On Monge-Ampère Type Equations Arising In Optimal Transportation Problems

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ABSTRACT

On Monge-Ampère Type Equations Arising In Optimal Transportation Problems

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In this dissertation we study Monge-Ampère type equations arising in optimal transportation problems. We introduce notions of weak solutions, and prove the stability of solutions, the comparison principle and the analogous maximum principle of Aleksandrov-Bakelman-Pucci. We also establish a quantitative estimate of Aleksandrov type for c-convex functions which generalizes the well known estimate of Aleksandrov proved for convex functions. These results are in turn used to give a positive answer for the solvability and uniqueness of the Dirichlet problems for any continuous boundary condition and for finite Borel measures provided the domains satisfy a so called c-strictly convex condition which we have introduced.

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To my parents,

in memory of my brother, Quang.

TABLE OF CONTENTS

ABSTRACT **iv**

ACKNOWLEDGEMENTS **v**

DEDICATION **vi**

1 Introduction **1**

2 Generalized Monge-Ampère Measures **5**
2.1 c-subdifferential and c-convexity 5
2.2 A Monge-Ampère measure associated with the cost function c 13

3 Notions Of Weak Solutions **21**
3.1 Aleksandrov Solutions 21
3.2 Viscosity Solutions 27

4 Maximum Principles **28**

5 Comparison principles **37**

6 Dirichlet Problems **45**
6.1 Homogeneous Dirichlet Problem 45
6.2 Nonhomogeneous Dirichlet Problem 51
6.3 Nonhomogeneous Dirichlet Problem with general right hand side 58

7 Second Boundary Value Problems **69**
7.1 Aleksandrov Solution ⇒ Brenier Solution 69
7.2 c^*-convexity and preliminary lemmas 73
7.3 Brenier solution ⇒ Aleksandrov solution 75

8 Aleksandrov Type Estimates **80**

A Perron Method For The Dirichlet Problems **84**

REFERENCES **92**

CHAPTER 1

Introduction

The problem of optimal transportation is to find an optimal map that pushes masses from one location to another, where the optimality depends upon the context of the problem. These types of problems appear in several forms and in various areas of mathematics and its applications: economics, probability theory, optimization, meteorology, and computer graphics. We refer to [**RR98**] for a detailed and complete description of the probabilistic approach of Kantorovitch to this problem and to the Preface to Volume I of this work for a large number of examples of applications in econometrics, probability, quality control, etc.

The mathematical formulation of the optimal transportation problem considered in this thesis originates with Gaspar Monge 1746–1818. Let $f, g \in L^1(\mathbb{R}^n)$ be nonnegative compactly supported with $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$, and let $d\mu = f\,dx$, $d\nu = g\,dx$. The Borel measurable map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is measure preserving with respect to μ and ν if $\mu(\phi^{-1}(E)) = \nu(E)$ for each Borel set $E \subset \mathbb{R}^n$. Let $\mathcal{S}(\mu,\nu)$ denote the class of all these measure preserving maps, and let $c : \mathbb{R}^n \to \mathbb{R}$ be a convex function[1], the cost function. The problem is minimizing the cost functional

$$\mathcal{C}(s) = \int_{\mathbb{R}^n} c(x - s(x))\, d\mu(x)^2 \tag{1.1}$$

[1] The convexity assumption is technical and to be able to use the tools of convex analysis.

[2] A more general cost function can be used: $c = c(x, y)$, but for simplicity we

among all $s \in \mathcal{S}(\mu, \nu)$, and the answer is given by the following theorem due to Caffarelli, Gangbo and McCann, see [**Caf96**], [**GM96**] and also [**Urb98**].

Theorem 1.1 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be C^1 and strictly convex, f, g and $\mathcal{C}$ as above. Then*

1. *there exists $t \in \mathcal{S}(\mu, \nu)$ such that $\mathcal{C}(t) = \inf_{s \in \mathcal{S}(\mu,\nu)} \mathcal{C}(s)$;*
2. *t is essentially unique, i.e., if the infimum is attained also at $\bar{t}$, then $t(x) = \bar{t}(x)$ for a.e. x in the supp(f);*
3. *t is essentially one to one, that is, there exists $t^* \in \mathcal{S}(\nu, \mu)$ such that $t^*(t(x)) = x$ for a.e. $x \in$ supp(f), and $t(t^*(y)) = y$ for a.e. $y \in$ supp(g);*
4. *there exists a c-convex function u such that t is given by the formula*

$$t(x) = x - (Dc)^{-1}(-Du(x)).$$

Monge's original problem is the case $c(x) = |x|$, and the minimizer is not unique, see [**EG99**, **TW01**] for recent results.

