal* $\mathfrak{P}$ *of* S*,* $\operatorname{ann}_S(M) \subset \mathfrak{P}$ *iff* $\operatorname{ann}_{S^G}(M) \subset \mathfrak{P} \cap S^G$.

(b) *For any ideal* $\mathfrak{b}$ *of* S*,* $\operatorname{depth}(\mathfrak{b}, M) = \operatorname{depth}(\mathfrak{b} \cap S^G, M)$ *and* $\operatorname{height}(\mathfrak{b}, M) = \operatorname{height}(\mathfrak{b} \cap S^G, M)$.

Proof. (a) Set $\mathfrak{p} = \mathfrak{P} \cap S^G$. If $\operatorname{ann}_S(M) \subset \mathfrak{P}$, then clearly $\operatorname{ann}_{S^G}(M) \subset \mathfrak{p}$. For the converse, assume that $\operatorname{ann}_{S^G}(M) \subset \mathfrak{p}$. Let $\mathfrak{P}'$ be a lift of $\mathfrak{p}$ in S containing $\operatorname{ann}_S(M)$ (by Lemma 2.1.2(c), such a lift exists). Then there exists $g \in G$ such that $g(\mathfrak{P}') = \mathfrak{P}$. Since $g(\operatorname{ann}_S(M)) = \operatorname{ann}_S(M)$, $\mathfrak{P}$ contains $\operatorname{ann}_S(M)$.

(b) Set $\mathfrak{a} = \mathfrak{b} \cap S^G$. If $\mathfrak{a}M = M$, then so is $\mathfrak{b}M = M$. Therefore we can assume that $\operatorname{depth}(\mathfrak{a}, M)$ is finite. Clearly $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{depth}(\mathfrak{b}, M)$. To prove the reverse inequality, suppose $\operatorname{depth}(\mathfrak{a}, M) = n$. Let $\{x_1, \ldots, x_n\}$ be a maximal M-regular sequence in $\mathfrak{a}$. Then $\mathfrak{a}$ consists of zero divisors of $\bar{M} = M/(\underline{x})M$. By Proposition 2.1.6, $\mathfrak{a} \subset \cup_{\mathfrak{Q} \in \operatorname{Ass}_S(\bar{M})} \mathfrak{Q}$. Thus $\mathfrak{a} \subset \cup_{\mathfrak{Q} \in \operatorname{Ass}_S(\bar{M})} \mathfrak{Q} \cap$

S^G. Now we apply Lemma 2.1.7 to find $\mathfrak{Q} \in \mathrm{Ass}_S(\bar{M})$ such that $\mathfrak{a} \subset \mathfrak{Q} \cap S^G$. By definition, $\mathfrak{Q} = \mathrm{ann}_S(\bar{m})$ for some nonzero element $\bar{m} \in \bar{M}$. But then $x_1, \ldots, x_n$ is a maximal M-regular sequence in $\mathfrak{Q}$; so $n = \mathrm{depth}(\mathfrak{Q}, M)$, by Theorem 2.2.3(b). By going up, there exist a prime $\mathfrak{P}$ in S containing $\mathfrak{b}$ such that $\mathfrak{Q} \cap S^G = \mathfrak{P} \cap S^G$. As before, there exists $g \in G$ such that $g(\mathfrak{P}) = \mathfrak{Q}$. Hence, $\mathrm{depth}(\mathfrak{a}, M) = \mathrm{depth}(\mathfrak{Q}, M) = \mathrm{depth}(\mathfrak{P}, M) \geq \mathrm{depth}(\mathfrak{b}, M)$.

Now for height. We first assume that $\mathfrak{a}$ and $\mathfrak{b}$ both are prime ideals in S^G and S respectively and $\mathrm{ann}_S(M) \subset \mathfrak{b}$ (and hence, $\mathrm{ann}_{S^G}(M) \subset \mathfrak{a}$). Our goal is to show that $\mathrm{height}(\mathfrak{a}, M) = \mathrm{height}(\mathfrak{b}, M)$, that is, $\mathrm{height}(\mathfrak{a}/\mathrm{ann}_{S^G}(M)) = \mathrm{height}(\mathfrak{b}/\mathrm{ann}_S(M))$. But Lemma 2.1.2(b) (Incomparability), applied to the integral extension $S^G/\mathrm{ann}_{S^G}(M) \hookrightarrow S/\mathrm{ann}_S(M)$, gives $\mathrm{height}(\mathfrak{b}/\mathrm{ann}_S(M)) \leq \mathrm{height}(\mathfrak{a}/\mathrm{ann}_{S^G}(M))$. The reverse inequality follows from Lemma 2.1.2(c) (Going Up) together with part(b) of Lemma 2.3.3.

For the general case, let $\mathfrak{P}$ be a prime ideal of S containing $\mathfrak{b} + \mathrm{ann}_S(M)$ such that $\mathrm{height}(\mathfrak{b}, M) = \mathrm{height}(\mathfrak{P}, M)$. Putting $\mathfrak{p} = \mathfrak{P} \cap S^G$, we note that $\mathfrak{p}$ contains $\mathfrak{a} + \mathrm{ann}_{S^G}(M)$. Thus by above discussion,

$$\mathrm{height}(\mathfrak{a}, M) \leq \mathrm{height}(\mathfrak{p}, M) = \mathrm{height}(\mathfrak{P}, M) = \mathrm{height}(\mathfrak{b}, M) .$$

Hence $\mathrm{height}(\mathfrak{a}, M) \leq \mathrm{height}(\mathfrak{b}, M)$. Conversely, let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a} + \mathrm{ann}_{S^G}(M)$ such that $\mathrm{height}(\mathfrak{a}, M) = \mathrm{height}(\mathfrak{p}, M)$. Then $\mathfrak{p} \supset (\mathfrak{b} + \mathrm{ann}_S(M)) \cap S^G$. Indeed, let $a + b \in S^G$ with $a \in \mathfrak{b}$ and $b \in \mathrm{ann}_S(M)$.

Then

$$(a+b)^{|G|} = \prod_{g \in G} g(a+b) = \prod_{g \in G} g(a) + c$$

with $c \in \operatorname{ann}_S(M)$. Clearly, $\prod_{g \in G} g(a) \in \mathfrak{a}$ and $c \in \operatorname{ann}_{S^G}(M)$. Therefore, $(a+b)^{|G|} \in \mathfrak{a} + \operatorname{ann}_{S^G}(M) \subset \mathfrak{p}$, and so $a+b \in \mathfrak{p}$, as desired. By Lemma 2.1.2(c), we may choose a prime $\mathfrak{P}$ of S with $\mathfrak{P} \cap S^G = \mathfrak{p}$ and $\mathfrak{P} \supset (\mathfrak{b} + \operatorname{ann}_S(M))$. Since $\operatorname{height}(\mathfrak{P}, M) = \operatorname{height}(\mathfrak{p}, M)$, we finally obtain $\operatorname{height}(\mathfrak{a}, M) = \operatorname{height}(\mathfrak{p}, M) = \operatorname{height}(\mathfrak{P}, M) \geq \operatorname{height}(\mathfrak{b}, M)$. $\square$

As a consequence, we obtain the following result.

Theorem 2.3.5. *Let M be a finitely generated left $S * G$-module. Then M is Cohen-Macaulay as S-module if and only if M is Cohen-Macaulay as an S^G-module.*

Proof. We use the Cohen-Macaulay criterion in Lemma 2.2.6(ii). Note that, by Lemmas 2.1.2 and 2.3.4(a), the maximal ideals of S^G containing $\operatorname{ann}_{S^G}(M)$ are precisely the ideals of the form $\mathfrak{m} \cap S^G$, where $\mathfrak{m}$ is a maximal ideal of S containing $\operatorname{ann}_S(M)$. Moreover, by Lemma 2.3.4(b), neither depth nor height on M changes when passing from $\mathfrak{m}$ to $\mathfrak{m} \cap S^G$. $\square$

Remark 2. Suppose M is a finitely generated $S*G$-module. Our main concern is to determine when M^G is Cohen-Macaulay as an S^G-module, given that M is a Cohen-Macaulay S-module. The above theorem enables us to replace $S*G$ by the group ring $S^G[G]$ and rephrase the question as follows:

Let M be an $S^G[G]$-module which is Cohen-Macaulay as an S^G module. When can we say that M^G is Cohen-Macaulay over S^G?

2.4 Background from homological algebra

In the Section 3.5, we make use of certain spectral sequences associated with "derived functors". We do not define categories and functors here; a good background reference is [Rot]. Since we are interested in categories of modules over a ring, all our categories are assumed to be abelian (in the sense of [Be_1, ch.2, sec. 1]). In the following examples, we introduce certain module categories that are important for our purposes, along with some notation. In this section, R denotes an arbitrary ring (associative with identity) and S will denote a commutative ring. Moreover, G will be a finite group acting on S, as in the previous sections.

Definition 2.4.1. For any ring R, R-mod is the category of all left R-modules. For two left R-modules M and N, the group of morphisms $\mathrm{Hom}_R(M,N)$ in R-mod is the group of all R-linear transformations $M \to N$. Similarly one defines category mod-R of right R-modules.

We are primarily interested in the categories $S*G$-mod and $S^G[G]$-mod. Note that any $S*G$-module is also a $S^G[G]$-module and any $S*G$-linear map is also $S^G[G]$-linear. In this sense, $S*G$-mod is a subcategory of $S^G[G]$-mod. Two more categories of which $S*G$-mod is a subcategory are S-mod and

S^G-mod. This situation is summarized in the following diagram, where all arrows are the so-called "restriction functors":

$$\begin{array}{ccc} S * G\text{-mod} & \longrightarrow & S\text{-mod} \\ \downarrow & & \downarrow \\ S^G[G]\text{-mod} & \longrightarrow & S^G\text{-mod} \end{array}$$

We give two further examples of functors which are very important for us.

Example 2.4.2. Our first example is the *fixed-point* functor $(\_)^G\colon S^G[G]\text{-mod} \to S^G\text{-mod}$. For any $S^G[G]$-module M, one defines $(M)^G$ as the collection of all G-invariant elements in M; this is easily seen to be an S^G-module. For simplicity, we will write M^G in place of $(M)^G$. Now, for any $S^G[G]$-linear homomorphism $f : M \to N$, $f(M^G) \subset N^G$. Therefore the restriction of f to M^G gives a morphism $f^G \in \mathrm{Hom}_{S^G}(M^G, N^G)$. By composing $(\_)^G$ with the restriction functor $S * G\text{-mod} \to S^G[G]\text{-mod}$, we also obtain a fixed point functor $(\_)^G\colon S * G\text{-mod} \to S^G\text{-mod}$.

Example 2.4.3. Our second example is the $\mathfrak{a}$-*torsion* functor $\Gamma_{\mathfrak{a}}(\_)\colon S\text{-mod} \to S\text{-mod}$ that is associated an ideal $\mathfrak{a}$ in a commutative ring S. Under this functor, the image of an S-module M is

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0, \text{ for some } n \in \mathbb{N}\}.$$

For any morphism $f\colon M \to N$ in S-mod, $\Gamma_{\mathfrak{a}}(f)$ is the restriction of f to $\Gamma_{\mathfrak{a}}(M)$. We will often consider the situation where $\mathfrak{a}$ is an ideal of S^G and M is an $S^G[G]$-module (or an $S * G$-module). In this case, $\Gamma_{\mathfrak{a}}(M)$ is defined by first restricting M to S^G, as in the above diagram.

We recall the (standard) notion of an exact functor.

Definition 2.4.4. A functor $\mathfrak{F}$ defined on R-mod is called *left exact* if, given an exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ in R-mod (i.e., f is injective and $\mathrm{Ker}(g) = \mathrm{Im}(f)$), the resulting sequence

$$0 \to \mathfrak{F}(M_1) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(M) \xrightarrow{\mathfrak{F}(g)} \mathfrak{F}(M_2)$$

is also exact. Right exactness can be defined in the same manner. A functor that is both right exact and left exact is called *exact*.

We will use $\circ$ to denote the composite of functors.

Lemma 2.4.5. *The fixed-point functor $(\,.\,)^G$ and the $\mathfrak{a}$-torsion functors $\Gamma_{\mathfrak{a}}$ defined above are left exact. Moreover, for any ideal $\mathfrak{a}$ of S^G, $(\,.\,)^G \circ \Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}} \circ (\,.\,)^G$ as functors $S^G[G]$-$\mathsf{mod} \to S^G$-mod.*

Proof. The commuting part can be verified directly from the expression

$$\Gamma_{\mathfrak{a}}(M^G) = \{m \in M \mid m \,\text{is}\; G\text{-invariant and for some}\, n,\ \mathfrak{a}^n m = 0\} = \Gamma_{\mathfrak{a}}(M)^G$$

for any $S^G[G]$-module M.

To show left exactness, let $0 \to L \xrightarrow{f} M \xrightarrow{f'} N$ be an exact sequence. Being restrictions of f, both $\Gamma_{\mathfrak{a}}(f)$ and f^G are injective. Also since $f'f = 0$, $\mathrm{Im}(\Gamma_{\mathfrak{a}}(f)) \subset \mathrm{Ker}(\Gamma_{\mathfrak{a}}(f'))$ and $\mathrm{Im}(f^G) \subset \mathrm{Ker}(f'^G)$. To complete the proof, let $m \in \mathrm{Ker}(\Gamma_{\mathfrak{a}}(f'))$ and $m' \in \mathrm{Ker}((f')^G)$; so $m \in \Gamma_{\mathfrak{a}}(M) \cap \mathrm{Ker}\, f'$ and $m' \in M^G \cap \mathrm{Ker}\, f'$. Then there exists l, $l' \in L$ such that $f(l) = m$ and $f(l') = m'$.

Since f is injective and $f(\mathfrak{a}^n l) = f(l' - g(l')) = 0$ for some value of n and for any $g \in G$, we conclude that $l \in \Gamma_{\mathfrak{a}}(L)$ and $l' \in L^G$. This completes the proof. $\square$

The above functors $(\,.\,)^G$ and $\Gamma_{\mathfrak{a}}$ are however usually not exact. In general, if a left exact functor $\mathfrak{F}$ on some module category R-mod fails to be exact, the "exactness defect" can be measured by the *right derived functors* $R^n(\mathfrak{F})$ $(n \geq 0)$. We briefly discuss the construction of these functors, referring to [Rot] for complete details. For this, we will need the concept of injective modules.

Definition 2.4.6. A left R-module I is called *injective* if given an injective R-module map $f\colon M \hookrightarrow N$ and any R-module map $g\colon M \to I$, there exists an R-module map $\phi\colon N \to I$ such that $\phi \circ f = g$. In other words, $\mathrm{Hom}_R(\,.\,, I)$ is an exact functor on R-mod.

Let M be any left R-module. It is known that M can be embedded into some injective module, say I_0. The cokernel of this embedding, I_0/M, can be embedded in to another injective module, I_1. Continuing in this way we construct a long exact sequence ,

$$I_* : 0 \to I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} \cdots \to I_n \xrightarrow{d^n} I_{n+1} \to \ldots$$

I_* is called an *injective resolution* of M. Note that $\mathrm{Ker}\, d^0 \simeq M$; so the above sequence is exact everywhere except at 0^{th} level. Now apply $\mathfrak{F}$ to this sequence and put $\delta^i = \mathfrak{F}(d^i)$ to get

$$0 \to \mathfrak{F}(I_0) \xrightarrow{\delta^0} \mathfrak{F}(I_1) \xrightarrow{\delta^1} \cdots \to \mathfrak{F}(I_n) \xrightarrow{\delta^n} \mathfrak{F}(I_{n+1}) \to .$$

One defines

$$R^n\mathfrak{F}(M) = \mathrm{Ker}(\delta^n)/\,\mathrm{Im}(\delta_{n-1}) \ .$$

The following two properties will be important for us.

- $R^0\mathfrak{F}(M) = \mathfrak{F}(M)$.

- Any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ gives rise to the a "natural" long exact sequence, $0 \to \mathfrak{F}(M_1) \to \mathfrak{F}(M) \to \mathfrak{F}(M_2) \xrightarrow{\delta^0} R^1\mathfrak{F}(M_1) \to \cdots \to R^n\mathfrak{F}(M_2) \xrightarrow{\delta^n} R^{n+1}\mathfrak{F}(M_1) \to \cdots$

We will now turn specifically to the fixed-point and torsion functors for $(\,.\,)^G$ and $\Gamma_{\mathfrak{a}}$.

Example 2.4.7. (a) Let M be a left module over $S^G[G]$. The n^{th} right derived functor of the G-fixed point functor $(\,.\,)^G$, evaluated on M, is called n^{th} *cohomology group* of M. The usual notation is

$$R^n(\,.\,)^G(M) = H^n(G, M) \ .$$

Note that for $s \in S^G$, the map $\sigma\colon M \to M$, $m \mapsto sm$ is a morphism in $S^G[G]$-mod, and hence it induces a map $\sigma_*^n\colon H^n(G,M) \to H^n(G,M)$ for each n. Letting s act on $H^n(G,M)$ via σ_*^n, we make $H^n(G,M)$ into a left S^G-module.

(b) The n^{th} right derived functor of the $\mathfrak{a}$-torsion functor, $R^n\Gamma_{\mathfrak{a}}$, is commonly denoted by $H^n_{\mathfrak{a}}(\,.\,)$; it is called the n^{th} *local cohomology* functor. If M is a left module over $S^G[G]$ then, for any $a \in S^G[G]$, the left multiplication map $\alpha\colon M \to M$, $m \mapsto am$ is an S^G-module morphism. Hence, as in (a) above, we obtain maps $\alpha^n_*\colon H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$ which make each $H^n_{\mathfrak{a}}(M)$ into a left $S^G[G]$-module.