The objective in this thesis is to study the following fully nonlinear pde of Monge-Ampère type arising in the problem of optimal transport:

$$g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)] = f(x) \qquad \text{in } \Omega, \quad (1.2)$$

where Ω is a bounded open set, $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ is positive, $f \in L^1_{\text{loc}}(\Omega)$ is nonnegative, and c^* is the Legendre-Fenchel transform of c, see (2.2). Solutions to this equation are understood in a weak sense and in a way parallel to the notion of weak solution to the Monge-Ampère equation this time with a notion of subdifferential associated with the cost function c, see Definition 3.1. Note that when the cost function is given by $c(x) = \frac{1}{2}|x|^2$ then (1.2) is reduced to the standard Monge-Ampère equation

$$g(Dv) \det D^2v(x) = f(x) \qquad \text{in } \Omega, \quad (1.3)$$

choose $c = c(x - y)$.

with $v(x) = \frac{1}{2}|x|^2 + u(x)$. A particular case of (1.3) is the Gaussian curvature equation arising in differential geometry. A fundamental difference between equation (1.2) and equation (1.3) is that the principle part of the latter has affine invariant structure whereas that of the first one no longer shares this nice property unless the cost function is quadratic. This fact presents serious difficulties in extending the classical results to the present setting since the analysis for the standard Monge-Ampère equation heavily relies on the affine invariant structure of the equation. Our main results in this thesis are notions of weak solutions, comparison and maximum principles for this equation, and the solvability of the Dirichlet problem in this class of solutions. Also, we establish a quantitative estimate of Aleksandrov type for c-convex functions.

The organization of the thesis is the following. In Chapter 2 we study the notions of convexity and subdifferential associated with the cost function c and define the notion of generalized Monge-Ampère measure associated with (1.2). The notions of weak solutions to (1.2) are given in Chapter 3 where we also prove a stability property, Corollary 3.1. Chapter 4 contains maximum principles extending to the present setting the Aleksandrov-Bakelman-Pucci estimate for the Monge-Ampère operator. Chapter 5 contains the proofs of the comparison principles. Chapters 4 and 5 have independent interest and are used later to solve the Dirichlet problem. In Chapter 6 we solve the Dirichlet problem for a class of domains strictly convex with respect to the cost function c, see Definition 6.2, first for the homogeneous case, Theorem 6.1, next for the case when the right hand side is a sum of deltas, Theorem 6.2, and finally for general right hand sides, Theorem 6.3 and Corollary 6.1. We also consider the second boundary value problems in Chapter 7 where we show that Aleksandrov solution and Brenier solution for the problems are equivalent when the target domain is c^*-convex. In Chapter 8 we establish a quantitative estimate of Aleksandrov type for c-convex functions which generalizes the well known estimate of Aleksandrov proved for convex functions. We end the thesis by an appendix showing that the standard Perron's method can be carried out to prove Theorem 6.2 provided that we assume in addition a subsolution to the problem exists.

CHAPTER 2

Generalized Monge-Ampère Measures

Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and Ω be an open set in $\mathbb{R}^n$.

2.1 c-subdifferential and c-convexity

[1]

Definition 2.1 *Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$. The c-subdifferential $\partial_c u(x)$ at $x \in \Omega$ is defined by*

$$\partial_c u(x) = \{p \in \mathbb{R}^n : u(z) \geq u(x) - c(z-p) + c(x-p), \forall z \in \Omega\}.$$

Also for $E \subset \Omega$ we define $\partial_c u(E) = \cup_{x \in E} \partial_c u(x)$.

Remark 2.1 If $c(x) = \frac{1}{2}|x|^2$, then it is clear that $p \in \partial_c u(x)$ if and only if $p \in \partial(u+c)(x)$, i.e., $\partial_c u(x) = \partial(u+c)(x)$ where ∂ denotes the standard subdifferential.

Definition 2.2 *A function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, is c-convex in Ω if there is a set $A \subset \mathbb{R}^n \times \mathbb{R}$ such that*

$$u(x) = \sup_{(y,\lambda) \in A} [-c(x-y) - \lambda] \text{ for all } x \in \Omega.$$

[1]Introduced in [**Die88**] and [**EN74**].

Proposition 2.1 *If* $c(x) = \frac{1}{2}|x|^2$*, then* u *is* c*-convex if and only if* $u + \frac{1}{2}|x|^2$ *is convex.*

PROOF: It follows immediately noticing that $c(x-y) = \frac{1}{2}\langle x-y, x-y\rangle$. □

Remark 2.2 The definition of c-convexity is not stable by linear operations. For example, from Proposition 2.1 it follows that the function $u(x) = 1 - \frac{1}{2}|x|^2$ is $|x|^2/2$-convex, but $2u(x) = 2 - |x|^2$ is not $|x|^2/2$-convex. However it can be proved, see [**Vil03**], that if u is c-convex then tu is also c-convex for every $t \in [0,1]$.

Remark 2.3 If u is a c-convex function in Ω, then there exists a function $\phi : W \to \mathbb{R}$ such that

$$u(x) = \sup_{y \in W} [-c(x-y) - \phi(y)] \text{ for all } x \in \Omega, \tag{2.1}$$

where W is the projection of A into $\mathbb{R}^n$, i.e., $W = \{y \in \mathbb{R}^n : (y, \lambda) \in A \quad \text{for some} \quad \lambda \in \mathbb{R}\}$. Indeed, for each $y \in W$ define $\phi(y) = \inf\{\lambda : (y, \lambda) \in A\}$. Then $\phi(y) > -\infty$, since otherwise there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $(y, \lambda_n) \in A$ and $\lambda_n \downarrow -\infty$. But then we get $-c(x-y) - \lambda_n \uparrow +\infty$ for all $x \in \Omega$, and therefore $u(x) = +\infty$ for all $x \in \Omega$ which contradicts the definition of c-convex function. So we have $\phi : W \to \mathbb{R}$.

Let us now prove (2.1). If $x \in \Omega$, then by the definition of ϕ we have

$$-c(x-y) - \phi(y) \geq -c(x-y) - \lambda \qquad \text{for all } (y, \lambda) \in A,$$

or

$$\sup_{y' \in W} [-c(x-y') - \phi(y')] \geq -c(x-y) - \lambda \qquad \text{for all } (y, \lambda) \in A.$$

Hence,

$$\sup_{y \in W} [-c(x-y) - \phi(y)] \geq u(x).$$

Now let $y \in W$. Given $\epsilon > 0$ we have

$$[-c(x-y) - \phi(y)] - \epsilon = -c(x-y) - [\phi(y) + \epsilon] < -c(x-y) - \lambda_y \leq u(x),$$

for some λ_y such that $(y, \lambda_y) \in A$. Hence $-c(x-y) - \phi(y) \le u(x)$ for each $y \in W$ and so

$$\sup_{y \in W} [-c(x-y) - \phi(y)] \le u(x).$$

This completes the remark.

In the following we shall consider the following conditions for the cost function c.

$(H1)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a C^1 and strictly convex function.

$(H2)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a strictly convex function and $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$.

$(H3)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$.

Notice that the function $c(x) = (1 + |x|^2)^{1/2}$ satisfies $(H1)$ and does not satisfy $(H2)$.

Proposition 2.2 *Suppose c satisfies $(H3)$. If $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and convex, then u is c-convex.*

Proof: Since every lower semicontinuous convex function is the supremum of affine functions, it is enough to assume that $u(z) = q \cdot z + b$. Since c is continuous and $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$, it follows sliding $-u$ in a parallel fashion that $-u + \lambda$ is a supporting hyperplane to c at some point for some λ. That is, there exist $x_u \in \mathbb{R}^n$ and $\lambda_u \in \mathbb{R}$ such that $c(x_u) = -u(x_u) + \lambda_u$ and $c(z) \ge -u(z) + \lambda_u$ for all $z \in \mathbb{R}^n$. Given $x \in \Omega$, let $y_x = x - x_u$ and $\lambda_x = -u(x) + u(x_u) - \lambda_u$. We have $u(x) = -c(x - y_x) - \lambda_x$ and $u(z) \ge -c(z - y_x) - \lambda_x$ for all $z \in \Omega$ since u is affine. Setting $A = \{(y_x, \lambda_x) : x \in \Omega\}$ we obtain the proposition. $\square$

Remark 2.4 Suppose c satisfies $(H3)$. If u is a c-convex function in Ω that is bounded in a neighborhood of $x_0 \in \Omega$, then $\partial_c u(x_0) \ne \emptyset$. Indeed, without loss of generality we can assume that the set $A \subset \mathbb{R}^n \times \mathbb{R}$ in the definition of u is closed. Then arguing as in Claim 2 of the proof of Lemma 2.1 below, there exists $(y, \lambda) \in A$ such that $u(x_0) = -c(x_0 - y) - \lambda$ and so $u(x) \ge -c(x-y) - \lambda = u(x_0) - c(x-y) + c(x_0 - y)$ for all $x \in \Omega$. Therefore $y \in \partial_c u(x_0)$.

If c satisfies $(H1)$, then we also notice that $\partial_c(-c(\cdot - y)) = \{y\}$ from Proposition 2.3(1).

Remark 2.5 Suppose c satisfies $(H3)$. It follows from the convexity of c that if u is c-convex and locally bounded in Ω, then u is locally Lipschitz in Ω. Indeed, let $K \subset \Omega$ be compact and $x_1, x_2 \in K$. From Remark 2.4, we have that $\partial_c u(x_i) \neq \emptyset$ for $i = 1, 2$. Let $y_1 \in \partial_c u(x_1)$. By Lemma 2.3, $|y_1| \leq R$, and since c is locally Lipschitz we have

$$u(x_2) - u(x_1) \geq -c(x_2 - y_1) + c(x_1 - y_1) \geq -C(K, R)\,|x_2 - x_1|.$$

Proposition 2.3 *Let u be a function defined on Ω, and suppose that x_0 is a point of differentiability of u, and $\partial_c u(x_0) \neq \emptyset$. Then we have*

1. *If $(H1)$ holds, then*

$$\partial_c u(x_0) = \{x_0 - (Dc)^{-1}(-Du(x_0))\}.$$

2. *If $(H2)$ holds, then*

$$\partial_c u(x_0) = \{x_0 - Dc^*(-Du(x_0))\},$$

where c^* is the Legendre-Fenchel transform[2] of c defined by

$$c^*(y) = \sup_{x \in \mathbb{R}^n} [x \cdot y - c(x)]. \tag{2.2}$$

PROOF: Suppose first that c satisfies $(H1)$. Let $p \in \partial_c u(x_0)$. Then $u(x) + c(x - p) \geq u(x_0) + c(x_0 - p)$ for all $x \in \Omega$ with equality at $x = x_0$. That is, $u(x) + c(x - p)$ attains a minimum at x_0 and therefore $Dc(x_0 - p) = -Du(x_0)$. Since c is C^1 and strictly convex, $(Dc)^{-1}$ exists on the image of Dc and we have $p = x_0 - (Dc)^{-1}(-Du(x_0))$.