(c) In view of the above considerations, for any left $S^G[G]$-module M and any ideal $\mathfrak{a}$ of S^G, we may consider the groups

$$H^p_{\mathfrak{a}}(H^q(G,M)) \qquad \text{and} \qquad H^p(G, H^q_{\mathfrak{a}}(M))$$

for $p, q \geq 0$. These groups will be important later on.

For our next result we need some preparations. Let $\alpha\colon R \to R'$ be any homomorphism of (not necessary commutative) rings. Then R' can be viewed as a (R, R')-bimodule and as a as a (R', R)-bimodule through α. Furthermore, any left R'-module M' can be viewed as left R-module via α; the resulting R-module is often called the "restricted module". Let M be any left R-module. Then the tensor product $R' \otimes_R M$ is a left R'-module, sometimes called the "induced module", and $\mathrm{Hom}_R(R', M)$ is another left R'-module, usually called the "co-induced module". For the detailed construction of these modules, see [Rot] or [Br, Chap. III.3]. These references also contain proofs for the following natural isomorphisms, the so-called "adjoint isomorphisms":

$$\mathrm{Hom}_R(M', M) \simeq \mathrm{Hom}_{R'}(M', \mathrm{Hom}_R(R', M)) \tag{2.4.1}$$

$$\mathrm{Hom}_R(M, M') \simeq \mathrm{Hom}_{R'}(R' \otimes_R M, M') \ . \tag{2.4.2}$$

One consequence of these isomorphisms is the following proposition.

Proposition 2.4.8. *Let I be an injective left $S^G[G]$-module and let $\mathfrak{a}$ be an ideal of S^G. Then:*

(a) *I and I^G are injective S^G-modules;*

(b) *$\Gamma_{\mathfrak{a}}(I)$ is injective S^G-module.*

Proof. (a) Since $S^G[G]$ is a free right S^G module, $S^G[G] \otimes_{S^G} (\,.\,)\colon S^G\text{-}\mathsf{mod} \to S^G[G]\text{-}\mathsf{mod}$ is an exact functor. Also since I is injective, $\mathrm{Hom}_{S^G[G]}(\,.\,, I)$ is an exact functor on $S^G[G]\text{-}\mathsf{mod}$. Thus, the composite $\mathrm{Hom}_{S^G[G]}(S^G[G] \otimes_{S^G} (\,.\,), I)$ is an exact functor on $S^G\text{-}\mathsf{mod}$. Now we apply the isomorphism (2.4.2), with $\alpha\colon R = S^G \hookrightarrow R' = S^G[G]$ the inclusion map, to see that $\mathrm{Hom}_{S^G}(\,.\,, I)$ is exact on $S^G\text{-}\mathsf{mod}$. Thus, I is an injective S^G-module.

For the second assertion of (a), consider the augmentation map $\omega : S^G[G] \to S^G$, $\omega(\sum_{g\in G} s_g g) = \sum_{g \in G} s_g$. If M is any left $S^G[G]$-module then the co-induced S^G-module $\mathrm{Hom}_{S^G[G]}(S^G, M)$ is naturally isomorphic with the module of G-fixed points M^G, via $f \mapsto f(1)$. Applying the isomorphism (2.4.1) with $\alpha = \omega$ and $M = I$, we get isomorphisms of functors on $S^G\text{-}\mathsf{mod}$,

$$\mathrm{Hom}_{S^G[G]}(\,.\,, I) \simeq \mathrm{Hom}_{S^G}(\,.\,, \mathrm{Hom}_{S^G[G]}(S^G, I)) \simeq \mathrm{Hom}_{S^G}(\,.\,, I^G) \ .$$

Since $\mathrm{Hom}_{S^G[G]}(\,.\,,I)$ is exact, this shows that $\mathrm{Hom}_{S^G}(\,.\,,I^G)$ is an exact functor on S^G-mod, I^G is an injective S^G-module.

(b) Let $\mathfrak{b}$ be an ideal of S^G and let $f:\mathfrak{b}\to\Gamma_{\mathfrak{a}}(I)$ be an S^G-linear map. By [Rot, Theorem 3.20], it is enough to show that there exists an element $m\in\Gamma_{\mathfrak{a}}(I)$ such that $f(s)=sm$ holds for all $s\in\mathfrak{b}$. Since I is injective as S^G-module, by part (a), there certainly exists an element $m'\in I$ with this property. Now, since S^G is Noetherian, $f(\mathfrak{b})$ is a finitely generated submodule of $\Gamma_{\mathfrak{a}}(I)$. Hence there exist a positive integer r such that $\mathfrak{a}^r f(\mathfrak{b})=0$. Further, $f(\mathfrak{b})$ is a submodule of the finitely generated S^G-module $S^G m'$. By the Artin-Rees Lemma [Bou$_1$, Cor. 1, Chap. III, §3.1], there exists a positive integer t such that for all $n\geq t$,

$$\mathfrak{a}^n(S^G m')\cap f(\mathfrak{b})=\mathfrak{a}^{n-t}(\mathfrak{a}^t(S^G m')\cap f(\mathfrak{b}))\ .$$

Therefore $\mathfrak{a}^{r+t}(S^G m')\cap f(\mathfrak{b})=\mathfrak{a}^r(\mathfrak{a}^t(S^G m')\cap f(\mathfrak{b}))\subset\mathfrak{a}^r f(\mathfrak{b})=0$. This implies that we can extend f to $h\colon\mathfrak{a}^{r+t}+\mathfrak{b}\to\Gamma_{\mathfrak{a}}(I)$ by defining

$$h(s+s')=s'm'$$

for all $s\in\mathfrak{a}^{r+t}$ and $s'\in\mathfrak{b}$. Indeed, if $s+s'=u+u'$ with $s,u\in\mathfrak{a}^{r+t}$ and $s',u'\in\mathfrak{b}$ then $(u'-s')m'=(s-u)m'\in\mathfrak{a}^{r+t}(S^G m')\cap f(\mathfrak{b})=0$. Once again we use injectivity of I over S^G to find $m\in I$ such that $h(s)=sm$ holds for all $s\in S^G$. We claim that $m\in\Gamma_{\mathfrak{a}}(I)$. Indeed, if $s\in\mathfrak{a}^{r+t}$ then $sm=h(s)=h(s+0)=0$. This completes the proof. $\square$

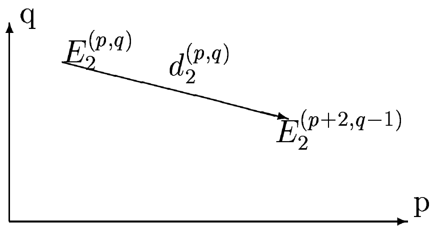

Figure 2.1: E_2-page of a spectral sequence.

We conclude this section with a brief introduction to spectral sequences. The spectral sequences considered here are sometimes called "third quadrant (homology)" or "first quadrant (cohomology)" spectral sequences. Again, we refer to [Rot] for details and more general versions.

Definition 2.4.9. A *spectral sequence* is a sequence of bigraded S-modules $E_r = \{E_r^{p,q}\}_{p,q\geq 0}$ $(r \geq 2)$ equipped with an S-linear maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ such that $d_r^{p,q} \circ d_r^{p-r,q-1+r} = 0$ and $E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{Im}(d_r^{p-r,q-1+r}) = H^{p,q}(E_r)$. We say that for a fixed pair p, q, $\{E_r^{p,q}\}$ converges to $E_\infty^{p,q}$ and write $E_r^{p,q} \Longrightarrow E_\infty^{p,q}$ if there exists a positive integer r dependent on p and q such that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty$.

We are particularly interested in $r = 2$. In this case the differential d_2 has bi-degree $(2, -1)$. Figure 1.1 shows the E_2-page of the spectral sequence.

We end this chapter with a remarkable theorem of Grothendieck which gives a convenient way to construct certain spectral sequences. The categories in the theorem can be any abelian categories with enough injective objects (i.e., every object embeds into an injective object), but for our purposes, it will be enough to think quite specifically of module categories. We refer to [Rot, Theorem 11.38] for a complete proof.

Theorem 2.4.10. *Let $\mathfrak{G} : \mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{F} : \mathfrak{B} \to \mathfrak{C}$ be functors with $\mathfrak{F}$ left exact such that, for each injective object I in $\mathfrak{A}$, the image $\mathfrak{G}(I)$ in $\mathfrak{B}$ satisfies $R^q\mathfrak{F}(\mathfrak{G}(I)) = 0$ for $q > 0$. Then, for each module A in $\mathfrak{A}$, there exists a spectral sequence*

$$E_2^{p,q} = R^p\mathfrak{F}(R^q\mathfrak{G}(A)) \Longrightarrow R^{p+q}(\mathfrak{F} \circ \mathfrak{G})(A)$$

CHAPTER 3

General Group Actions

In this chapter, we consider the following question: Suppose G is a finite group acting on the commutative ring S and M is an $S * G$-module which is Cohen-Macaulay as an S-module, when is M^G Cohen-Macaulay as an S^G-module? As we saw in Theorem 2.3.5, if M is a Cohen-Macaulay S-module then M is also Cohen-Macaulay S^G-module. Therefore, later in Section 3.4 of this chapter, we will replace S by S^G and assume that S is pointwise fixed under G. Most of the literature on invariant theory is concerned with the special case of linear actions, as explained in the Introduction; see, e.g., [H_1, H_2, N_1, N_2, ST, C]. In this chapter, we will consider an arbitrary group actions.

We assume throughout this chapter that

S	is a commutative ring,
G	is a finite group acting on S,
S^G	is the subring of G-invariants in S,
$S * G$	is the skew group ring that is associated with the given G-action on S,
M	is a left $S * G$-module, and
M^G	denotes the submodule of G-invariant elements of M.

Further hypotheses will be explicitly mentioned whenever they are needed.

3.1 Trace map

The relative trace map is defined for any subgroup H of G. Let τ be a set of representatives for the cosets gH of H in G.

Definition 3.1.1. The *relative trace map* $\mathrm{tr}_{G/H}\colon M^H \to M^G$ is defined by $\mathrm{tr}_{G/H}(m) = \sum_{g\in\tau} g(m) \quad (m \in M^H)$. The image of $\mathrm{tr}_{G/H}$ is denoted by M_H^G. Thus,

$$M_H^G = \{\mathrm{tr}_{G/H}(m) \mid m \in M^H\}.$$

It is easy to check that $\mathrm{tr}_{G/H}$ does not depend on the choice of τ. In the following lemma we will explore the elementary properties of the trace map. In particular, we will show that M_H^G is an S^G-submodule of M^G. We use ${}^gH$ to denote the conjugate subgroup gHg^{-1} for $g \in G$.

Lemma 3.1.2. *Let M, S, G be as above. Then:*

(a) $\mathrm{tr}_{G/H}$ *is a homomorphism of S^G-modules. Thus M_H^G is an S^G- submodule of M^G.*

(b) *If K is any subgroup of H, then $\mathrm{tr}_{G/K} = \mathrm{tr}_{G/H}\,\mathrm{tr}_{H/K}$ Consequently, $M_K^G \subset M_H^G$.*

(c) *For any $m \in M^G$, $\mathrm{tr}_{G/H}(m) = [G : H]m$*

(d) *For any g in G, $M_H^G = M_{gH}^G$.*

Proof. (a) Since the trace map is defined by summation, it is additive. To show S^G- linearity, let $r \in S^G$ and let $m \in M$. Choose a set τ of cosets representatives of H in G Then,

$$\mathrm{tr}_{G/H}(rm) = \sum_{g\in\tau} g(rm) = \sum_{g\in\tau} rg(m) = r\,\mathrm{tr}_{G/H}(m).$$

Thus $\mathrm{tr}_{G/H}$ is S^G-linear. Since an image of any module homomorphism is a module, the second statement follows.

(b) Let τ be a set of cosets representatives of H in G and μ be a set of cosets representatives of K in H. It is clear that we can choose $\gamma = \{gh|g \in \tau,\, h \in \mu\}$ as a set of cosets representatives of K in G. Therefore, $\mathrm{tr}_{G/K}(m) = \sum_{gh\in\gamma} gh(m) = \sum_{g\in\tau} g(\sum_{h\in\mu} h(m)) = \mathrm{tr}_{G/H}(\mathrm{tr}_{H/K}(m))$. The second assertion follows from this identity.

(c) If $m \in M^G$, then $gm = m$ for all $g \in G$. Therefore, $\mathrm{tr}_{G/H}(m) = [G : H]m$.

(d) Let $m \in M_H^G$. Then there exists $m' \in M^H$ such that $\mathrm{tr}_{G/H}(m') = m$. Now for any $ghg^{-1} \in gHg^{-1}$, $ghg^{-1}(gm') = gm'$ shows that $gm' \in M^{{}^gH}$. Also, $\{gtg^{-1} \mid t \in \tau\}$ is a set of coset representatives for ${}^gH$ in G. Thus, $\mathrm{tr}_{G/{}^gH}(gm') = \sum_{t\in\tau} gtg^{-1}gm' = g\,\mathrm{tr}_{G/H}(m') = m$, we get $M_H^G \subset M_{{}^gH}^G$. By the symmetry of the situation, we get $M_H^G = M_{{}^gH}^G$. $\square$

Applying property (a) to the special case $M = S$, we deduce that $S_H^G = \mathrm{tr}_{G/H}(S^H)$ is an ideal of S^G.

Definition 3.1.3. For any additive subgroup N of M, we define the *inertia group* $I_G(N)$ as follows,

$$I_G(N) = \{g \in G \mid (g-1)(M) \subset N\}.$$

In the following lemma we justify the word subgroup in the definition. We also list elementary properties.

Lemma 3.1.4. *With the above notations we have the following:*

(a) *$I_G(N)$ is a subgroup of G.*

(b) *If $N_1 \subset N_2$ then $I_G(N_1) \subset I_G(N_2)$.*

(c) *For $H \subset G$, $I_G(N) \cap H = I_H(N)$.*

Proof. (a) Since G is finite, it suffices to show that if $g, h \in I_G(N)$ then $gh \in I_G(N)$. But this follows from the calculation

$$(gh-1)(m) = (g-1)(h-1)(m) + (g-1)(m) + (h-1)(m) \ ,$$

because all three terms on the right belong to N.

(b) and (c) are clear from the definition. □

In the following lemma we derive an identity which will be used later with a special case $M = S$. We will use the notation $K\backslash G/H$ to denote a set of double coset representatives $\{KgH\}$ of (K, H) in G.

Lemma 3.1.5. *For any $m \in M^H$,*

$$\mathrm{tr}_{G/H}(m) \equiv \sum_{g \in I_G(N)\backslash G/H} [I_G(N) : I_{{}^gH}(N)]\, g(m) \pmod{N}$$

Proof. Write G as a disjoint union $G = \bigcup_g I_G(N)gH$ with g running over $I_G(N)\backslash G/H$, and for each g, let τ_g be a set of the coset representatives of H in the double coset $I_G(N)gH$. Then,

$$\mathrm{tr}_{G/H}(m) = \sum_{g \in I_G(N)\backslash G/H} \sum_{g' \in \tau_g} g'(m) \ .$$

Now each g' has the form $g' = agh$ with $a \in I_G(N)$ and $h \in H$. Thus, we have for any $m \in M^H$, $g'(m) = agh(m) = ag(m) \equiv g(m) \pmod{N}$. Thus the above formula yields,

$$\mathrm{tr}_{G/H}(m) \equiv \sum_{g \in I_G(N)\backslash G/H} |\tau_g| g(m) \pmod{N} \ .$$

We claim that $|\tau_g| = [I_G(N) : I_{{}^gH}(N)]$. To show this, let $g_1,\, g_2 \in I_G(N)gH$. Then $g_1 = a_1gh_1,\, g_2 = a_2gh_2$ for some $a_1, a_2 \in I_G(N)$ and $h_1,\, h_2 \in H$. Then $g_1H = g_2H \Leftrightarrow a_1\, {}^gH = a_2\, {}^gH \Leftrightarrow a_1\, I_{{}^gH}(N) = a_2\, I_{{}^gH}(N)$, where the last equivalence uses Lemma 3.1.4(c). This proves the claimed formula for $|\tau_g|$, and hence the lemma. $\square$

The proof of the following lemma is based on an earlier proof that was communicated to M. Lorenz by Don Passman (e-mail of Oct. 18, 2000). The special case where S is an affine algebra over a field is covered by [Ke$_2$, Satz 4.7]. Let $I_G(\mathfrak{Q}) = \{g \in G \mid (g-1)(S) \subset \mathfrak{Q}\}$ be the inertia group of an ideal $\mathfrak{Q}$ of S.

Lemma 3.1.6. *For any prime ideal $\mathfrak{Q}$ of S,*

$$\mathfrak{Q} \supseteq S_H^G \Longleftrightarrow [I_G(\mathfrak{Q}) : I_{{}^gH}(\mathfrak{Q})] \in \mathfrak{Q} \quad \textit{for all } g \in G$$

Proof. By Lemma 3.1.5 the implication $\Leftarrow$ is clear. For $\Rightarrow$, assume that $\mathfrak{Q} \supseteq S_H^G$. Since $S_H^G = S_{{}^gH}^G$ by Lemma 3.1.2(d), It suffices to show that

$$[I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \in \mathfrak{Q}$$

.