To prove (2) we need the following definition.

[2]See [**RW98**, Chapter 11].

Definition 2.3 *The function* $u : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ *is subdifferentiable at* $x_0 \in \Omega$ *if* $u(x_0)$ *is finite and there exists* $z \in \mathbb{R}^n$ *such that*

$$u(x_0 + v) \geq u(x_0) + v \cdot z + o(|v|)$$

as $|v| \to 0$. *Let us denote by* $M_u(x_0)$ *the set of* z*'s satisfying the property above.*

Suppose that $(H2)$ holds, and let $p \in \partial_c u(x_0)$ with x_0 a point of differentiability of u. Then

$$u(x_0) - c(x_0 + v - p) + c(x_0 - p) \leq u(x_0 + v) \leq u(x_0) + v \cdot Du(x_0) + o(|v|).$$

Hence $c(x_0 + v - p) \geq c(x_0 - p) + v \cdot (-Du(x_0)) + o(|v|)$ as $|v| \to 0$, and so $-Du(x_0) \in M_c(x_0 - p)$. From [**GM96**, Corollary A.2] we get that $x_0 - p = Dc^*(-Du(x_0))$ and the proposition follows. □

Note that if u is a c-convex function and u is differentiable at $x_0 \in \Omega$, then from Remark 2.4 we get $\partial_c u(x_0) \neq \emptyset$.

Remark 2.6 Suppose c is strictly convex satisfying c and c^* are $C^2(\mathbb{R}^d)$[3] and $u : \Omega \to \mathbb{R}$ has a second derivative $D^2u(x_0)$ at x_0. Then if $\partial_c u(x_0) \neq \emptyset$, we have

$$I + D^2c^*\left(-Du(x_0)\right) D^2u(x_0) \text{ is diagonalizable with nonnegative eigenvalues.} \tag{2.3}$$

Indeed, let $p \in \partial_c u(x_0)$, then $u(x) + c(x - p) \geq u(x_0) + c(x_0 - p)$ for all $x \in \Omega$. Hence $Du(x_0) + Dc(x_0 - p) = 0$ and by Taylor's theorem $D^2u(x_0) + D^2c(x_0 - p) \geq 0$. So from Proposition 2.3(2) we get

$$D^2u(x_0) + D^2c\left(Dc^*\left(-Du(x_0)\right)\right) \geq 0. \tag{2.4}$$

On the other hand, from [**Roc97**, Corollary 23.5.1 and Theorem 26.1] we have that $Dc^*\left(Dc(x)\right) = x$ for every $x \in \mathbb{R}^n$ and $Dc\left(Dc^*(y)\right) = y$ for every y in the image of Dc. Differentiating these equations yields

$$D^2c^*\left(Dc(x)\right)\, D^2c(x) = I, \text{ and } D^2c\left(Dc^*(y)\right)\, D^2c^*(y) = I$$

---

[3] If c is $C^2(\mathbb{R}^n)$ and $D^2c(x)$ is positive definite for all x, then $c^* \in C^2(\mathbb{R}^n)$, $D^2c^*(x)$ is positive definite for all x, and $(c^*)^* = c$, see [**RW98**, Example 11.9, p. 480].

for every x in $\mathbb{R}^n$ and every y in the image of Dc. From this we derive that for any y in the image of Dc we have $D^2c^*(y)$ is invertible with $[D^2c^*(y)]^{-1} = D^2c\,(Dc^*(y))$, and letting $y = -Du(x_0)$, we obtain from (2.4) that

$$D^2u(x_0) \geq -[D^2c^*\,(-Du(x_0))]^{-1} \tag{2.5}$$

Since c^* is convex, the symmetric matrix $D^2c^*\,(-Du(x_0))$ is positive definite as it is invertible. Therefore, (2.5) implies (2.3).

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^d$ be an open set and suppose that $(H3)$ holds. If $u_n : \Omega \to \mathbb{R}$ is a sequence of c-convex functions such that $u_n \to u$ locally uniformly in Ω with u locally bounded in Ω, then u is c-convex in Ω.*

PROOF: By definition of c-convexity we have

$$u_n(x) = \sup_{(y,\lambda)\in A_n} [-c(x-y) - \lambda] \qquad \forall x \in \Omega,$$

for each n. Since c is continuous on $\mathbb{R}^d$, without loss of generality we can assume that each A_n is a closed subset of $\mathbb{R}^d \times \mathbb{R}$.