To simplify notation, put $I = I_G(\mathfrak{Q})$ and let P denote a Sylow p-subgroup of $I \cap H = I_H(\mathfrak{Q})$, where $p \geq 0$ is the characteristic of the commutative domain $S/\mathfrak{Q}$. (Here $P = \{1\}$ if $p = 0$.) Then our desired conclusion, $[I : I \cap H] \in \mathfrak{Q}$, is equivalent with

$$[I : P] \in \mathfrak{Q} \ .$$

Furthermore, our assumption $\mathfrak{Q} \supseteq S^G_H$ entails that $\mathfrak{Q} \supseteq S^G_P$, by Lemma 3.1.2(b). Thus, leaving H for P, we may assume that $H = P$ is a p-subgroup of I. Let $D = \{g \in G \mid g(\mathfrak{Q}) = \mathfrak{Q}\}$ denote the decomposition group of $\mathfrak{Q}$; so $I \leq D$. We claim that

$$\mathfrak{Q} \supseteq S^D_P \ .$$

To see this, choose $r \in S$. Then that $r \in g(\mathfrak{Q})$ for all $g \in G \setminus D$ but $r \notin \mathfrak{Q}$. Then $s = \prod_{g \in D} g(r)$ also belongs to $\bigcap_{g \in G \setminus D} g(\mathfrak{Q})$ but not to $\mathfrak{Q}$ and, in addition, $s \in S^D$. Now assume that, contrary to our claim, there exists an element $f \in S^P$ so that $\mathrm{tr}_{D/P}(f) \notin \mathfrak{Q}$. Then $\mathrm{tr}_{D/P}(sf) = s\,\mathrm{tr}_{D/P}(f) \in \bigcap_{g \in G \setminus D} g(\mathfrak{Q}) \setminus \mathfrak{Q}$. Hence, $\mathrm{tr}_{G/P}(sf) = \mathrm{tr}_{G/D}(\mathrm{tr}_{D/P}(sf)) = \mathrm{tr}_{D/P}(sf) + \sum_{1 \neq g \in G/D} g\,\mathrm{tr}_{D/P}(sf) \notin \mathfrak{Q}$, a contradiction.

By the claim, we may replace G by D, thereby reducing to the case where $\mathfrak{Q}$ is G-stable. (Note that I is unaffected by this replacement.) So G acts on $S/\mathfrak{Q}$ with kernel I, P is a p-subgroup of I, and $S^G_P \subseteq \mathfrak{Q}$. Thus, for all $r \in S^P$, $0 \equiv \mathrm{tr}_{G/P}(r) \equiv [I : P] \cdot \sum_{g \in G/I} g(r) \pmod{\mathfrak{Q}}$. Our desired conclusion, $[I : P] \in \mathfrak{Q}$, will follow if we can show that $\sum_{g \in G/I} g(r) \notin \mathfrak{Q}$ holds for some $r \in S^P$. But $\sum_{g \in G/I} g$ induces a nonzero endomorphism on $S/\mathfrak{Q}$, by linear independence of automorphisms of $K = \mathrm{Fract}(S/\mathfrak{Q})$; so $\sum_{g \in G/I} g(s) \notin \mathfrak{Q}$ holds for some $s \in S$. Putting $r = \prod_{h \in P} h(s)$, we have $r \in S^P$ and $r \equiv s^{|P|} \pmod{\mathfrak{Q}}$. Since

$|P|$ is 1 or a power of $p = \operatorname{char} K$, we obtain $\sum_{g\in G/I} g(r) \equiv \sum_{g\in G/I} g(s^{|P|}) \equiv \left(\sum_{g\in G/I} g(s)\right)^{|P|} \equiv 0 \pmod{\mathfrak{Q}}$, as required. $\square$

3.2 Height formula

For any collection $\mathcal{X}$ of subgroups of G, we define the ideal $S^G_{\mathcal{X}}$ of S^G by

$$S^G_{\mathcal{X}} = \sum_{H\in\mathcal{X}} S^G_H \ .$$

By Lemma 3.1.2(b),(d), we can assume that $\mathcal{X}$ is closed under G-conjugation and under taking subgroups without changing $S^G_{\mathcal{X}}$.

Moreover, for any subgroup $H \leq G$, we define

$$I_S(H) = \sum_{h\in H} (h-1)(S)S \ .$$

Thus, $I_S(H)$ is an ideal of S, and $\mathfrak{Q} \supseteq I_S(H)$ is equivalent with $H \leq I_G(\mathfrak{Q})$.

Lemma 3.2.1. *Assume that S has characteristic p, a positive prime, and let $\mathcal{X}$ be a collection of subgroups of G that is closed under G-conjugation and under taking subgroups. Then*

$$\operatorname{height} S^G_{\mathcal{X}} = \inf\{\operatorname{height} I_S(P) \mid P \textit{ is a p-subgroup of } G,\ P \notin \mathcal{X}\}$$

Proof. One has $\operatorname{height} S^G_{\mathcal{X}} = \inf_{\mathfrak{q}} \operatorname{height} \mathfrak{q} = \inf_{\mathfrak{Q}} \operatorname{height} \mathfrak{Q}$, where $\mathfrak{q}$ runs over the prime ideals of S^G containing $S^G_{\mathcal{X}}$ and $\mathfrak{Q}$ runs over the primes of S containing $S^G_{\mathcal{X}}$. Here, the first equality is just the definition of height, while the second equality is a consequence of Lemma 2.3.3.

By Lemma 3.1.6,

$$\mathfrak{Q} \supseteq S^G_{\mathcal{X}} \Longleftrightarrow p \,\big|\, [I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \text{ for all } H \in \mathcal{X}.$$

Since $I_H(\mathfrak{Q}) = I_G(\mathfrak{Q}) \cap H$ belongs to $\mathcal{X}$ for $H \in \mathcal{X}$, the latter condition just says that the Sylow p-subgroups of $I_G(\mathfrak{Q})$ do not belong to $\mathcal{X}$ or, equivalently, some p-subgroup $P \leq I_G(\mathfrak{Q})$ does not belong to $\mathcal{X}$. Therefore,

$$\mathfrak{Q} \supseteq S^G_{\mathcal{X}} \Longleftrightarrow \mathfrak{Q} \supseteq \bigcap_{P \leq G \text{ a } p\text{-subgroup, } P \notin \mathcal{X}} I_S(P) \ ,$$

which implies the asserted height formula. □

3.3 Annihilators of cohomology classes

Recall that M denotes a module over the skew group ring $S * G$. Recall further that the G-cohomology $H^*(G, M)$ is a left module over S^G; see Example 2.4.7(a).

The following lemma generalizes [Ke$_3$, Corollary 2.4].

Lemma 3.3.1. *The ideal* S^G_H *of* S^G *annihilates the kernel of the restriction map* $\operatorname{res}^G_H : H^*(G, M) \to H^*(H, M)$.

Proof. The action of $S^G = H^0(G, S)$ on $H^*(G, M)$ described in Example 2.4.7(a)can also be interpreted as coming from the cup product

$$H^0(G, S) \times H^*(G, M) \stackrel{\cup}{\longrightarrow} H^*(G, S \otimes_{\mathbb{Z}} M) \stackrel{\cdot}{\longrightarrow} H^*(G, M) \ ,$$

where the map denoted by $\cdot$ comes from the G-equivariant map $S \otimes_{\mathbb{Z}} M \to M$, $r \otimes m \mapsto rm$; see, e.g., [Br, Exerc. 1 on p. 114]. Furthermore, the relative trace map $\mathrm{tr}_{G/H} : S^H \to S^G$ is identical with the corestriction map $\mathrm{cor}_H^G : H^0(H,S) \to H^0(G,S)$; cf. [Br, p. 81]. Thus, the transfer formula for cup products ([Br, (3.8) on p. 112]) gives, for $s \in S^H$ and $x \in H^*(G,M)$,

$$\mathrm{tr}_{G/H}(s)x = \cdot(\mathrm{tr}_{G/H}(s) \cup x) = \cdot(\mathrm{cor}_H^G(s \cup \mathrm{res}_H^G(x))) \ .$$

Therefore, if $\mathrm{res}_H^G(x) = 0$ then $\mathrm{tr}_{G/H}(s)x = 0$. $\square$

We summarize the material of this section in the following proposition. For convenience, we write $\mathrm{res}_P^G(\,.\,) = \,.\,\big|_P$.

Proposition 3.3.2. *Assume that S has characteristic p, and let M be an $S * G$-module. Then, for any $x \in H^*(G,M)$,*

$$\mathrm{height}\,\mathrm{ann}_{S^G}(x) \geq \inf\{\mathrm{height}\, I_S(P) \mid P \textit{ a p-subgroup of } G,\ x\big|_P \neq 0\} \ .$$

Proof. Let $\mathcal{X}$ denote the splitting data of x, that is, $\mathcal{X} = \{H \leq G \mid x\big|_H = 0\}$; cf. [CoR]. By Lemma 3.3.1, $\mathrm{ann}_{S^G}(x) \supseteq S^G_{\mathcal{X}}$, and by Lemma 3.2.1, $\mathrm{height}\, S^G_{\mathcal{X}} = \inf\{\mathrm{height}\, I_S(P) \mid P$ is a p-subgroup of G, $x\big|_P \neq 0\}$. The proposition follows.

$\square$

3.4 Depth estimates

In this section, we will assume that S is Noetherian and that G acts trivially on S. Thus, $S = S^G$ and the skew group ring $S * G$ is the ordinary group ring $S[G]$. Throughout, $\mathfrak{a}$ will denote an ideal of S. The $S[G]$-module M will be assumed to be finitely generated. Hence, M is Noetherian as an S-module, and likewise for the submodule M^G of G-invariant elements of M.

Our goal is to find estimates on the $\operatorname{depth}(\mathfrak{a}, M^G)$ in terms of $\operatorname{depth}(\mathfrak{a}, M)$. We then apply this results to get a sufficient condition for M^G to be Cohen-Macaulay, in Corollary 3.4.4.

We recall from Theorem 2.2.3 that

$$\operatorname{depth}(\mathfrak{a}, M) = \inf_i \{ i \mid \operatorname{Ext}^i_S(S/\mathfrak{a}, M) \neq 0 \} \ .$$

In the following lemma we replace 'Ext' by local cohomology.

Lemma 3.4.1. *Let $\mathfrak{a}$ be an ideal in the ring S and let $H^i_{\mathfrak{a}}$ denote the i-th local cohomology functor with respect to $\mathfrak{a}$. Then, for any finitely generated S-module M,*

$$\operatorname{depth}(\mathfrak{a}, M) = \inf\{ i \mid H^i_{\mathfrak{a}}(M) \neq 0 \} \ .$$

Proof. This is [BS, Theorem 6.2.7]. $\square$

Since M is a Noetherian S-module, all $H^q(G, M)$ are Noetherian S-modules as well. Indeed, $H^q(G, M)$ is the q^{th} cohomology group of the cochain complex

$\mathrm{Hom}_{S[G]}(F_*, M) = \mathrm{Hom}_S(F_*, M)^G$, where F_* is any free resolution of the trivial $S[G]$-module S. Since S is Noetherian, all F_q may be chosen to be finitely generated. Thus, each $\mathrm{Hom}_S(F_q, M)^G$ is finitely generated over S, and hence so is the subquotient $H^q(G, M)$. Therefore, the above lemma applies to the S-module $H^q(G, M)$ and gives

$$\mathrm{depth}(\mathfrak{a}, H^q(G, M)) = \inf\{i \mid H^i_{\mathfrak{a}}(H^q(G, M)) \neq 0\} \ .$$

These S-modules $H^p_{\mathfrak{a}}(H^q(G, M))$ feature as the E_2^{pq}-terms of a certain spectral sequence due to Ellingsrud and Skjelbred [ES]. In fact, two related spectral sequences are constructed in [ES] which we describe now. We take the functors $\Gamma_{\mathfrak{a}}$ and $(\,.\,)^G$ from Examples 2.4.2 and 2.4.3 and apply Theorem 2.4.10, to obtain two Grothendieck spectral sequences as follows.

- By restriction, we may view $\Gamma_{\mathfrak{a}}$ as a functor $\Gamma_{\mathfrak{a}} : S[G]\text{-}\mathsf{mod} \to S[G]\text{-}\mathsf{mod}$. By Lemma 2.4.5, this functor is left exact, and Proposition 2.4.8(b) says that $\Gamma_{\mathfrak{a}}(I)$ is injective whenever I is an injective (left) $S[G]$-module. Therefore, $H^q(G, \Gamma_{\mathfrak{a}}(I)) = 0$ holds for all $q > 0$; see the remarks on the construction of right derived functors by means of injective resolutions in Chapter 1. Theorem 2.4.10 now yields a spectral sequence

$$\mathcal{E}_2^{p,q} = H^p(G, H^q_{\mathfrak{a}}(M)) \Longrightarrow R^{p+q}((\,.\,)^G \circ \Gamma_{\mathfrak{a}})(M) \ .$$

- Similarly, by Lemma 2.4.5, the fixed point functor $(\,.\,)^G : S[G]\text{-}\mathsf{mod} \to S\text{-}\mathsf{mod}$ is left exact and from Proposition 2.4.8(a), we know that I^G is injective over S for any injective $S[G]$-module I. Hence, $H^q_{\mathfrak{a}}(I^G)$ for all $q > 0$. Again, by Theorem 2.4.10, we obtain a third quadrant (cohomology) spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{a}}(H^q(G, M)) \Longrightarrow R^{p+q}(\Gamma_{\mathfrak{a}} \circ (\,.\,)^G)(M) \ .$$

- Finally, by Lemma 2.4.5, we have a canonical isomorphism $\Gamma_{\mathfrak{a}}(M^G) \simeq \Gamma_{\mathfrak{a}}(M)^G$; so the right hand sides of both spectral sequences are isomorphic. For simplicity, we write

$$H^n_{\mathfrak{a}}(G, M) = R^n(\Gamma_{\mathfrak{a}} \circ (\,.\,)^G)(M) \simeq R^n((\,.\,)^G \circ \Gamma_{\mathfrak{a}})(M) \ .$$

To summarize, we have the following spectral sequences:

$$\begin{array}{l} E_2^{p,q} = H^p_{\mathfrak{a}}(H^q(G, M)) \\ \qquad\qquad\qquad\qquad\qquad\Longrightarrow H^{p+q}_{\mathfrak{a}}(G, M) \\ \mathcal{E}_2^{p,q} = H^p(G, H^q_{\mathfrak{a}}(M)) \end{array} \tag{3.4.1}$$

We use this to obtain estimates for $\operatorname{depth}(\mathfrak{a}, M^G)$ in the following lemma.

Lemma 3.4.2. *Let $\mathfrak{a}$ be an ideal of S and let M be a finitely generated $S[G]$-module. Put $m = \operatorname{depth}(\mathfrak{a}, M)$ and $h_{\mathfrak{a}} = \inf_{q>0}\{q + \operatorname{depth}(\mathfrak{a}, H^q(G, M))\}$. Then:*

(a) **lower bound:** $\operatorname{depth}(\mathfrak{a}, M^G) \geq \min\{m, h_{\mathfrak{a}} + 1\}$,

(b) **upper bound:** *Assume that $H^{p_0}_{\mathfrak{a}}(H^{q_0}(G, M)) \neq 0$ for some $p_0 \geq 0$, $q_0 > 0$ with $s = p_0 + q_0 < m$. Assume further that $H^{s+1-\ell}_{\mathfrak{a}}(H^{\ell}(G, M)) = 0$*

holds for $\ell = 1, \ldots, q_0 - 1$ *and* $H^{s-1-\ell}_{\mathfrak{a}}(H^\ell(G, M)) = 0$ *holds for* $\ell > q_0$. *Then* $\operatorname{depth}(\mathfrak{a}, M^G) \le s + 1$.

Proof. Since $m = \operatorname{depth}(\mathfrak{a}, M)$, $H^q_{\mathfrak{a}}(M) = 0$ for $q < m$, and so the $\mathcal{E}$-sequence in (3.4.1) implies that $H^n_{\mathfrak{a}}(G, M) = 0$ for $n < m$. Therefore, the E-sequence satisfies

$$E^{p,q}_\infty = 0 \quad \text{if } p + q < m. \tag{3.4.2}$$

Furthermore, $E^{p,0}_2 = H^p_{\mathfrak{a}}(M^G)$; so

$$\operatorname{depth}(\mathfrak{a}, M^G) = \inf\{p \mid E^{p,0}_2 \neq 0\} \ .$$

Finally,

$$h_{\mathfrak{a}} = \inf\{p + q \mid q > 0, E^{p,q}_2 \neq 0\} \ .$$

To prove (a), assume that $p < \min\{m, h_{\mathfrak{a}} + 1\}$. Then $E^{p,0}_\infty = 0$, by (3.4.2), and $E^{i,j}_r = 0$ for $j > 0$, $i + j < p$, $r \ge 2$. Recall that the differential d_r of E_r has bidegree $(r, 1 - r)$. Thus, $E^{p,0}_r$ has no nontrivial boundaries and consists entirely of cycles. This shows that $E^{p,0}_2 = E^{p,0}_3 = \cdots = E^{p,0}_\infty$, and hence $E^{p,0}_2 = 0$. Thus, (a) is proved.

For (b), we check that $E^{s+1,0}_2 \neq 0$. Our hypotheses imply that, at position (p_0, q_0), all incoming differentials d_r $(r \ge 2)$ are 0 as well as all outgoing d_r $(r \ge 2, r \neq q_0+1)$. Therefore, $E^{p_0,q_0}_{q_0+1} = E^{p_0,q_0}_2$ and $E^{p_0,q_0}_\infty = E^{p_0,q_0}_{q_0+2} = \operatorname{Ker}(d^{p_0,q_0}_{q_0+1})$. The former implies that $E^{p_0,q_0}_{q_0+1} \neq 0$, by hypothesis on (p_0, q_0), and the latter shows that $d^{p_0,q_0}_{q_0+1}$ is injective, because $E^{p_0,q_0}_\infty = 0$ by (3.4.2). Thus, $d^{p_0,q_0}_{q_0+1}$ embeds

$E_{q_0+1}^{p_0,q_0}$ into $E_{q_0+1}^{s+1,0}$, forcing the latter to be nonzero. Hence, $E_2^{s+1,0}$ is nonzero as well, as desired. $\square$

For future reference, we note the following simple lemma.

Lemma 3.4.3. *Assume that ${}_SM$ is Cohen-Macaulay and that $\sqrt{\mathfrak{a}} \supseteq \operatorname{ann}_S M^G$. Then* $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{height}(\mathfrak{a}, M) \geq \operatorname{height}(\mathfrak{a}, M^G)$.

Proof. Note that $\sqrt{\mathfrak{a}} \supseteq \operatorname{ann}_S M^G \supseteq \operatorname{ann}_S M$. This implies that $\operatorname{height}(\mathfrak{a}, M) \geq \operatorname{height}(\mathfrak{a}, M^G)$. Indeed, for every prime $\mathfrak{p}$ of S containing $\mathfrak{a}$, we clearly have $\operatorname{height}(\mathfrak{p}/\operatorname{ann}_S M) \geq \operatorname{height}(\mathfrak{p}/\operatorname{ann}_S M^G)$, and hence

$$\begin{aligned} \operatorname{height}(\mathfrak{a}, M) &= \inf\{\operatorname{height}(\mathfrak{p}/\operatorname{ann}_S M) \mid \mathfrak{p} \supseteq \mathfrak{a}\} \\ &\geq \inf\{\operatorname{height}(\mathfrak{p}/\operatorname{ann}_S M^G) \mid \mathfrak{p} \supseteq \mathfrak{a}\} = \operatorname{height}(\mathfrak{a}, M^G) \ . \end{aligned}$$

Further, $\operatorname{height}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{a}, M)$, because ${}_SM$ is Cohen-Macaulay. The lemma follows. $\square$

We now give a sufficient condition for M^G to be Cohen-Macaulay. Here $\dim_S M$ stands for $\dim(S/\operatorname{ann}_S M)$.

Corollary 3.4.4. *Assume that ${}_SM$ is Cohen-Macaulay. If $H^q(G, M) = 0$ holds for $0 < q < \dim_S M - 1$ then M^G is a Cohen-Macaulay S-module as well. In particular, this holds whenever the group order is a unit in S.*

Proof. Let $\mathfrak{a}$ be an ideal of S with $\mathfrak{a} \supseteq \operatorname{ann}_S M^G$. Since $H^q(G, M) = 0$ for $0 < q < \dim_S M - 1$, the value of $h_{\mathfrak{a}}$ in Lemma 3.4.2 satisfies $h_{\mathfrak{a}} \geq$

$\dim_S M - 1$. Also, $\dim M \geq \text{height}(\mathfrak{a}, M) \geq \text{height}(\mathfrak{a}, M^G)$, by Lemma 3.4.3. Thus, Lemma 3.4.2(a) gives $\text{depth}(\mathfrak{a}, M^G) \geq \text{height}(\mathfrak{a}, M^G)$, which proves that M^G is Cohen-Macaulay.

For the last assertion, just note that the group order $|G|$ annihilates $H^q(G, M)$ for all $q > 0$. Thus if the group order is invertible in S then $H^q(G, M) = 0$ holds for all $q > 0$. $\square$

Note that the condition $H^q(G, M) = 0$ for $0 < q < \dim_S M - 1$ is vacuous if $\dim_S M \leq 2$. For $\dim_S M = 3$, the condition becomes $H^1(G, M) = 0$. The latter holds, for example, whenever M is a G-permutation module without $|G|$-torsion; explicitly, as G-module, $M \cong \bigoplus_H \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M(H)\right)$, where H runs over certain subgroups of G and each $M(H)$ is an H-submodule of M^H such that $|H|m = 0$, $m \in M(H)$ implies $m = 0$.

3.5 Depth formula

We continue to work under the hypotheses of Section 3.4 and use the same notation.

In view of Corollary 3.4.4, we may concentrate on the case where M has non-vanishing positive G-cohomology. The following proposition is a version of results of Kemper, see [Ke_1, Corollary 1.6] and [Ke_2, Kor. 1.18].

Proposition 3.5.1. *Assume that ${}_SM$ is Cohen-Macaulay and that* $\sqrt{\mathfrak{a}} \supseteq$

$\operatorname{ann}_S M^G$. *Furthermore, assume that, for some* $r \geq 0$, $H^q(G, M) = 0$ *holds for* $0 < q < r$ *and* $\mathfrak{a}x = 0$ *for some* $0 \neq x \in H^r(G, M)$. *Then* $\operatorname{depth}(\mathfrak{a}, M^G) = \min\{r + 1, \operatorname{depth}(\mathfrak{a}, M)\}$.

Remark 3. By Lemma 3.4.3, $\operatorname{height}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{a}, M)$ holds in the above formula.

Proof of Proposition 3.5.1. Our hypothesis $\mathfrak{a}x = 0$ for some $0 \neq x \in H^r(G, M)$ is equivalent with $H^0_{\mathfrak{a}}(H^r(G, M)) \neq 0$; so $\operatorname{depth}(\mathfrak{a}, H^r(G, M)) = 0$. The asserted equality is trivial for $r = 0$, since $\operatorname{depth}(\mathfrak{a}, M^G) = \operatorname{depth}(\mathfrak{a}, M) = 0$ holds in this case. Thus we assume that $r > 0$. Then, in the notation of Lemma 3.4.2, we have $r = h_{\mathfrak{a}}$. Therefore, by part (a) of the lemma, we have

$$\operatorname{depth}(\mathfrak{a}, M^G) \geq \min\{r + 1, \operatorname{depth}(\mathfrak{a}, M)\} \ .$$

To prove the reverse inequality, note that Lemma 3.4.3 and Theorem 2.2.3(a) give $\operatorname{depth}(\mathfrak{a}, M) \geq \operatorname{depth}(\mathfrak{a}, M^G)$. Thus, it suffices to show that $\operatorname{depth}(\mathfrak{a}, M^G) \leq r + 1$ if $\operatorname{depth}(\mathfrak{a}, M) > r + 1$. For this, we quote Lemma 3.4.2(b) with $p_0 = 0$ and $q_0 = r$ (so $s = r$). $\square$

3.6 The Sylow subgroup of G

In this section, we focus on rings of invariants S^G rather than modules. Throughout, S is assumed to be Noetherian as an S^G-module. As we remarked earlier, this hypothesis is satisfied whenever S is an affine k-algebra for some commutative Noetherian ring k and G acts on S by k-algebra automorphisms. Furthermore, we assume that S has characteristic p, a positive prime; that is, $pS = \{0\}$. We let P denote a fixed Sylow p-subgroup of G.

Put

$$\mu = \mu(G, S) = \inf\{r > 0 \mid H^r(G, S) \neq 0\} \ .$$

Proposition 3.6.1. *Put $\mathcal{P} = \{P' \leq P \mid \text{height}\, I_S(P') \leq \mu + 1\}$. If S and S^G are both Cohen-Macaulay and $\mu < \infty$ then the restriction map*

$$\text{res}^G_{\mathcal{P}} \colon H^\mu(G, R) \to \prod_{P' \in \mathcal{P}} H^\mu(P', S)$$

is injective.

Proof. Let $0 \neq x \in H^\mu(G, S)$ be given and put $\mathfrak{a} = \text{ann}_{S^G}(x)$. Then, by Proposition 3.3.2,

$$\text{height}\, \mathfrak{a} \geq \inf\{\text{height}\, I_S(P') \mid P' \text{ a } p\text{-subgroup of } G,\ x\big|_{P'} \neq 0\} \ .$$

Since S^G is Cohen-Macaulay, $\text{height}\, \mathfrak{a} = \text{depth}\, \mathfrak{a}$. Finally, Proposition 3.5.1 with $M =_{S^G} S$ (note that M is Cohen-Macaulay, by Theorem 2.3.5) gives $\text{depth}\, \mathfrak{a} \leq \mu + 1$. Thus, there exists a p-subgroup P' of G with $x\big|_{P'} \neq 0$ and $\text{height}\, I_S(P') \leq \mu + 1$. Note that both the condition $x\big|_{P'} \neq 0$ and the value of $\text{height}\, I_S(P')$ are preserved upon replacing P' by a conjugate ${}^gP'$ with $g \in G$. Therefore, we may assume that $P' \in \mathcal{P}$, which proves the proposition. $\square$

3.7 Bireflections

We continue to use the notations and hypotheses of Section 3.6.

Following [Ke$_2$], we will call an element $g \in G$ a *bireflection* on S if height $I_S(\langle g \rangle) \leq 2$.

Corollary 3.7.1. *Assume that S and S^G are both Cohen-Macaulay. Let H denote the subgroup of G that is generated by all elements of G whose order is coprime with p and all bireflections in P. Then $S^G = S^G_H$.*

Proof. First note that H is a normal subgroup of G and G/H is a p-group. Thus, if $S^G \neq S^G_H$ or, equivalently, $\widehat{H}^0(G/H, S^H) \neq 0$ then also $H^1(G/H, S^H) \neq 0$; see [Br, Theorem VI.8.5]. In view of the exact sequence

$$0 \to H^1(G/H, S^H) \longrightarrow H^1(G, S) \xrightarrow{\operatorname{res}^G_H} H^1(H, S)$$

(see [Ba, 35.3]) we further obtain $H^1(G, S) \neq 0$. Thus, $\mu = 1$ holds in Proposition 3.6.1 and every $P' \in \mathcal{P}$ consists of bireflections. Therefore, $P' \subseteq H$ and Proposition 3.6.1 implies that $\operatorname{res}^G_H \colon H^1(G, S) \to H^1(H, S)$ is injective, contradicting the above exact sequence. Therefore, we must have $S^G = S^G_H$. □

Corollary 3.7.2. *Assume that S and S^G are both Cohen-Macaulay. If $\mathbb{F}_p = \mathbb{Z} \cdot 1_S$ is a G-module direct summand of S then $G = H$. In particular, if G is a p-group then G is generated by bireflections.*

Proof. If $\operatorname{tr}_{G/H} : S^H \to S^G$ is onto then we can write $1 = \operatorname{tr}(a) = \sum G/Hg(a)$. Since $\mathbb{F}_p$ is a direct summand of S, $a = u + r$ with $u \in \mathbb{F}_p$ and $r \in M$. Applying

$\mathrm{tr}_{G/H}$ on both sides, we get $1 = [G : H]u + \mathrm{tr}_{G/H}(r)$. Since $\mathbb{F}_p$ is a G-summand, $1 = [G : H]u$. Now by definition of H, $[G : H]$ is a power of p. This shows that $[G : H] = 1$ and $G = H$. The second assertion is immediate from this. $\square$

CHAPTER 4

Multiplicative Actions

Let G be a finite group acting by automorphisms on a $\mathbb{Z}$-lattice (free abelian group) A of rank n. In this case, A is often called a *G-lattice*. We extend the G-action on A action linearly to the group algebra $S = k[A]$ over a commutative ring k; so G acts on $S = k[A]$ by k-algebra automorphisms. Actions of this type are called *m*ultiplicative; the terms *e*xponential or *purely monomial* are also found in the literature. The term "purely monomial" refers to the fact that, by choosing a $\mathbb{Z}$-basis for A, one can identify $k[A]$ with the Laurent polynomial algebra $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the elements of A correspond to the monomials in the variables x_i and their inverses. Traditionally, the group operation in A is written as addition, $+$. On the other hand, when viewing $A \subset k[A]$, the lattice A becomes a multiplicative group.

The algebra S^G of G-invariant elements in S is called the *algebra of mul-*

tiplicative invariants. It is a standard fact that the group algebra $S = k[A]$ is Cohen-Macaulay precisely if the base ring k is Cohen-Macaulay; see [BH, Theorems 2.1.3 and 2.1.9]. In this chapter, we assume throughout that k is a field, although much of the material generalizes directly to the case where k is a Cohen-Macaulay ring. The above notations remain valid throughout the chapter. We will investigate the following question :

When is S^G Cohen-Macaulay ?

4.1 Passage to an effective lattice

Let

$$A^G = \{a \in A \mid g(a) = a \ \forall g \in G\}$$

denote the sublattice of G-invariants in A and let $\overline{} \colon A \twoheadrightarrow A/A^G$ denote the canonical map. Extend this map to a map

$$\overline{} \colon S = k[A] \to \overline{S} = k[A/A^G] \ .$$

Definition 4.1.1. We say that A is an *effective* G-lattice if $A^G = 0$.

Lemma 4.1.2. *$\overline{A} = A/A^G$ is an effective G-lattice.*

Proof. Let $\overline{a} \in \overline{A}^G$, with $a \in A$. Fix $g \in G$ and set $b = g(a) - a \in A^G$. Then $|G|b = \operatorname{tr} b = \operatorname{tr}(a) - \operatorname{tr}(a) = 0$. Since A^G is torsion free, $b = 0$. Therefore $g(a) = a$ for all $g \in G$ or $a \in A^G$. This shows that $\overline{A}^G = 0$. □

Let $G(a)$ denote the G-orbit $\{g(a) \mid g \in G\}$ of $a \in A$. The *orbit sum* of a is the element of $S = k[A]$ that is defined by

$$\sigma(a) = \sum_{b \in G(a)} b \ .$$

Note that $\sigma(a)$ belong to $S^G = k[A]^G$.

The following lemma reduces the general case of the Cohen-Macaulay problem for S^G to the case of the effective G-lattice $\overline{A} = A/A^G$. The lemma and its proof have been given to me by my advisor Martin Lorenz.

Lemma 4.1.3. *With the above notations, S^G is Cohen-Macaulay if and only if $\overline{S}^G = k[\overline{A}]^G$ is Cohen-Macaulay.*

Proof. We begin by constructing an auxiliary G-lattice $\widetilde{A}$ as follows. View G as acting on the $\mathbb{Q}$-vector space

$$A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$$

and let $\rho\colon A_{\mathbb{Q}} \twoheadrightarrow {A_{\mathbb{Q}}}^G$ denote the *Reynolds operator*, $\rho(v) = |G|^{-1} \sum_{g \in G} g(v)$. Define

$$\widetilde{A} = A + \rho(A) \subseteq A_{\mathbb{Q}} \ .$$

We make the following claims:

(i) $\widetilde{A} \simeq \widetilde{A}^G \oplus \overline{A}$ as G-lattices;

(ii) $A \subseteq \widetilde{A}$ and $\widetilde{A} = \dot{\bigcup}_{t \in \mathcal{T}} (t + A)$ (disjoint union) for some finite subset $\mathcal{T} \subseteq \widetilde{A}^G$.

To prove this, note that ρ and $\pi = 1 - \rho$ are orthogonal idempotents in $\mathrm{End}_{\mathbb{Q}[G]}(A_{\mathbb{Q}})$. Therefore, $A_{\mathbb{Q}} = \rho(A_{\mathbb{Q}}) \oplus \pi(A_{\mathbb{Q}}) = {A_{\mathbb{Q}}}^G \oplus \pi(A_{\mathbb{Q}})$ and $A \subseteq \rho(A) \oplus \pi(A) \subseteq |G|^{-1}A \subseteq A_{\mathbb{Q}}$. Consequently,

$$\widetilde{A} = A + \rho(A) = \rho(A) \oplus \pi(A)$$

and $\rho(A) = \widetilde{A}^G$. Moreover, $\pi(A) \cong A/\operatorname{Ker}_A(\pi) = A/A^G$. This proves Claim (i). Since $\widetilde{A}/A \subseteq (|G|^{-1}A)/A$, the index of A in $\widetilde{A}$ is finite, and since $\widetilde{A} = A + \widetilde{A}^G$, a transversal, $\mathcal{T}$, for A in $\widetilde{A}$ can be chosen from $\widetilde{A}^G$. Claim (ii) follows.

Now, turning to group rings, Claims (i) and (ii) above immediately imply that

$$k[\widetilde{A}] = \bigoplus_{t \in \mathcal{T}} tk[A] \simeq k[\overline{A}] \otimes_k k[\widetilde{A}^G]$$

and t can chosen to be in $\rho(A) \subset A^G$. By taking G-invariants, we obtain

$$k[\widetilde{A}]^G = \bigoplus_{t \in \mathcal{T}} tk[A]^G \simeq k[\overline{A}]^G \otimes_k k[\widetilde{A}^G] \ ;$$

so $k[\widetilde{A}]^G$ is finite and free over $k[A]^G$ and a Laurent polynomial algebra over $k[\overline{A}]^G$. In view of [BH, Theorems 2.1.3 and 2.1.9 and Exercise 2.1.23], we conclude that

$$\begin{aligned} k[A]^G \text{ is Cohen-Macaulay} &\Leftrightarrow k[\widetilde{A}]^G \text{ is Cohen-Macaulay} \\ &\Leftrightarrow k[\overline{A}]^G \text{ is Cohen-Macaulay} . \end{aligned}$$

This completes the proof of the lemma. $\square$

4.2 Cohomology $H^*(G, S)$

Definition 4.2.1. Let H be a subgroup of G. A left $k[G]$-module M is called *induced* from H to G if there exist a $k[H]$-module N such that $M \simeq k[G] \otimes_{k[H]} N$. In this case, we will write $M = N{\uparrow}_H^G$.

Since G is assumed finite, there is a natural isomorphism $\mathrm{Hom}_{k[H]}(k[G], N) \simeq N{\uparrow}_H^G$; see [Br, Proposition III(5.9)].