Claim 1. If $\{y_k\}$ and $\{\lambda_k\}$ are sequences such that there exists constants A, B, r with

$$A \leq -c(x_0 - y_k) - \lambda_k \tag{2.6}$$

and

$$-c(x - y_k) - \lambda_k \leq B \tag{2.7}$$

for all $x \in B(x_0, r) \Subset \Omega$ and for all k, then $\{y_k\}$ is a bounded sequence, and consequently $\{\lambda_k\}$ is also bounded.

Suppose by contradiction that $\{y_k\}$ is unbounded. Passing through a subsequence we can assume that $|y_k| \to +\infty$. Let $v_k := x_0 - y_k$. Since $|v_k| \to +\infty$, we may assume that $|v_k| > 1$ for all k sufficiently large. Setting $\zeta_k := 1 - \dfrac{r}{|v_k|}$, we have $\zeta_k \to 1$. Applying (2.7) at $x = x_0 + (\zeta_k - 1)v_k$ and using (2.6) we get

$$\begin{aligned} B &\geq -c(x_0 + (\zeta_k - 1)v_k - y_k)) - \lambda_k = -c(\zeta_k v_k) - \lambda_k \\ &\geq -c(\zeta_k v_k) + c(x_0 - y_k) + A = -c(\zeta_k v_k) + c(v_k) + A. \end{aligned}$$

Hence

$$B - A \geq c(v_k) - c(\zeta_k v_k). \tag{2.8}$$

Since c is convex, this difference can be bounded using a subgradient $p_k \in \partial c(\zeta_k v_k)$:

$$B - A \geq \langle p_k, v_k - \zeta_k v_k\rangle = \langle p_k, (1-\zeta_k)v_k\rangle = r\,\langle p_k, \frac{v_k}{|v_k|}\rangle. \tag{2.9}$$

On the other hand, being p_k a subgradient also implies that

$$c(0) \geq c(\zeta_k v_k) + \langle p_k, 0 - \zeta_k v_k\rangle. \tag{2.10}$$

Since $|v_k| \to +\infty$, we have that $\zeta_k > 0$ and dividing (2.10) by $\zeta_k|v_k| \to +\infty$ yields

$$\liminf_{k\to\infty}\langle p_k, \frac{v_k}{|v_k|}\rangle \geq \liminf_{k\to\infty} \frac{c(\zeta_k v_k)}{|\zeta_k v_k|}.$$

The assumption $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$ implies that both these limits diverge, yielding a contradiction with (2.9). Therefore y_k is bounded and since c is continuous we get from (2.6) that λ_k is also bounded and Claim 1 is proved.

Claim 2. For each $x \in \Omega$ there exists $N_x \in \mathbb{N}$ and a sequence

$$(y_n(x), \lambda_n(x)) \in A_n$$

such that $u_n(x) = -c(x - y_n(x)) - \lambda_n(x)$ for all $n \geq N_x$.

Let $r_x \in (0,1)$ such that $B(x, r_x) \Subset \Omega$ and u is bounded on $\bar{B}(x, r_x)$. Then since $u_n \to u$ uniformly on $\bar{B}(x, r_x)$, there exist constants $M_x > 0$ and $N_x \in \mathbb{N}$ such that

$$-M_x < u_n(z) < M_x \quad \forall z \in \bar{B}(x, r_x), \quad \text{and} \quad \forall n \geq N_x.$$

Since u_n is c-convex, for each n we can find a sequence $\{(y_n^k(x), \lambda_n^k(x))\}_{k=1}^{\infty} \subset A_n$ satisfying

$$u_n(x) = \lim_{k\to\infty}[-c(x - y_n^k(x)) - \lambda_n^k(x)],$$

$$u_n(x) - 1 \leq -c(x - y_n^k(x)) - \lambda_n^k(x).$$

Hence if $n \geq N_x$, then

$$-M_x - 1 \leq -c(x - y_n^k(x)) - \lambda_n^k(x),$$

and

$$-c(z - y_n^k(x)) - \lambda_n^k(x) \leq u_n(z) \leq M_x \quad \forall z \in \bar{B}(x, r_x).$$

Therefore from Claim 1, there exist $(y_n(x), \lambda_n(x)) \in A_n$ and a subsequence $\{(y_n^{k_j}(x), \lambda_n^{k_j}(x))\}_{j=1}^{\infty}$ such that $(y_n^{k_j}(x), \lambda_n^{k_j}(x)) \to (y_n(x), \lambda_n(x))$ as $j \to \infty$. Therefore,

$$u_n(x) = \lim_{j \to \infty} [-c(x - y_n^{k_j}(x)) - \lambda_n^{k_j}(x)] = -c(x - y_n(x)) - \lambda_n(x)$$

and Claim 2 is proved.

Claim 3. Let

$$B_x = \{(y, \lambda) \in \mathbb{R}^d \times \mathbb{R} : (y, \lambda) = \lim_{j \to \infty} (y_{n_j}(x), \lambda_{n_j}(x)), \text{ for some subsequence } n_j\},$$

and set $A = \cup_{x \in \Omega} B_x$. We claim that

$$u(z) = \sup_{(y,\lambda) \in A} [-c(z - y) - \lambda] \qquad \forall z \in \Omega. \tag{2.11}$$

Let $z \in \Omega$ and choose $r_z \in (0, 1)$ as above. Then $\bar{B}(z, r_z) \subset \Omega$ and as before we have

$$-M(z) < u_n(x) < M(z) \quad \forall x \in \bar{B}(z, r_z), \qquad \forall n \geq N_z.$$

If $(y_n(z), \lambda_n(z))$ is the sequence in Claim 2, we have that for any $n \geq N_z$

$$-M(z) < -c(z - y_n(z)) - \lambda_n(z), \text{and } -c(x - y_n(z)) - \lambda_n(z) < M(z) \forall x \in \bar{B}(z, r_z).$$

We conclude from Claim 1 that $\{y_n(z)\}_{n=N_z}^{\infty}$ and $\{\lambda_n(z)\}_{n=N_z}^{\infty}$ are bounded. Hence, there exist $(y^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}$ and a subsequence $\{(y_{n_k}(z), \lambda_{n_k}(z))\}_{k=1}^{\infty}$ of $\{(y_n(z), \lambda_n(z))\}_{n=N_z}^{\infty}$ such that $(y_{n_k}(z), \lambda_{n_k}(z)) \to (y^*, \lambda^*)$. Therefore, $(y^*, \lambda^*) \in B_z \subset A$ and

$$u(z) = \lim_{k \to \infty} u_{n_k}(z) = \lim_{k \to \infty} [-c(z - y_{n_k}(z)) - \lambda_{n_k}(z)] = -c(z - y^*) - \lambda^*.$$

Thus to prove (2.11) it is enough to show that

$$-c(z-y)-\lambda \leq u(z) \qquad \forall (y,\lambda)\in A.$$

Indeed, let $(y,\lambda)\in A$. Then $(y,\lambda)\in B_x$ for some $x\in\Omega$ and hence there exists a subsequence $\{(y_{n_j}(x),\lambda_{n_j}(x))\}_{j=1}^{\infty}$ of $\{(y_n(x),\lambda_n(x))\}_{n=N_x}^{\infty}$ such that $(y_{n_j}(x),\lambda_{n_j}(x))\to(y,\lambda)$. We have

$$u_{n_j}(z) = -c(z-y_{n_j}(z))-\lambda_{n_j}(z) \geq -c(z-y_{n_j}(x))-\lambda_{n_j}(x) \quad \forall j.$$

Letting $j\to\infty$ and since $(y_{n_j}(x),\lambda_{n_j}(x))\to(y,\lambda)$, we then get

$$u(z)\geq -c(z-y)-\lambda.$$

This completes the proof of the lemma. □

2.2 A Monge-Ampère measure associated with the cost function c

In this subsection we define a generalized Monge-Ampère measure, and to do it we need the following lemma, which is a generalization of a classical lemma of Aleksandrov.

Lemma 2.2 *Suppose that either (H1) or (H2) holds. Let $X\subset\mathbb{R}^n$ be a nonempty bounded set and $u: X\to\mathbb{R}\cup\{+\infty\}$, not identically $+\infty$, be bounded from below on X. Then the Lebesgue measure of the set*

$$\tilde{S}=\{p\in\mathbb{R}^n: \quad p\in\partial_c u(x_1)\cap\partial_c u(x_2) \;\; \textit{for some} \;\; x_1,x_2\in X, \;\; x_1\neq x_2\}$$

is zero.

PROOF: Define for each $y\in\mathbb{R}^n$,

$$u^*(y)=\sup_{x\in X}[-c(x-y)-u(x)].$$

Since $c \in C(\mathbb{R}^n)$, X is bounded, $u \not\equiv +\infty$ and u is bounded from below on X we get $u^* : \mathbb{R}^n \to \mathbb{R}$. Moreover, as c is locally Lipschitz on $\mathbb{R}^n$, it is clear that u^* is also locally Lipschitz on $\mathbb{R}^n$. Hence, if we let $E = \{x \in \mathbb{R}^n : u^* \text{ is not differentiable at } x\}$, then $|E| = 0$. We shall show that $\tilde{S} \subset E$. Indeed, let $p \in \tilde{S}$ then $p \in \partial_c u(x_1) \cap \partial_c u(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$. Hence,

$$u(z) \geq u(x_1) - c(z-p) + c(x_1 - p) \quad \forall z \in X,$$

$$u(z) \geq u(x_2) - c(z-p) + c(x_2 - p) \quad \forall z \in X.$$

Therefore,

$$-c(x_1 - p) - u(x_1) \geq -c(z-p) - u(z) \quad \forall z \in X,$$

$$-c(x_2 - p) - u(x_2) \geq -c(z-p) - u(z) \quad \forall z \in X.$$

Thus, $u^*(p) = -c(x_1 - p) - u(x_1)$ and $u^*(p) = -c(x_2 - p) - u(x_2)$. Moreover, by definition of u^* we have $u^*(z) \geq -c(x_1 - z) - u(x_1) \quad \forall z \in \mathbb{R}^n$, and $u^*(z) \geq -c(x_2 - z) - u(x_2) \quad \forall z \in \mathbb{R}^n$. So

$$u^*(z) \geq u^*(p) - c(x_1 - z) + c(x_1 - p) \quad \forall z \in \mathbb{R}^n$$

and

$$u^*(z) \geq u^*(p) - c(x_2 - z) + c(x_2 - p) \quad \forall z \in \mathbb{R}^n.$$