Fix a set of representatives τ for the G-orbits in A; so

$$A = \dot{\bigcup}_{a \in \tau} G(a) \ ,$$

where the union is a disjoint union. For each $a \in A$, let $G_a = \{g \in G \mid g(a) = a\}$ denote the stabilizer of a in G. Then

$$S = k[A] \cong \bigoplus_{a \in \tau} k{\uparrow}_{G_a}^G$$

as $k[G]$-modules, where k is the "trivial" $k[G_a]$-module, that is, every element of G_a acts as the identity on k. The Eckman-Shapiro Lemma [Br, Proposition III(6.2)] gives

$$H^*(G, k[A]) \cong \bigoplus_{a \in \tau} H^*(G_a, k) \ .$$

Here, we use finiteness of G for the fact that $H^*(G, \,.\,)$ commutes with direct sums; see [Br, Proposition VIII(4.6)].

Now assume that G acts *fixed-point freely* on A, that is, $G_a = \{1\}$ holds for all $0 \neq a \in A$. Then the above equation reduces to

$$H^*(G, k[A]) \cong H^*(G, k) \qquad (* > 0)$$

Therefore, if G acts fixed-point freely on A then the constant $\mu = \mu(G, k[A])$ introduced in Section 3.6 is given by

$$r_k(G) = \inf\{r > 0 \mid H^r(G, k) \neq 0\} \ .$$

It is known that $r_k(G) < \infty$ if and only if $\operatorname{char} k$ divides the order $|G|$; see [Be$_2$, Theorem 4.1.3].

Before stating and proving the main result of this section, we record the following simple lemma determining the height of the ideals $I_S(H)$ for subgroups $H \leq G$ in the present setting of multiplicative actions.

Lemma 4.2.2. *For any subgroup $H \leq G$,* $\operatorname{height} I_S(H) = \operatorname{rank} A - \operatorname{rank} A^H$.

Proof. By definition, the ideal $I_S(H)$ of $S = k[A]$ is generated by the elements $h(a) - a$, or else by the elements $h(a)a^{-1} - 1$ for $h \in H$, $a \in A$. Thus, $S/I_S(H) \cong k[A/[H, A]]$, where we have put $[H, A] = \langle h(a)a^{-1} \mid h \in H, a \in A\rangle \leq A$. Consequently, $\operatorname{height} I_S(H) = \dim S - \dim S/I_S(H) = \operatorname{rank} A - \operatorname{rank} A/[H, A]$. Finally, since the group algebra $\mathbb{Q}[H]$ is semisimple, $A \otimes \mathbb{Q} = (A^H \otimes \mathbb{Q}) \oplus ([H, A] \otimes \mathbb{Q})$; so $\operatorname{rank} A/[H, A] = \operatorname{rank} A^H$. $\square$

The following proposition is the main result of this section.

Proposition 4.2.3. *Assume that the action of G on $\overline{A}$ is fixed-point free and that $p = \operatorname{char} k$ divides $|G|$. If $\operatorname{rank}(\overline{A}) \geq r_k(G) + 2$ then S^G is not Cohen-Macaulay.*

Proof. By Lemma 4.1.3, we can assume without loss of generality that $\overline{A} = A$; so G acts fixed-point freely on A. In particular, by Lemma 4.2.2 above, we have height $I_S(H) = \operatorname{rank} A$ for every subgroup $1 \neq H \leq G$. Now assume that $\operatorname{rank} A \geq r_k(G) + 2$ but $S = k[A]$ is Cohen-Macaulay. To derive a contradiction, we will apply Proposition 3.6.1. As we pointed out above, $\mu = r_k(G) < \infty$. Let P be a fixed Sylow p-subgroup of G and define the set $\mathcal{P}$ as in Proposition 3.6.1. By the foregoing, $\mathcal{P}$ consists of the identity subgroup alone. Therefore, the product $\prod_{P' \in \mathcal{P}} H^\mu(P', S)$ in Proposition 3.6.1 is the trivial group, and hence $H^\mu(G, S)$ must be trivial as well, contradicting the definition of μ. □

The following corollary settles the case of groups of order $p = \operatorname{char} k$.

Corollary 4.2.4. *Suppose $|G| = p = \operatorname{char} k$. Then S^G is Cohen-Macaulay if and only if $\operatorname{rank}(\overline{A}) \leq 2$. In particular, if $p > 3$ and G acts non-trivially on A then S^G is never Cohen-Macaulay.*

Proof. Since $|G| = p$, the G-action on $\overline{A}$ is actually fixed-point free. Also, $r_k(G) = 1$. Thus, by Proposition 4.2.3, if $\operatorname{rank}(\overline{A}) \geq 3$ then S^G is not Cohen-Macaulay. On the other hand, if $\operatorname{rank}(\overline{A}) \leq 2$, then $k[\overline{A}]^G$ is certainly Cohen-

Macaulay; see [BH, Exercise 2.2.30(a)]. Thus, in view of Lemma 4.1.3, we obtain that S^G is Cohen-Macaulay iff $\operatorname{rank}(\overline{A}) \leq 2$, proving the first assertion of the corollary.

Since $\mathrm{GL}_n(\mathbb{Z})$ for $n \leq 2$ does not contain a subgroup of order $p > 3$, the second assertion also follows. □

Here is a simple explicit example.

Example 4.2.5. Suppose that $\operatorname{char} k = 2$ and $|G| = \{\pm Id_{n\times n}\}$ acts canonically on the lattice $A = \mathbb{Z}^n$. Then $k[A]^G$ is Cohen-Macaulay if and only if $n \leq 2$. Since $A = \overline{A}$ for the given action, this is immediate from Corollary 4.2.4.

4.3 Cyclic Sylow subgroups

The notation employed in the previous section remains in effect. In particular, we assume that $p = \operatorname{char} k$ divides the order $|G|$ of G, and we let P denote a fixed Sylow p-subgroup of G. Our main interest in this section is on the case where P is cyclic. The following lemma, however, is valid generally, even for arbitrary G-actions on a commutative ring S.

Lemma 4.3.1. *Suppose that G has a subgroup H such that $S^G_H = S^G$ and S^H is Cohen-Macaulay. Then S^G is Cohen-Macaulay as well.*

Proof. By virtue of our hypothesis $S^G_H = S^G$, the relative trace map $\operatorname{tr}_{G/H} \colon S^H \to S^G$ is surjective; so we may fix an element $c \in S^H$ with $\operatorname{tr}_{G/H}(c) = 1$. Define the map $\rho\colon S^H \to S^G$ by $\rho(s) = \mathrm{tr}_{G/H}(sc)$. Then ρ is a "Reynolds operator", that is, ρ is S^G-linear and the restriction of ρ to S^G is the identity map on S^G. Since S^H is integral over the subring S^H, we may apply a result of Hochster and Eagon [HE] (see also [BH, Theorem 6.4.5]) to conclude that the Cohen-Macaulay property descends from S^H to S^G, as we have claimed. □

Note that the hypothesis $S_H^G = S^G$ in the lemma is certainly satisfied whenever the index $[G : H]$ is invertible in S. For, in this case, the element $c = [G : H]^{-1} \in S^H$ satisfies $\mathrm{tr}_{G/H}(c) = 1$. Our main application of these observations in Chapter 4 will via the following

Corollary 4.3.2. *If S^P is Cohen-Macaulay then so is S^G.*

The converse of this corollary is false in general:

Example 4.3.3. Let $G = S_p$ denote the symmetric group on p symbols and let $A = \mathbb{Z}^p$. Then G acts on A by permuting the canonical basis $\{x_1, \ldots, x_p\}$ of A. By the classical theorem on elementary symmetric functions, the corresponding multiplicative invariant algebra is given by

$$k[A]^G = k[x_1^{\pm 1}, \ldots, x_p^{\pm 1}]^{S_p} = k[s_1, \ldots, s_{p-1}, s_p^{\pm 1}] \ ,$$

where the s_i denote the elementary symmetric functions. Thus, $k[A]^G$ is certainly Cohen-Macaulay. On the other hand, the Sylow p-subgroup P is cyclic of order p, generated by the p-cycle $(1, 2, \ldots, p)$. If $p > 3$ then P is not generated by a bireflection on A. Therefore, Corollary 3.7.2 shows that $k[A]^P$ is not Cohen-Macaulay.

For any finite group G, $O^p(G)$ denotes the intersection of all normal subgroups N of G so that G/N is a p-group. In other words, $O^p(G)$ is the subgroup of G that is generated by all p'-elements.

Theorem 4.3.4. *Assume that $O^p(G) \neq G$ and that P is cyclic. Then S^G is Cohen-Macaulay if and only if P is generated by a bireflection on A.*

Proof. First assume that S^G is Cohen-Macaulay. Let H denote the subgroup of G that is generated by $O^p(G)$ and all bireflections in P. Then Corollary 3.7.2 implies that $G = H$. It follows that $G/O^p(G) = P/P \cap O^p(G)$ is generated by the images of the bireflections in P. Since P is cyclic, it follows that P is generated by a bireflection.

Conversely, if P is generated by a bireflection on A then A/A^P has rank at most 2. Therefore, the invariant algebra $k[A/A^P]^P$ is Cohen-Macaulay; see [BH, Exercise 2.2.30(a)]. Lemma 4.1.3 gives that $k[A]^P$ is Cohen-Macaulay, and Corollary 4.3.2 further implies that $k[A]^G$ is Cohen-Macaulay. $\square$

CHAPTER 5

Multiplicative Invariants of Rank 3 Groups

If A is any G-lattice of rank at most 2 then the multiplicative invariant algebra $k[A]^G$ is a normal domain of dimension at most 2, and hence $k[A]^G$ is Cohen-Macaulay; see [BH, Exercise 2.2.30(a)]. As we have seen in Example 4.2.5, this does not hold in rank 3. Our aim in this chapter is to classify all multiplicative invariant algebras $k[A]^G$ with $\operatorname{rank} A = 3$ into two classes, Cohen-Macaulay and non-Cohen-Macaulay. In view of Lemma 4.1.3, this will entail an answer to the Cohen-Macaulay problem for all G-lattices A so that $\overline{A} = A/A^G$ has rank at most 3.

Thus, for the rest of this chapter, we assume that

$$A = \mathbb{Z}^3$$

and

$$G \text{ is a finite subgroup of } \mathrm{GL}_3(\mathbb{Z})$$

acting canonically on A. As is customary in group theory, we shall use the exponential notation a^g $(a \in A, g \in G)$ for the G-action on A, and likewise for the extended action on the group algebra $S = k[A]$. In other words, we think of the elements of A as integer row vectors on which the matrices in G act by right multiplication. Writing $a = (1,0,0), b = (0,1,0), c = (0,0,1)$ for the canonical basis of A, the group algebra $S = k[A]$ can be thought of as the Laurent polynomial algebra

$$S = k[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \ .$$

For example, the matrix $g = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_3(\mathbb{Z})$ acts on these generators via $a^g = a$, $b^g = c^{-1}$ and $c^g = b$.

Throughout, k will denote a field of characteristic p; it will turn out that only the case $p = 2$ needs to be considered in detail.

Our approach to the classification is to proceed case-by-case, using the machinery explained in the previous chapters along with explicit computations. To this end, we use the list of all finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$, up to conjugation, that is given in reference [T]. There are 73 such groups; one group listed in [T] is superfluous, as has been pointed out in [AGr]. Borrowing a convention from [HK], we will denote the group W_i on the page j in [T] by

$$W_i(j) \ .$$

(Occasionally, if there is no ambiguity, we will drop the page number j from the notation.) For example, $W_5(198)$ is the group listed as W_5 on the page 198 in [T]. It is claimed there that $W_5(198)$ has order 24, while the actual order is 12, as was pointed out in [AGr]; so the group $W_5(198)$ should be disregarded.

With this notation, our findings can be summarized in the following

Theorem 5.0.1. (a) *If* $\operatorname{char} k \neq 2$ *then all multiplicative invariants* $k[A]^G$ *in dimension* 3 *are Cohen-Macaulay.*

(b) *Assume that* $\operatorname{char} k = 2$. *Then, with exception of the groups conjugate to one of the following*

- $W_5(173)$ *(order* 2*),*
- $W_2(174)$ *and* $W_4(174)$ *(both cyclic of order* 4*),*
- $W_3(184)$ *and* $W_4(184)$ *(both cyclic of order* 6*),*

and the possible exception of the conjugates of

$$W_{14}(174)\ (\simeq C_2 \times C_2) \quad \textit{and} \quad W_{10}(191)\ (\simeq A_4)$$

all finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$ *have Cohen-Macaulay multiplicative invariant algebras.*

Thus far, we have not been able to settle the cases $W_{14}(174)$ and $W_{10}(191)$. Here, $W_{14}(174)$ is (up to conjugation) the Sylow 2-subgroup of $W_{10}(191)$. Thus, once $W_{14}(174)$ will be shown to be Cohen-Macaulay, which we expect, then it will follow that $W_{10}(191)$ is Cohen-Macaulay as well.

5.1 Some reductions

We first show that the five exceptions listed part (b) of Theorem 5.0.1 do indeed have non-Cohen-Macaulay multiplicative invariant algebras in characteristic 2.

Proposition 5.1.1. *Assume that $p = 2$. If G is conjugate to one of the following*

- $W_5(173)$ *(order 2),*
- $W_2(174)$ *and* $W_4(174)$ *(both cyclic of order 4),*
- $W_3(184)$ *and* $W_4(184)$ *(both cyclic of order 6),*

then the invariant algebra $k[A]^G$ is not Cohen-Macaulay.

Proof. Let P denote a Sylow 2-subgroup of G. By Theorem 4.3.4, it suffices to show that P is not generated by bireflection. Recall that bireflections are matrices $g \in \mathrm{GL}_3(\mathbb{Z})$ so that $\mathrm{rank}(g - Id) \leq 2$. (It follows from Lemma 4.2.2 that this condition is equivalent with the earlier definition of bireflections given in Section 3.7.) Since powers and conjugates of bireflections are again bireflections, it suffices to exhibit, for each of the groups $W_5(173), \ldots, W_4(184)$, a generator of P that is not a bireflection.

In each of the cases $W_5(173), W_3(184), W_4(184)$, the Sylow 2-subgroup P is generated by $-Id_{3\times 3}$, clearly not a bireflection. For $W_2(174)$ and $W_4(174)$, [T] gives the generators $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$, respectively. Both are not bireflections. $\square$

We remark that the groups in the above proposition are indeed the *only* finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$ that cannot be generated by bireflections.

Our next proposition gives a number of sufficient conditions for Cohen-Macaulayness of the invariant algebra. In particular, part (a) of Theorem 5.0.1 is covered by part (a) below.

Proposition 5.1.2. *Each of the following conditions entails that the invariant algebra $k[A]^G$ is Cohen-Macaulay:*

(a) $p = \operatorname{char} k \neq 2$;

(b) *the order $|G|$ is odd;*

(c) *G has non-trivial fixed points in A, that is, $A^G \neq 0$;*

(d) *G can be generated by reflections, that is, by matrices $g \in \mathrm{GL}_3(\mathbb{Z})$ so that* $\operatorname{rank}(g - Id) = 1$.

Proof. (a) The order of $|G|$ is a divisor of 48; see [T]. Thus, if $p \neq 2, 3$ then $k[A]^G$ is Cohen-Macaulay by Corollary 3.4.4. Assume now that $p = 3$. Then

P has order 3. Hence, by [T, Proposition 3], P is conjugate to one of the following two groups:

$$W_1(173) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \right\rangle \qquad \text{or} \qquad W_2(173) = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$$

Both these matrices are bireflections; so $k[A]^P$ is Cohen-Macaulay by Corollary 4.2.4, and Corollary 4.3.2 further implies that $k[A]^G$ is Cohen-Macaulay. This proves (a).

(b) Assume that $|G|$ is odd. If $p \neq 2$ then $k[A]^G$ is Cohen-Macaulay by part (a), and if $p = 2$ then the same conclusion follows from Corollary 3.4.4 or Corollary 4.3.2.

(c) If $A^G \neq 0$ then $\overline{A} = A/A^G$ has rank at most 2. Therefore, $k[\overline{A}]^G$ is Cohen-Macaulay, by [BH, Exercise 2.2.30(a)]. Lemma 4.1.3 now gives that $k[A]^G$ is Cohen-Macaulay.

(d) This is a consequence of the main result of [Lo_1] which asserts that multiplicative invariants $k[A]^G$ of reflection groups G (in any rank) are affine normal semigroup algebras. Cohen-Macaulayness of $k[A]^G$ now follows from a result of Hochster; see [BH, Theorem 6.3.5(a)]. $\square$

In view of part (a) above, we will henceforth assume that

$$p = \operatorname{char} k = 2\ .$$

The remaining sections each are devoted to finite subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of a given even order.

We conclude this section with a simple lemma that will be useful in our explicit computations. The lemma is independent of the particular setting of multiplicative actions and holds more generally.

Lemma 5.1.3. *Let G be a finite group acting faithfully on a commutative integral domain S.*

(a) *Let $S_0 \subseteq S^G$ be a Noetherian subring such that S is free of rank r as S_0-module. Then $|G|$ divides r and S^G is finitely generated as S_0-module, with minimal number of generators $\geq r/|G|$. Moreover, S^G is free as S_0-module if and only if it can be generated by $r/|G|$ elements.*

(b) *Let $H \subseteq G$ be a subgroup of G and suppose that $S^H = \sum_{i=1}^m \eta_i S^G$ where $m = [G : H]$ is the index of H in G. Then $\{\eta_i\}_1^m$ is a free S^G-module basis of S^H.*

Proof. Denote by K the quotient field of S. The action of G can be extended to K by putting $(a/b)^g = a^g/b^g$.

(a) The fact that S^G is finitely generated as S_0-module follows from Noetherianness of S_0. Let m denote the minimal number of module generators and write $S^G = \sum_{i=1}^m x_i S_0$. Let F denote the quotient field of S_0; so $F \subseteq K$. The subring $S^G F = \sum_{i=1}^m x_i F \subseteq K$ has dimension at most m over F and is contained in the fixed field K^G. Therefore, $K^G = S^G F$ and $[K^G : F] \leq m$. Similarly, $K = SF$ and our freeness hypothesis on S over S_0 entails that

$[K : F] = r$. Since $[K : K^G] = |G|$, by Galois theory, we conclude that $r = |G|[K^G : F] \leq |G|m$. This proves the estimate $m \geq r/|G|$. Moreover, equality holds here if and only if $[K^G : F] = m$, which in turn is equivalent to $\{x_i\}_1^m$ being F-independent or, equivalently, S_0-independent. This proves (a).

(b) The fixed field K^H contains the subring $S^H K^G$ and $[K^H : K^G] = [G : H]$, by Galois theory. This entails that $K^H = S^H K^G$. Now assume that $S^H = \sum_{i=1}^m \eta_i S^G$ with $m = [G : H]$. Then $K^H = S^H K^G = \sum_{i=1}^m \eta_i K^G$; so $\{\eta_i\}_1^m$ is a K^G-basis of K^H. Therefore, $\{\eta_i\}_1^m$ is surely S^G-independent, proving the lemma. □

5.2 Groups of order 2

There are 5 conjugacy classes of subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 2.

Proposition 5.2.1. *If $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ is a group of order 2, then S^G is Cohen-Macaulay if and only if $G \neq \langle -Id_{3\times 3}\rangle = W_5(173)$.*

Proof. We already know that the multiplicative invariants of $W_5(173)$ are not Cohen-Macaulay; see Proposition 5.1.1. The remaining 4 conjugacy classes of subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 2 are represented by the groups with the following generators:

$$\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{or} \qquad \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

These matrices all have fixed points. Thus for those four groups, S^G is Cohen-Macaulay by Proposition 5.1.2(c). $\square$

5.3 Order 4

There are 14 conjugacy classes of subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 4; these are represented by the groups $W_i(174)$ for $i = 1, \ldots, 14$.

Proposition 5.3.1. *If* $G = W_i(174)$ $(i \neq 14)$, *then* S^G *is Cohen-Macaulay if and only if* $i \neq 2$ *or* 4.

Proof. By Proposition 5.1.1, we know that the multiplicative invariants of $W_2(174)$ and of $W_4(174)$ are not Cohen-Macaulay. We consider the cases $G = W_i(174)$ $(i \neq 2, 4, 14)$ in turn.

• The following 7 groups have fixed points; so the ring of invariants of these groups are Cohen-Macaulay by Proposition 5.1.2(c):

(i) $W_1(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

(ii) $W_3(174)$ has fixed points $(2, 1, 1)\mathbb{Z}$.

(iii) $W_7(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

(iv) $W_9(174)$ has fixed points $(1, 0, 0)\mathbb{Z}$.

(v) $W_{10}(174)$ has fixed points $(0, 1, 1)\mathbb{Z}$.

(vi) $W_{13}(174)$ has fixed points $(0,1,1)\mathbb{Z}$.

It remains to show that if $G = W_i(174)$ where i = 5, 6, 8, 11, 12 then S^G is Cohen-Macaulay. We will do this case by case.

- Let $G = W_5(174) = \left\langle g = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right), h = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)\right\rangle$. Clearly,

$$W_5(174) \subset H = \left\langle \left(\begin{smallmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{smallmatrix}\right) \right\rangle$$

Observe that

$$S^H = S_0 = k[u, v, w] \ ,$$

where we have put $u = a + a^{-1}$, $v = b + b^{-1}$ and $w = c + c^{-1}$. Thus,

$$\begin{aligned} S_0 \subset S^G \subset k[a^{\pm1},\, b^{\pm1},\, c^{\pm1}] &= (S_0 \oplus aS_0) \otimes (S_0 \oplus bS_0) \otimes (S_0 \otimes cS_0) \\ &= S_0 \oplus aS_0 \oplus bS_0 \oplus cS_0 \oplus abS_0 \oplus acS_0 \oplus bcS_0 \oplus abcS_0 \ . \quad (5.3.1) \end{aligned}$$

Let $f \in S^G$. Then there exists $f_i \in S_0$ $(0 \le i \le 7)$, such that $f = f_0 + af_1 + bf_2 + cf_3 + abf_4 + acf_5 + bcf_6 + abcf_7$. Comparing $f^g = f$ and substituting $b^g = b^{-1} = v + b$, $c^g = c^{-1} = w + c$, we get the following equation:

$$vf_2 + wf_3 + a(vf_4 + wf_5 + vwf_7) + b(wf_6) + c(vf_6) + ab(wf_7) + ac(vf_7) = 0 \ .$$

This implies that $f_6 = f_7 = 0$ and $vf_2 = wf_3 = vwf_1'$, $vf_4 = wf_5 = vwf_2'$ for suitable $f_1', f_2' \in S_0$. Thus,

$$f = f_0 + af_1 + (bw + cv)f_1' + (abw + acv)f_2' \qquad (5.3.2)$$

Applying h and using $f^h = f$, we get $f_1 = f_2' = 0$ This shows that $S_0 \subset S^G \subset S_0 + (bw+cv)S_0 \subset S^G$. Thus $S^G = S_0 + (bw+cv)S_0$. Since the index of G in H is

2, Lemma 5.1.3(b) implies that this sum is direct. Thus S^G is Cohen-Macaulay. Alternatively, this follows from the observation that $G = G_1 \times G_2$, where G_1 acts by inversion on a and trivially on $\langle b, c\rangle$, while G_2 acts by inversion on $\langle b, c\rangle$ and is trivial on a. Therefore, $S^G = k[a^{\pm 1}]^{G_1} \otimes_k k[b^{\pm 1}, c^{\pm 1}]^{G_2}$. Since both factors are Cohen-Macaulay, being normal domains of rank 1 and 2, S^G is Cohen-Macaulay as well.

- When $i = 6$, $W_6(174) = \left\langle g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$. Note that g is identical with the matrix denoted by g in the case of $W_5(174)$. Let $f \in S^{W_6}$. Since $g \in W_6 \subset H$, we can still deduce equation (5.3.2) from the condition $f^g = f$. Applying $f^h = f$, we further derive $f = f_0 + (avw + buw + cuv)f_1$. Thus $S^{W_6} = S_0 \oplus (avw + buw + cuv)S_0$. Thus, as above, Lemma 5.1.3 implies that S^{W_6} is Cohen-Macaulay.

- Suppose $G = W_8$:

$$W_8(174) = \left\langle g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle .$$

Take $G' = \left\langle \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$. Then G' is a group of order 16 containing W_8 and $S^{G'} = S_0 = k[u, s, t]$, where $s = v + w$ and $t = vw$ with u, v, w as above. As $S_0 \subset S^{W_8} \subset S$ and S is a free module over S_0 with basis $1, a, av, b, bv, c, cv, ab, abv, bc, bcv, ac, acv, abc, abcv$, any $f \in S^{W_8}$ is an S_0-linear combination of this basis. Using $f^g = f, f^h = f$ we get

$$f = f_0 + (uv + as)f_1 + (bs + bv + cv)f_2 + (btu + cvs + abs^2 + abvs + acvs)f_3$$

Thus $S^{W_8} = S_0+(uv+as)S_0+(bs+bv+cv)S_0+(btu+cvs+abs^2+abvs+acvs)S_0$. Since the index of W_8 in G is 4, Lemma 5.1.3(b) shows that this is a direct sum, and so S^{W_8} is Cohen-Macaulay.

- Suppose $G = W_{11}$:

$$W_{11}(174) = \left\langle g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle .$$

W_{11} is also a subgroup of G' defined for the case $i = 8$ above; so we can use the methods and notations developed there. If $f \in S^{W_{11}}$ then $f^g = f$, $f^h = f$ implies that $f = f_a + (u + as + bu + cu)f_1 + (bs + bv + cv)f_2 + (cvs + abs + abv + acv + bcu)f_3$. Thus $S^{W_{11}} = S_0 \oplus (u + as + bu + cu)S_0 \oplus (bs + bv + cv)S_0 \oplus (cvu + abs + abv + acv + bcu)S_0$. As above, this shows that $S^{W_{11}}$ is Cohen-Macaulay.

- Now we consider the case $G = W_{12}$:

$$W_{12}(174) = \left\langle g = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle .$$

Denote by H the subgroup of W_{12} generated by h. It is easy to see that $S^H = k[\alpha, \beta, \gamma^{\pm 1}, \delta]$, where, $\alpha = a + a^{-1}$, $\beta = b^{-1} + c^{-1}$, $\gamma = bc$, $\delta = ab + a^{-1}c$. One can easily check that α satisfies the monic quadratic polynomial

$$t^2 - (\beta\delta + \beta^2)t + \gamma^{-1}\beta^2$$

over $A = k[\beta, \gamma^{\pm 1}, \delta]$. Therefore, $S^H = A \oplus \alpha A$. Since $H \subset W_{12}$, $S^{W_{12}} \subset S^H$. Now g acts on S^H as follows: $\beta^g = \delta$, $\delta^g = \beta$, $\gamma^g = \gamma^{-1}$ and $\alpha^g = \alpha$. Put

$B = k[\beta+\delta, \gamma+\gamma^{-1}, \beta\delta]$. Then $B \subset S^{W_{12}} \subset S^H$ and $A = B \oplus \delta B \oplus \gamma B \oplus \delta\gamma B$. Thus,

$$S^H = B \oplus \delta B \oplus \gamma B \oplus \alpha B \oplus \delta\gamma B \oplus \delta\alpha B \oplus \gamma\alpha B \oplus \alpha\delta\gamma B.$$

Now let $f \in S^{W_{12}} \subset S^H$. Then there exist $f_0, \ldots, f_7 \in B$ such that

$$f = f_0 + \delta f_1 + \gamma f_2 + \alpha f_3 + \delta\gamma f_4 + \delta\alpha f_5 + \gamma\alpha f_6 + \alpha\delta\gamma f_7.$$

Now using $f^g = f$, we obtain relations among f_i. Solving these relation gives,

$$f = f_0 + [\delta(\gamma+\gamma^{-1}) + \gamma(\delta+\beta)]f_1' + \alpha f_3 + [\delta\alpha(\alpha+\alpha^{-1}) + \gamma\alpha(\beta+\delta)]f_2',$$

for some f_1', $f_2' \in B$. Thus $S^G = B + [\delta(\gamma+\gamma^{-1}) + \gamma(\delta+\beta)]B + \alpha B + [\delta\alpha(\alpha + \alpha^{-1}) + \gamma\alpha(\beta+\delta)]B$. Since the index of H in W_{12} is 2, this is a direct sum decomposition, by a Galois theoretic argument analogous to the one used in the proof of Lemma 5.1.3. Thus, $S^{W_{12}}$ is a free module over a Cohen-Macaulay algebra, and hence it is Cohen-Macaulay. $\square$

5.4 Order 6

There are 10 conjugacy classes of subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 6, with representative groups $W_i(184)$ for $i = 1, \ldots, 10$.

Proposition 5.4.1. *If $G = W_i(184)$, then S^G is Cohen-Macaulay if and only if $i \neq 3$ or 4*

Proof. As we have pointed out in Proposition 5.1.1, the invariant algebras of $G = W_i(184)$ for $i = 3$ or 4 are not Cohen-Macaulay.

For the remaining groups $G = W_i(184)$, $i \neq 3, 4$, let P denote the Sylow 2-subgroup of G as usual. In each case, one easily checks that $P \neq \langle -Id_{3\times 3}\rangle$. Thus, Proposition 5.2.1 implies that $k[A]^P$ is Cohen-Macaulay. Hence, by Corollary 4.3.2, $k[A]^G$ is Cohen-Macaulay for these groups. □

5.5 Groups of order 12

Proposition 5.5.1. *If the order of G is* 12*, then S^G is Cohen-Macaulay, with the possible exception of groups conjugate to $W_{10}(191) \simeq A_4$.*

Proof. Representative groups for the 11 conjugacy classes of subgroups of order 12 in $\mathrm{GL}_3(\mathbb{Z})$ are the groups $W_i(193)$ with $i = 1, \ldots, 11$. These groups are isomorphic to $C_6 \times C_2$ ($i = 1$), the dihedral group D_{12} ($2 \leq i \leq 8$) and the alternating group A_4 ($9 \leq i \leq 11$). In each case, the Sylow 2-subgroup P of G is isomorphic to the Klein four-group $C_2 \times C_2$. Therefore, P is not conjugate to the two non-Cohen-Macaulay cases of order 4, $W_2(174)$ and $W_2(174)$, both of which are cyclic; see Proposition 5.3.1. To conclude that S^P is Cohen-Macaulay (and hence S^G as well), it suffices to rule out the possibility that P is conjugate to the as yet unsettled potential non-Cohen-Macaulay case $W_{14}(174)$.

Note that $W_{14}(174) \subset \mathrm{SL}_3(\mathbb{Z})$. Therefore, if $W_i(193)$ contains an element of determinant -1 then the 2-part of this element will then also have determinant -1, and hence P cannot be conjugate to $W_{14}(174)$. This observation takes care of the groups $W_i(193)$ with $i \neq 2, 9, 10, 11$; see [T, Corollary on p. 191].

For the remaining groups $W_i(193)$ with $i = 2, 9, 10, 11$, we note that, up to conjugation, the Sylow 2-subgroups are given by:

- $W_8(174) \in Syl_2(W_2(191))$;
- $W_6(174) \in Syl_2(W_9(191))$;
- $W_{12}(174) \in Syl_2(W_{11}(191))$;
- $W_{14}(174) \in Syl_2(W_{10}(191))$;

Thus, except for the case of $W_{10}(191)$, the Sylow 2-subgroup has a Cohen-Macaulay invariant algebra, as desired. □

5.6 Orders 8 and 24

Proposition 5.6.1. *The invariants S^G for subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 8 are all Cohen-Macaulay. Consequently, the same conclusion holds for all subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ of order 24.*

Proof. By Corollary 4.3.2, we may concentrate on groups of order 8, since groups of order 24 have a Sylow 2-subgroup of order 8. There are 14 conjugacy classes of subgroups of order 8 in $\mathrm{GL}_3(\mathbb{Z})$, represented by the groups $W_i(187)$ $(i = 1, \ldots, 14)$. We will consider these groups in turn.

- The groups $W_8(187) = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \right\rangle$ and $W_{12}(187) = \left\langle \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right) \right\rangle$ have non-trivial fixed points; so Proposition 5.1.2(c) applies.

- It turns out that all subgroups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ isomorphic to $C_2 \times C_2 \times C_2$ are generated by reflections, and hence their rings of invariants S^G are Cohen-Macaulay, by Proposition 5.1.2(d). These are the groups $W_3(187)$, $W_4(187)$, $W_5(187)$ and $W_6(187)$.

- Next, we treat the groups $W_1 = W_1(187)$, $W_7 = W_7(187)$, $W_9 = W_9(187)$ and $W_{10} = W_{10}(187)$. To this end, let

$$g = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right) \qquad \text{and} \qquad h = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right) .$$

Then $W_1 = \langle -Id_{3\times 3}, h\rangle$, $W_7 = \langle g, h\rangle$, $W_9 = \langle g, -h\rangle$, and $W_{10} = \langle -g, -h\rangle$. Note that all these groups are contained in the group

$$G = \left\langle \left(\begin{smallmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{smallmatrix}\right), h \right\rangle .$$

This is the group $W_1(194)$ of order 16. The invariants of G are easy to calculate directly. Alternatively, since G is a reflection group, we may use the methods of [Lo$_1$]. The result is that S^G is a polynomial algebra:

$$S^G = S_0 = k[u, s, t] ,$$

where $u = a + a^{-1}$, $s = b + c + b^{-1} + c^{-1}$ and $t = (b + b^{-1})(c + c^{-1})$. In addition, we put $v = b + b^{-1}$. Then one has the following decomposition of S as S_0-module:

$$S = S_0 \oplus vS_0 \oplus aS_0 \oplus avS_0 \oplus bS_0 \oplus bvS_0 \oplus cS_0 \oplus cvS_0 \oplus abS_0$$
$$\oplus\, abvS_0 \oplus acS_0 \oplus acvS_0 \oplus bcS_0 \oplus bcvS_0 \oplus abcS_0 \oplus abcvS_0\ . \qquad (5.6.1)$$

Thus, when considering an invariant $f \in S^H$ with $H \in \{W_1, W_7, W_9, W_{10}\}$, we will assume that f is written in the form

$$\begin{aligned} f &= f_0 + vf_1 + af_2 + avf_3 + bf_4 + bvf_5 + cf_6 + cvf_7 + abf_8 + abvf_9 + acf_{10} \\ &\quad + acvf_{11} + bcf_{12} + bcvf_{13} + abcf_{14} + abcvf_{15}. \end{aligned} \qquad (5.6.2)$$

with $f_i \in S_0 \subseteq S^H$. Our goal will be to show that S^H, as module over S_0, can be generated by 2 elements. Lemma 5.1.3(b) then implies that S^H is indeed free over S_0, and hence Cohen-Macaulay. We now consider the four groups $H \in \{W_1, W_7, W_9, W_{10}\}$ separately.