Hence, we obtain $x_1, x_2 \in \partial_h(u^*, \mathbb{R}^n)(p)$ where we denote $h(x) = c(-x)$ for every $x \in \mathbb{R}^n$. Note that h satisfies the same assumptions as c. Then by Proposition 2.3 we must have $p \in E$ since $x_1 \neq x_2$. The proof is complete. □

Remark 2.7 *Suppose c satisfies $(H3)$. Let Ω be a bounded open set in $\mathbb{R}^n$ and $u \in C(\Omega)$ be c-convex. Then*

$$p \in \partial_c u(x) \textit{ if and only if } x \in \partial_h(u^*, \mathbb{R}^n)(p),$$

where $h(x) = c(-x)$ and $u^*(y) = \sup_{x \in \Omega} [-c(x-y) - u(x)]$.

PROOF: It follows by the argument in Lemma 2.2 that if $p \in \partial_c u(x)$ then $x \in \partial_h(u^*, \mathbb{R}^n)(p)$. Now if $x \in \partial_h(u^*, \mathbb{R}^n)(p)$, then $u^*(y) \geq u^*(p) - c(x-y) + c(x-p)$ for all $y \in \mathbb{R}^n$. This gives by the definition of u^* that for each $y \in \mathbb{R}^n$, $u^*(y) \geq -c(z-p) - u(z) - c(x-y) + c(x-p)$ for all $z \in \Omega$. Picking $y \in \partial_c u(x)$ which is nonempty by Remark 2.4, then as $u^*(y) = -c(x-y) - u(x)$ we obtain $u(z) \geq u(x) - c(z-p) + c(x-p)$ for all $z \in \Omega$. That is, $p \in \partial_c u(x)$ as desired. □

Corollary 2.1 *Let Ω be an open set in $\mathbb{R}^n$ and suppose that either $(H1)$ or $(H2)$ holds. Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ be such that on any bounded open set $U \Subset \Omega$, u is not identical to $+\infty$ and bounded from below. Then the Lebesgue measure of the set*

$$S = \{p \in \mathbb{R}^n : \quad \textit{there exist} \quad x, y \in \Omega, x \neq y \quad \textit{and} \quad p \in \partial_c u(x) \cap \partial_c u(y)\}$$

is zero.

PROOF: We can write $\Omega = \cup_k \Omega_k$ where $\Omega_k \subset \Omega_{k+1}$ are open and $\bar{\Omega}_k \subset \Omega$ are compact. If $p \in S$ then there exist $x, y \in \Omega, x \neq y$ with $u(z) \geq u(x) - c(z-p) + c(x-p) \;\; \forall z \in \Omega$, and $u(z) \geq u(y) - c(z-p) + c(y-p) \;\; \forall z \in \Omega$. Since Ω_k increases, $x, y \in \Omega_m$ for some m. That is, if

$$S_m = \{p \in \mathbb{R}^n : \quad \exists \, x, y \in \Omega_m, x \neq y \quad \text{and} \quad p \in \partial_c(u, \Omega_m)(x) \cap \partial_c(u, \Omega_m)(y)\},$$

then we have $p \in S_m$, i.e., $S \subset \cup_{m=1}^{\infty} S_m$. But by the assumptions and Lemma 2.2 we get $|S_m| = 0$ for all m. Hence the proof is complete. □

Lemma 2.3 *Suppose c satisfies $(H3)$. Let $u : \Omega \to \mathbb{R}$ be a locally bounded function in Ω. If $K \subset \Omega$ is compact, then there exists $R > 0$, depending only on K and the L^∞-norm of u over a small neighborhood of K, such that*

$$\partial_c u(K) \subset B(0, R).$$

PROOF: Indeed, assume that this is not true. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ and $p_n \in \partial_c u(x_n)$ with $|p_n| > n$. Hence $u(x) \geq u(x_n) - c(x -$

$p_n) + c(x_n - p_n) \quad \forall x \in \Omega$, and since u is locally bounded, there exists $M > 0$ such that

$$-c(x - p_n) + c(x_n - p_n) \leq M \quad \forall x \in K_\delta \text{ and } \forall n \in \mathbb{N}, \tag{2.12}$$

where $\delta = \frac{1}{2}\min\{\text{dist}(x, \partial\Omega), 1\}$ and $K_\delta = \{x \in \Omega : \text{dist}(x, K) \leq \delta\}$. Let $v_n = x_n - p_n$. Since $|v_n| \to +\infty$, we may assume $|v_n| > 1 \quad \forall n$. Setting $\zeta_n = 1 - \frac{\delta}{|v_n|}$ and evaluating (2.12) at $x = x_n + (\zeta_n - 1)v_n \in K_\delta$ yields

$$M \geq -c(x_n + (\zeta_n - 1)v_n - p_n) + c(x_n - p_n) = c(v_n) - c(\zeta_n v_n).$$

Applying the argument used after the inequality (2.8) yields a contradiction. This proves the lemma. □

Lemma 2.4 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in C(\Omega)$, and $\mathcal{B} = \{E \subset \Omega : \partial_c u(E)$ is Lebesgue measurable$\}$. We have*

(i) If $K \subset \Omega$ is compact, then $\partial_c u(K)$ is closed. Moreover, if $(H3)$ holds then $\partial_c u(K)$ is compact.