$\underline{W_1 = \langle -Id_{3\times 3}, h\rangle}$: Let $f \in S^{W_1}$, written in the form (5.6.2). Then $f^h = f$ gives the following relations:

(i) $f_5s^2 + f_4s + f_1s + f_5t = 0$, $f_9t + f_8s + f_3s + f_9s^2 = 0$,

(ii) $f_{12}s + f_7s + f_6 + f_{13}s^2 + f_{13}t + f_4 = 0$, $f_6 + f_5s + f_4 = 0$,

(iii) $f_{13}s + f_5 + f_{12} + f_7 = 0$, $f_4 + f_5s = 0$, $f_5 + f_7 = 0$,

(iv) $f_8 + f_{15}t + f_{10} + f_{15}s^2 + f_{11}s + f_{14}s = 0$,

(v) $f_{10} + f_9 s + f_8 = 0$, $f_{14} + f_9 + f_{11} + f_{15}s = 0$, $f_9 + f_{11} = 0$,

(vi) $f_9 s + f_8 = 0$, $f_{13}s = 0$, $f_{15}s = 0$.

Similarly, $f^{-Id} = f$ gives the following relations:

(i) $f_9 ut + f_{14}ut + f_{11}ut + f_7 t + f_6 s + f_{10}us + f_{12}t + f_2 u + f_5 t = 0$,

(ii) $f_9 t + f_{14}t + f_{11}t + f_{10}s = 0$, $f_8 u + f_{15}ut + f_{13}t + f_{12}s + f_{14}us = 0$,

(iii) $f_{13}t + f_{10}u + f_{15}ut = 0$, $f_9 u + f_{14}u + f_{12} = 0$, 6. $f_{10}u + f_5 s + f_{13}t + f_6 + f_3 u + f_8 u + f_4 + f_{15}ut + f_9 us = 0$,

(iv) $f_{14}u + f_{11}u + f_{12} + f_{15}us + f_{13}s = 0$, $f_{15}t + f_{14}s = 0$,

(v) $f_{15}t = 0$, $f_{14} = 0$, $f_{14} + f_{15}s = 0$, $f_8 + f_{10} + f_9 s + f_{15}t = 0$,

Solving these equations yields $f_i = 0$ $(i = 3, 6, 8, 9, 10, 11, 12, 13, 14, 15)$, $f_1 s = f_5 t = stf'$ for some $f' \in S_0 = k[u, s, t]$ and $f_4 = f_5 s = f_7 s = 0$. Therefore, $S^{W_1(187)} = S_0 + (vt + bs^2 + bvs + cvs)S_0$, whence S^{W_1} is Cohen-Macaulay.

$\underline{W_7 = \langle g, h\rangle}$: For $f \in S^{W_7}$, the condition $f^g = f$ is equivalent to the following relations: $f_3 = f_{15} = 0$, $f_9 = f_{11}, f_8 + f_9 s = f_{10}$, $f_4 + f_6 + f_7 s + f_8 u = 0$, $f_1 s = f_2 u$. These relations and the relations derived above (case W_1) from $f^h = f$ give us $f_i = 0$ $(i = 6, 8, 9, 10, 11, 12, 13, 14, 15)$, $f_5 = f_7$, $f_4 = sf_7$, and $f_1 s = f_7 t = f_2 u = stuf'$ for some $f' \in S_0$. Therefore, $f = f_0 + (vut + ast +$

$bs^2u + bvsu + cvsu)f'$, and so $S^{W_7} = S_0 + (vut + ast + bs^2u + bvsu + cvsu)S_0$, as desired.

$\underline{W_9 = \langle g, -h\rangle}$: Let $f \in S^{W_9}$. As in the case of W_7, the condition $f^g = f$ amounts to following relations: $f_3 = f_{15} = 0$, $f_9 = f_{11}$, $f_8 + f_9 s = f_{10}$, $f_4 + f_6 + f_7 s + f_8 u = 0$, $f_1 s = f_2 u$. Similar relations result from $f^{-h} = f$. When combined with the above relations, they give $f_i = 0$ $(i = 4, \ldots, 15)$ and $f_3 u = f_2 s = usf$ for some $f \in S_0$. Thus $f = f_0 + (uv + as)f'$ for some $f' \in S_0$. Thus, as before, we conclude that S^{W_9} is Cohen-Macaulay.

$\underline{W_{10} = \langle -g, -h\rangle}$: For an invariant $f \in S^{W_{10}}$, the condition $f^{-g} = f$ amounts to the following relations: $f_{13} = f_{15} = 0$, $f_9 + f_{11} + f_{14} = 0$, $f_8 + f_{10} +_{f9} s = 0$, $f_{12} + f_7 + f_5 = 0$, $f_9 s + f_8 + f_3 = 0$, $f_4 + f_6 + f_5 s = 0$, $f_6 = f_1$. These relations and similar relations derived from $f^{-h} = f$ give:

$f_i = 0$ for $i = 1, 3, 6, 8, 10, 11, 12, 13, 14, 15$, $f_7 = f_5$, $f_4 = f_5 s$, $f_5 t = f_2 u$.

Therefore $f = f_0 + (at + bsu + bvu + cvu)f'$. Since the index of W_{10} in G is 2, $S^{W_{10}}$ is Cohen-Macaulay.

• It remains to consider the groups W_2, W_{11}, W_{13} and W_{14}, where $W_i = W_i(187)$. For this, we put

$$g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The groups W_2, W_{11}, W_{13} and W_{14} are generated by $\{g, -Id\}$, $\{g, -h\}$,

$\{-g, -h\}$ and $\{-g, h\}$ respectively. All of them are subgroups of the reflection group $W_2(195)$ of order 16, whose ring of invariants (strictly) contains the polynomial subalgebra

$$S_0 = k[r, s, t] \ ,$$

where $r = a^2bc + (a^2bc)^{-1}$, $s = b + b^{-1} + c + c^{-1}$ and $t = (b + b^{-1})(c + c^1)$. Put $u = a^2bc$, $v = b + b^{-1}$ and $w = c + c^{-1}$. Then the ring $S = k[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ is a free S_0-module:

$$\begin{aligned} S &= S_0 + vS_0 + aS_0 + avS_0 + bS_0 + bvS_0 + cS_0 + cvS_0 + abS_0 \\ &\quad + abvS_0 + acS_0 + acvS_0 + bcS_0 + bcvS_0 + abcS_0 + abcvS_0 + a^2bcS_0 \\ &\quad + a^2bcvS_0 + a^3bcS_0 + a^3bcvS_0 + a^2b^2cS_0 + a^2b^2cvS_0 + a^2bc^2S_0 \\ &\quad + a^2bc^2vS_0 + a^3bcbS_0 + a^3bcbvS_0 + a^3bc^2S_0 + a^3bc^2vS_0 \\ &\quad + a^2b^2c^2S_0 + a^2b^2c^2vS_0 + a^3b^2c^2S_0 + a^3b^2c^2vS_0 \end{aligned}$$

Thus any invariant $f \in S^G$, where G is one of the 4 subgroups W_2, W_{11}, W_{13} or W_{14}, can be written as follows:

$$\begin{aligned} f &= f_0 + vf_1 + af_2 + avf_3 + bf_4 + bvf_5 + cf_6 + cvf_7 + abf_8 + abvf_9 \\ &\quad + acf_{10} + acvf_{11} + bcf_{12} + bcvf_{13} + abcf_{14} + abcvf_{15} \\ &\quad + uf_{16} + uvf_{17} + uaf_{18} + uavf_{19} + ubf_{20} \\ &\quad + ubvf_{21} + ucf_{22} + ucvf_{23} + uabf_{24} + uabvf_{25} \\ &\quad + uacf_{26} + uacvf_{27} + ubcf_{28} + ubcvf_{29} + uabcf_{30} + uabcvf_{31} \end{aligned}$$

with $f_i \in S_0$. Our goal will be to show, for each of the groups $G \in \{W_2, W_{11}, W_{13}, W_{14}\}$ that the invariant algebra S^G can be generated as S_0-module by 4 elements. Then, applying Lemma 5.1.3(a) with $r = 32$ and $|G| = 8$, we deduce that S^G is free over S_0, and hence S^G is Cohen-Macaulay.

$\underline{G = W_2 = \langle g, -Id_{3\times 3}\rangle}$: Suppose $f \in S^G$. Using $f^g = f^{-Id} = f$ gives us the following relations:

(i) $f_4 s + f_5 s^2 + f_1 s + f_5 t = 0,\ f_9 s + f_2 + f_8 = 0,$

$f_{13} t + f_4 + f_7 s + f_{13} s^2 + f_{12} s + f_6 = 0,$

(ii) $f_5 s + f_4 + f_6 = 0,\ f_{17} s + f_{21} t + f_{21} s^2 + f_{20} s = 0,$

(iii) $f_{29} s + f_{23} + f_{28} + f_{21} = 0,\ f_{23} + f_{21} = 0,\ f_5 + f_{12} + f_{13} s + f_7 = 0,$

$f_{15} s + f_8 + f_{14} = 0,\ f_{30} + f_{26} + f_{27} s = 0,$

(iv) $f_5 s + f_4 = 0,\ f_{10} + f_2 + f_3 s = 0,\ f_5 + f_7 = 0,\ f_{11} s + f_{10} + f_{14} = 0,$

$f_{24} + f_{25} s + f_{18} = 0,\ f_{13} s = 0,$

(v) $f_{22} + f_{28} s + f_{29} t + f_{23} s + f_{20} + f_{29} s^2 = 0,\ f_{20} + f_{21} s = 0,\ f_{31} + f_{27} = 0,$

(vi) $f_{21} s + f_{20} + f_{22} = 0,\ f_{31} + f_{25} = 0,\ f_{27} + f_{19} = 0,\ f_{11} + f_{15} = 0,\ f_{25} + f_{19} = 0,$

$f_{31} s + f_{30} + f_{24} = 0,\ f_{19} s + f_{26} + f_{18} = 0,\ f_{15} + f_9 = 0,$

(vii) $f_3 + f_{11} = 0,\ f_3 + f_9 = 0,\ f_{12} t + f_5 t + f_{23} rt + f_{22} rs + f_6 s + f_{28} rt + f_{16} r +$

$f_7 t + f_{21} rt = 0,\ f_2 + f_{30} r^2 + f_{30} + f_{14} r = 0,$

(viii) $f_{28}rs + f_{12}s + f_{20}r + f_{13}t + f_{29}rt = 0$, $f_{13}t + f_{22}r + f_{29}rt = 0$, $f_{28}t + f_{21}t + f_{23}t + f_{22}s = 0$, $f_{28} = 0$, $f_{28} + f_{29}s = 0$, $f_{12} + f_{28}r + f_{21}r = 0$,

(ix) $f_{10}r + f_{26}r^2 + f_{26} + f_8 = 0$, $f_2 + f_{30} + f_{18}r = 0$, $f_{13}t + f_{17}r + f_{20}r + f_6 + f_4 + f_5s + f_{22}r + f_{29}rt + f_{21}rs = 0$, $f_{24} + f_{10} + f_{24}r^2 + f_8r = 0$,

(x) $f_{23}r + f_{28}r + f_{13}s + f_{12} + f_{29}rs = 0$, $f_2r + f_{18}r^2 + f_{14} + f_{18} = 0$, $f_{14} + f_{18} + f_{30}r = 0$, $f_{28}r = 0$, $f_{29}t + f_{21}s + f_{20} + f_{22} = 0$,

(xi) $f_3 + f_{19}r + f_{31} = 0$, $f_{25} + f_{11} + f_{27}r = 0$, $f_9 + f_{27} + f_{25}r = 0$, $f_{15} + f_3r + f_{19} + f_{19}r^2 = 0$, $f_{31}r + f_{15} + f_{19} = 0$, $f_{24} + f_{10} + f_{26}r = 0$,

(xii) $f_{24}r + f_{26} + f_8 = 0$, $f_{27}r^2 + f_9 + f_{11}r + f_{27} = 0$, $f_{11} + f_9r + f_{25}r^2 + f_{25} = 0$, $f_{15}r + f_31r^2 + f_{31} + f_3 = 0$,

These equations imply that $f_i = 0$ for $i = 6, 12, 13, 16, 17, 20, 21, 22, 23, 28,$

29

$f_4 = f_5s = f_7s$, $f_1s = f_5t$, $f_3 = f_9 = f_{11} = f_{15} = f_{19}(r+1)$, $f_{19} = f_{25} = f_{27} = f_{31}$, $f_2 = f_14 = f_{30}(r+1)$, $f_{18} = f_{30}$, $f_{24} = f_{26}$, $f_{24}(r+1) = f_8$, $f_8 = f_{10}$, $f_3s + f_2 + f_8$, $f_{24} + f_{25}s + f_{18}$,

Solving this gives us,

$f = f_0 + (tv + s^2b + svb + svc)x + (sac + srab + uva + sab + uvac + va + rva + uvab + baus + srac + uvabc + caus)y + (rac + rab + a + ac + rvabc + ab + uabc + ua + vab + rvab + vac + rvac + vabc + uab + ra + abc + rabc + uac)z.$

Therefore,

$S^{W_2} = S_0 + (tv + s^2b + svb + svc)S_0 + (sac + srab + uva + sab + uvac + va + rva + uvab + baus + srac + uvabc + caus)S_0 + (rac + rab + a + ac + rvabc + ab + uabc + ua + vab + rvab + vac + rvac + vabc + uab + ra + abc + rabc + uac)S_0$.

Hence S^{W_2} is Cohen-Macaulay.

$\underline{G = W_{11} = \langle g, -h\rangle}$: Again, let $f \in S^G$. From $f^g = f^{-h} = f$ we get the following relations:

(i) $f_4s + f_1s + f_5t + f_5s^2 = 0$, $f_2 + f_8 + f_9s = 0$, $f_{12}s + f_{13}s^2 + f_{13}t + f_6 + f_7s + f_4 = 0$,

(ii) $f_6 + f_5s + f_4 = 0$, $f_{20}s + f_{17}s + f_{21}s^2 + f_{21}t = 0$, $f_{10} + f_3s + f_2 = 0$,

(iii) $f_{20} + f_{29}s^2 + f_{22} + f_{29}t + f_{28}s + f_{23}s = 0$, $f_{21}s + f_{20} = 0$, $f_{20} + f_{21}s + f_{22} = 0$, $f_{18} + f_{25}s + f_{24} = 0$,

(iv) $f_{14} + f_8 + f_{15}s = 0$, $f_4 + f_5s = 0$, $f_5 + f_{13}s + f_7 + f_{12} = 0$, $f_7 + f_5 = 0$,

(v) $f_9 + f_3 = 0$, $f_{30} + f_{26} + f_{27}s = 0$, $f_{14} + f_{11}s + f_{10} = 0$, $f_{26} + f_{19}s + f_{18} = 0$, $f_{15} + f_9 = 0$,

(vi) $f_3 + f_{11} = 0$, $f_{19} + f_{25} = 0$, $f_{21} + f_{29}s + f_{28} + f_{23} = 0$, $f_{23} + f_{21} = 0$,

(vii) $f_{24} + f_{30} + f_{31}s = 0$, $f_{11} + f_{15} = 0$, $f_{25} + f_{31} = 0$,

(viii) $f_{27} + f_{19} = 0$, $f_{31} + f_{27} = 0$, $f_{29}s = 0$, $f_{13}s = 0$,

(ix) $f_{17}rs+f_{12}t+f_7t+f_23rt+f_5t+f_{13}st+f_{21}rt+f_1s+f_{20}rs+f_4s+f_{21}rs^2+f_{16}r+f_5s^2+f_{29}rst+f_{28}rt=0$, $f_{30}r^2+f_2+f_{31}s+f_{30}+f_{14}r+f_{31}r^2s+f_{15}rs=0$,

(x) $f_6+f_4+f_{28}rs+f_7s+f_{22}r+f_{12}s+f_{23}rs+f_{13}s^2+f_{13}t+f_{29}rs^2+f_{29}rt=0$, $f_{21}rs+f_4+f_{20}r+f_6+f_{13}t+f_{29}rt+f_5s=0$, $f_{17}s+f_{29}st+f_{21}t+f_{28}t+f_{23}t+f_{21}s^2+f_{20}s=0$,

(xi) $f_{27}r^2s+f_{10}r+f_{10}+f_{26}r^2+f_{11}rs+f_{26}+f_{27}s=0$, $f_{29}s^2+f_{20}+f_{29}t+f_{22}+f_{28}s+f_{23}s=0$, $f_{29}t+f_{20}+f_{21}s+f_{22}=0$,

(xii) $f_{29}t+f_{20}+f_{21}s+f_{22}=0$, $f_{18}+f_{14}+f_{15}s+f_{30}r+f_{31}rs=0$, $f_{24}r^2+f_{25}s+f_8r+f_8+f_9rs+f_{24}+f_{25}r^2s=0$,

(xiii) $f_{21}rs+f_{20}r+f_{22}r+f_{13}t+f_{29}rt+f_{17}r+f_6+f_4+f_5s=0$, $f_{23}r+f_5+f_7+f_{28}r+f_{12}+f_{13}s+f_{29}rs=0$,

(xiv) $f_{28}r+f_5+f_{21}r+f_{12}+f_7=0$, $f_{31}+f_{15}r+f_3+f_31r^2=0$,

(xv) $f_{30}+f_2+f_3s+f_{19}rs+f_{18}r=0$, $f_{19}s+f_{14}+f_3rs+f_{19}r^2s+f_{18}+f_{18}r^2+f_2r=0$, $f_{11}s+f_{26}r+f_{27}rs+f_{26}+f_{10}=0$,

(xvi) $f_9r+f_9+f_{25}+f_{25}r^2=0$, $f_{27}+f_{27}r^2+f_{11}+f_{11}r=0$, $f_{19}+f_{15}+f_{31}r=0$,

(xvii) $f_{21}+f_{28}+f_{23}+f_{29}s=0$, $f_{21}+f_{23}+f_{28}=0$,

(xviii) $f_{24}r+f_{24}+f_{25}rs+f_9s+f_8=0$, $f_{15}+f_{19}r^2+f_3r+f_{19}=0$,

(xix) $f_{25}+f_{25}r+f_9=0,\ f_{27}r+f_{11}+f_{27}=0,\ f_{31}+f_{19}r+f_3=0,\ f_{29}r=0,$

(xx) $f_{29}rs+f_{13}s+f_{28}r=0$

Solving these equations we get, $f_i=0$ for $i=6,12,13,17,20,21,22,23,$ $28,29,$ $f_7=f_5, f_4=f_5s, f_1s=f_5t, f_1s=f_{16}r,$ $f_{19}=f_{27}=f_{25}=f_{31},$ $f_{19}(1+r)=f_3=f_9=f_{11}=f_{15},$ $f_{10}=f_8,$ $f_{26}=f_{24},$ $f_{30}=f_{18},$ $f_{14}=f_2,$ $f_2+f_8=f_{19}(1+r)s,$ $f_{18}+f_{24}=f_{19}s,$ $f_2+f_{18}(1+r)=f_{19}s,$ $f_8+f_{24}(1+r)=f_{19}s.$

This shows that

$f=f_0+(vrt+cvrs+brs^2+ust+bvrs)x+(ua+uab+uac+ar+abc+abr+acr+ay+ab+uabc+ac+abcr)y+(uav+uabs+uacv+uabcv+uabv+avr+as+abcs+abvr+uacs+abrs+abv+abcvr+acv+acvr+abcv+av+acrs)z,$

where $rstx=f_1s=f_5t=f_{16}r,$ $y=f_{18}$ and $z=f_{19}$ are in S_0.

In other words

$S^{W_{11}}=S_0+(vrt+cvrs+brs^2+ust+bvrs)S_0+(ua+uab+uac+ar+abc+abr+acr+ay+ab+uabc+ac+abcr)S_0+(uav+uabs+uacv+uabcv+uabv+avr+as+abcs+abvr+uacs+abrs+abv+abcvr+acv+acvr+abcv+av+acrs)S_0.$

Hence $S^{W_{11}}$ Cohen-Macaulay.

$\underline{G=W_{13}=\langle -g,-h\rangle}$: The conditions $f^{-g}=f^{-h}=f$ lead to the following relations:

(i) $f_7t+f_{16}r+f_{23}rt+f_1s+f_{17}rs=0,\ f_{11}rs+f_{27}r^2s+f_{27}st+f_{10}r+f_{26}r^2+f_{26}t+f_2=0,$

(ii) $f_4+f_6+f_7s+f_{23}rs+f_{22}r=0,\ f_{29}rt+f_5s+f_{13}t+f_{20}r+f_6+f_{21}rs+f_4=0,$

(iii) $f_{17}s+f_{23}t=0,\ f_6+f_{17}r+f_{22}r=0,\ f_{23}+f_{21}=0,\ f_{23}+f_{21}+f_{28}=0,$

$f_{20}+f_{22}+f_{23}s=0.$

(iv) $f_{28}r+f_{13}s+f_{29}rs=0,\ f_{29}r=0,\ f_{28}r+f_5+f_7+f_{21}r+f_{12}=0,$

(v) $f_{18}+f_{10}+f_{27}rs+f_{26}r+f_{11}s=0,\ f_{27}t+f_{27}r^2+f_3+f_{11}r=0,\ f_{27}r+$

$f_{11}+f_{19}=0,$

(vi) $f_{30}t+f_{14}r+f_{10}+f_{31}st+f_{15}rs+f_{30}r^2+f_{31}r^2s=0,\ f_9+f_{25}r+f_{31}=0,$

$f_{30}+f_{24}r+f_9s+f_{25}rs+f_8=0,$

(vii) $f_{25}r^2+f_9r+f_{25}t+f_{15}=0,\ f_{14}+f_{25}r^2s+f_9rs+f_{25}st+f_{24}t+f_{24}r^2+f_8r=0,$

(viii) $f_{19}r+f_3+f_{25}=0,\ f_3s+f_{19}rs+f_{24}+f_{18}r+f_2=0,\ f_{19}t+f_{19}r^2+f_3r+f_9=0,$

(ix) $f_{19}st+f_{18}r^2+f_3rs+f_2r+f_{19}r^2s+f_8+f_{18}t=0,$

(x) $f_{15}+f_{27}+f_{31}r=0,$

$f_{14}+f_{30}r+f_{31}rs+f_{26}+f_{15}s=0,$

(xi) $f_{15}r+f_{11}+f_{31}r^2+f_{31}t=0,\ f_{20}+f_{29}t+f_{21}s+f_{22}=0,\ f_{23}r+f_7+f_5=0,$

(xii) $f_7t+f_{16}r+f_{23}rt+f_1s+f_{17}rs=0,$

(xiii) $f_{11}rs+f_{27}r^2s+f_{27}st+f_{10}r+f_{26}r^2+f_{26}t+f_2=0,\ f_4+f_6+f_7s+f_{23}rs+$

$f_{22}r=0,$

(xiv) $f_{29}rt + f_5s + f_{13}t + f_{20}r + f_6 + f_{21}rs + f_4 = 0,\ f_{17}s + f_{23}t = 0,$

(xv) $f_6 + f_{17}r + f_{22}r = 0,\ f_{23} + f_{21} = 0,$

(xvi) $f_{23} + f_{21} + f_{28} = 0,\ f_{20} + f_{22} + f_{23}s = 0,\ f_{29}s = 0,\ f_{28}r + f_{13}s + f_{29}rs = 0,$

(xvii) $f_{29}r = 0,\ f_{28}r + f_5 + f_7 + f_{21}r + f_{12} = 0,$

(xviii) $f_{18} + f_{10} + f_{27}rs + f_{26}r + f_{11}s = 0,\ f_{27}t + f_{27}r^2 + f_3 + f_{11}r = 0,$

(xix) $f_{27}r + f_{11} + f_{19} = 0,\ f_{30}t + f_{14}r + f_{10} + f_{31}st + f_{15}rs + f_{30}r^2 + f_{31}r^2s = 0,$

(xx) $f_9 + f_{25}r + f_{31} = 0,\ f_{30} + f_{24}r + f_9s + f_{25}rs + f_8 = 0,$

(xxi) $f_{25}r^2 + f_9r + f_{25}t + f_{15} = 0,\ f_{14} + f_{25}r^2s + f_9rs + f_{25}st + f_{24}t + f_{24}r^2 + f_8r = 0,$

(xxii) $f_{19}r + f_3 + f_{25} = 0,\ f_3s + f_{19}rs + f_{24} + f_{18}r + f_2 = 0,$

(xxiii) $f_{19}t + f_{19}r^2 + f_3r + f_9 = 0,\ f_{19}st + f_{18}r^2 + f_3rs + f_2r + f_{19}r^2s + f_8 + f_{18}t = 0,$

(xxiv) $f_{15} + f_{27} + f_{31}r = 0,\ f_{14} + f_{30}r + f_{31}rs + f_{26} + f_{15}s = 0,\ f_{15}r + f_{11} + f_{31}r^2 + f_{31}t = 0,$

(xxv) $f_{20} + f_{29}t + f_{21}s + f_{22} = 0,\ f_{23}r + f_7 + f_5 = 0,\ f_{22} = 0,$

Which shows that,

$f_i = 0$ for $i = 2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22, 24, 25, 26, 27$

$, 28, 29, 30, 31.$ and $f_7t + f_{16}r + f_1s = 0,\ f_{21} + f_{23} = 0,\ f_{20} = f_{21}s,\ f_{17}s + f_{21}t = 0,$

$f_6 + f_{17}r = 0,\ f_5 + f_7 + f_{21}r = 0.$

Thus $S^{W_{13}}$ is generated by four generators, and so it is Cohen-Macaulay.

$G = W_{14} = \langle -g, h\rangle$: Comparing $f^{-g} = f^h = f$ for any $f \in S^G$, we find the following relations:

(i) $f_7t + f_{23}rt + f_{16}r + f_1s + f_{17}rs = 0.$

(ii) $f_{26}r^2 + f_{27}st + f_{27}r^2s + f_{10}r + f_{26}t + f_{11}rs - f_2 = 0$, 3. $f_7s + f_6 + f_{22}r - f_4 + f_{23}rs = 0$

(iii) $f_6 + f_{20}r + f_{21}rs + f_{29}rt + f_4 + f_5s + f_{13}t = 0.$

(iv) $f_{17}s + f_{23}t = 0,\ f_{22}r + f_6 + f_{17}r = 0,\ f_{22} = 0.$

(v) $f_5 + f_7 + f_{23}r = 0,\ f_{20} + f_{29}t + f_{21}s = 0.$

(vi) $f_{23}s - f_{20} = 0,\ f_3r + f_9 + f_{19}t + f_{19}r^2 = 0.$

(vii) $f_{31}r^2+f_{15}r+f_{31}t-f_{11} = 0,\ f_{27}+f_{15}+f_{31}r = 0,\ f_7+f_5+f_{21}r+f_{12}+f_{28}r = 0.$

(viii) $f_2r+f_{18}t-f_8+f_3rs+f_{18}r^2+f_{19}st+f_{19}r^2s = 0$, 8. $f_{30}r^2+f_{14}r-f_{10}+f_{30}t+f_{15}rs+f_{31}r^2s+f_{31}st = 0,\ f_{24}t-f_{14}+f_{24}r^2+f_8r+f_9rs+f_{25}r^2s+f_{25}st = 0,$

(ix) $f_{15}s + f_{31}rs + f_{30}r - f_{26} + f_{14} = 0,\ f_{19} + f_{11} + f_{27}r = 0.$

(x) $f_{26}r - f_{18} + f_{27}rs + f_{10} + f_{11}s = 0,\ f_{11}r + f_{27}t - f_3 + f_{27}r^2 = 0,\ f_{29} = 0.$

(xi) $f_9 + f_{25}r - f_{31} = 0,\ f_{30} + f_8 + f_{24}r + f_{25}rs + f_9s = 0,\ f_{23} + f_{21} + f_{28} = 0.$

(xii) $f_{15} + f_9 r + f_{25} r^2 + f_{25} t = 0,\ f_{13} s + f_{28} r + f_{29} rs = 0.$

(xiii) $f_{25} + f_{19} r + f_3 = 0,\ f_{18} r + f_3 s + f_2 - f_{24} + f_{19} rs = 0.\ f_{21} + f_{23} = 0$,

These combined with $f^h = f$ gives,

$f_i = 0$ for $i = 1, 3, 6, 9, 11, 12, 13, 15, 17, 19, 20, 21, 22, 23, 25, 27, 28, 29, 31.$

$f_4 = f_5 s = s f_7 = 0,\ f_7 t = f_{16} r,\ f_8 = f_{30} + f_{24} r = 0,\ f_2 = f_{18} r + f_{24} = 0,$ $f_{18} t = f_{30},\ f_{26} = f_{24} t,$

$f_{26} r = f_{10} + f_{30} t,\ f_{30} r = f_{26} + f_{14} = 0,\ f_{26} r = f_{18} + f_{10} = 0$

which shows that

$f = f_0 + (cvr + ut + brs + bvr)x + (uab + a + uact)y + (uabct + ar)z$

Thus $S^{W_{14}} = S_0 + (cvr + ut + brs + bvr)S_0 + (uab + a + uact)S_0 + (uabct + ar)S_0$ has 4 generators as S_0-module, and hence S_{14}^W is Cohen-Macaulay.

□

5.7 Groups of order 16 and 48

There are 2 non-conjugate subgroups of order 16 and 3 non-conjugate subgroups of order 48.

Proposition 5.7.1. *If $G \subset \mathrm{GL}_3(\mathbb{Z})$ is a group of order 16 or 48, then S^G is Cohen-Macaulay.*

Proof. All groups of order 16, the conjugates of the groups $W_1(194)$ and

$W_2(195)$, are reflection groups. Hence their invariant algebras are Cohen-Macaulay by Proposition 5.1.2.

If G is of order 48, then it has a 2-Sylow subgroup of order 16. In view of Corollary 4.3.2, this implies that S^G is Cohen-Macaulay. $\square$

REFERENCES

[AM] A. Adem and R.J. Milgram, *Cohomology of Finite Groups*, Springer-Verlag, Berlin, 1994.

[AGr] E. Ascher and H. Grimmer, *Comment on a paper by Tahara on the finite subgroups of* $\mathrm{GL}(3,\mathbb{Z})$, Nagoya Math. J. **48** (1972), 203.

[AG] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1959), 749–765.

[Ba] A. Babakhanian, *Cohomological Methods in Group Theory*, Marcel Dekker, New York, 1972.

[Be_1] D.J. Benson, *Representations and Cohomology*, Vol. I, Cambridge Univ. Press, Cambridge, 1995.

[Be_2] D.J. Benson, *Representations and Cohomology*, Vol. II, Cambridge Univ. Press, Cambridge, 1995.

[Bou_1] N. Bourbaki, *Commutative Algebre, chap. 1-7*, Springer Verlag , New York, 1989.

[Bou_2] N. Bourbaki, *Groupes et algèbres de Lie*, chap. 4–6, Hermann, Paris, 1968.

[BS] M.P. Brodmann and R. Y. Sharp, *Local Cohomology*, Cambridge University Press, Cambridge, 1998.

[Br] K.S. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[C...] H. E. A. Campbell, A. V. Geramita, I. P. Hughes, R. J. Shank, D. L. Wehlau, *Non-Cohen-Macaulay vector invariants and a Noether bound for Gorenstein ring of invariants*, Canad. Math. Bull. **42** (1999), 155–161.

[CHR] S.U. Chase, D.K. Harrison, and A. Rosenberg, *Galois Theory and Cohomology of Commutative Rings*, Memoirs of the Amer. Math. Soc., No. 52, Amer. Math. Soc., Providence, 1965.

[C] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **67** (1955), 778–782.

[Co] I. S. Cohen, *On the stucture and ideal theory of complete local rings* , Trns. Amer. Math. Soc.**59** (1946), 54–106.

[CoR] L. Corry and S. Rosset, *The splitting data of cohomology classes*, Arch. Math. **44** (1985), 418–423.

[CR] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley & Sons, New York, 1962.

[HE] M. Hochster and J.A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[E] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geunetry*, Springer Verlag,New York, 1995.

[ES] G. Ellingsrud and T. Skjelbred, *Profondeur d'anneaux d'invariants en caracteristique p*, Compos. Math. **41** (1980), 233–244.

[F_1] D. R. Farkas, *Multiplicative invariants* , L'Enseignement Math. 30 (1984), 141-157.

[F_2] D. R. Farkas, *Reflection groups and multiplicative invariants* , Rocky Mt. J. 16 (1986), 215-222.

[F_3] D. R. Farkas, *Toward multiplicative invariant theory*, "Group Actions on Rings" (S. Montgomery, ed.), Amer. Math. Soc., Providence, RI, Contemporary Math.43 (1985), 69-80.

[H_1] D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473–534.

[H_2] D. Hilbert, *Über die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313–373.

[HE] M. Hochster and J.A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[HK] M. Hajja and Ming-chang Kang, *Three Dimensional Purely monomial Group Actions*, J. Algebra **170** (1994), 806-860.

[Ke_1] G. Kemper, *On the Cohen-Macaulay property of modular invariant rings*, J. Algebra **215** (1999), 330–351.

[Ke_2] G. Kemper, *Die Cohen-Macaulay Eigenschaft in der modularen Invariantentheorie*, Habilitationsschrift, Universität Heidelberg, 1999.

[Ke_3] G. Kemper, *The depth of invariant rings and cohomology* (with an appendix by Kay Magaard), IWR Preprint **99-25**, Heidelberg, 1999.

[Lo_1] M. Lorenz, *Multiplicative invariants and semigroup algebras*, Algebras and Representation Theory **4**, 293-304 (2001).

[Lo_2] M. Lorenz, *Class groups of multiplicative invariants*, J. Algebra **177** (1995), 242–254.

[Lo_3] M. Lorenz, *Regularity of multiplicative invariants*, Commun. in Algebra **24** (1996), 1051–1055.

[LP] M. Lorenz and J. Pathak, *On Cohen Macaulay ring of Invariants*, J. Algebra **245** (2001), 247-264.

[M] F.S.Macaulay, *The algebraic theory of modular system*, Cambridge Univ. Press, 1916.

[Nak] H. Nakajima, *Invariants of finite abelian groups generated by transvections*, Tokyo J. Math. **3** (1980), 201–204.

[N_1] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1916), 89–92; reprinted in *Collected Papers*, pp. 181–184, Springer-Verlag, New York, 1983.

[N_2] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Charakteristik p*, Nachr. Ges. Wiss. Göttingen (1926), 28–35; reprinted in *Collected Papers*, pp. 485–492, Springer-Verlag, New York, 1983.

[Ros] J. E. Roseblade, *Prime ideals in group algebras of polycyclic groups*, Proc. London Math. Soc. **36** (1978), 385–447; *Corrigenda*, Proc. London Math. Soc. **38** (1979), 216–218.

[Rot] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, Orlando, 1979.

[Sa] D. J. Saltman, *Multiplicative field invariants*, J. Algebra **106** (1987), 221–238.

[Sl] N. J. A. Sloane, *Error correcting codes and invariant theory: New applications of a nineteenth-century technique*, Amer. Math. Monthly **84** (1977), 82–107.

[Sm_1] L. Smith, *Polynomial invariants of finite groups, a survey of recent developments*, Bull. Amer. Math. Soc. **34** (1997), 211–250.

[Sm_2] L. Smith, *Homological codimension of modular rings of invariants and the Koszul complex.*, J. Math. Kyoto Univ. **38-4**(1998), 727–747.

[ST] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6**(1954), 274–304.

[St] B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, Wien, 1993.

[T] Ken-Ichi Tahara, *On the Finite subgroups of* $\mathrm{GL}(3, \mathbb{Z})$, Nagoya Math J, **41** (1971) 169-209.