(ii) $\mathcal{B}$ contains all closed subsets and all open subsets of Ω.

(iii) If either $(H1)$ or $(H2)$ holds, then $\mathcal{B}$ is a σ-algebra on Ω containing all Borel subsets of Ω. Moreover,

$$|\partial_c u(\Omega \setminus E)| = |\partial_c u(\Omega) \setminus \partial_c u(E)| \qquad \forall E \in \mathcal{B}.$$

Proof: (i). Let K be a compact subset of Ω, $\{p_n\}_{n=1}^\infty \subset \partial_c u(K)$, and suppose $p_n \to p$. We shall show that $p \in \partial_c u(K)$. For each n, since $p_n \in \partial_c u(K)$ we have $p_n \in \partial_c u(x_n)$ for some $x_n \in K$. But K is compact, so there exist $x \in K$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. We have

$$u(z) \geq u(x_{n_k}) - c(z - p_{n_k}) + c(x_{n_k} - p_{n_k}) \quad \forall z \in \Omega,$$

passing to limit and since u and c are continuous we obtain

$$u(z) \geq u(x) - c(z - p) + c(x - p) \quad \forall z \in \Omega.$$

So $p \in \partial_c u(x) \subset \partial_c u(K)$. Hence, $\partial_c u(K)$ is closed. The second statement then follows from Lemma 2.3.

(ii). Let E be a closed subset of Ω. Then we can write $E = \cup_{n=1}^{\infty} K_n$ where K_n are compact. Therefore, $\partial_c u(E) = \partial_c u(\cup_{n=1}^{\infty} K_n) = \cup_{n=1}^{\infty} \partial_c u(K_n)$. By (i), each $\partial_c u(K_n)$ is Lebesgue measurable. So $\partial_c u(E)$ is measurable, i.e., $E \in \mathcal{B}$. The proof is identical if E is open.

(iii). Suppose $\{E_i\}_{i=1}^{\infty} \subset \mathcal{B}$. Since $\partial_c u(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} \partial_c u(E_i)$ we then get $\partial_c u(\cup_{i=1}^{\infty} E_i)$ is Lebesgue measurable. So $\cup_{i=1}^{\infty} E_i \in \mathcal{B}$. We also have $\Omega \in \mathcal{B}$ by (ii). Now suppose $E \in \mathcal{B}$, we shall show that $\Omega \setminus E \in \mathcal{B}$. Indeed, we have

$$\partial_c u(\Omega \setminus E) = [\partial_c u(\Omega) \setminus \partial_c u(E)] \cup [\partial_c u(\Omega \setminus E) \cap \partial_c u(E)].$$

By Corollary 2.1, the second set on the right hand side has measure zero. So $\partial_c u(\Omega \setminus E)$ is Lebesgue measurable and $|\partial_c u(\Omega \setminus E)| = |\partial_c u(\Omega) \setminus \partial_c u(E)|$. Also since $\mathcal{B}$ is a σ-algebra and by (ii) $\mathcal{B}$ contains all closed subsets of Ω we get $\mathcal{B}$ contains all Borel subsets of Ω. □

From Lemma 2.4 we then define

Definition 2.4 *Let g be a locally integrable function which is positive a.e. on $\mathbb{R}^n$. Suppose that c satisfies either (H1) or (H2), and Ω is an open set in $\mathbb{R}^n$. Then for each given function $u \in C(\Omega)$, the generalized Monge-Ampère measure of u associated with the cost function c and the weight g is the Borel measure defined by*

$$\omega_c(g, u)(E) = \int_{\partial_c u(E)} g(y)\, dy$$

for every Borel set $E \subset \Omega$. When $g \equiv 1$, we simply write the measure as $\omega_c(u)$.

Remark 2.8 The σ-additivity of $\omega_c(g, u)$ follows from Corollary 2.1 and the argument from [**Gut01**, Theorem 1.1.13].

Remark 2.9 If c satisfies $(H2)$ and $u \in C(\Omega)$ is c-convex in Ω, then we know from Lemma 2.4(i) that $\partial_c u(K)$ is compact for every compact set $K \subset \Omega$. Therefore the measure $\omega_c(g, u)$ is finite on compact subsets of Ω, and so $\omega_c(g, u)$

is a regular measure. Namely, it has the following regularity properties

$$\omega_c(g,u)(E) = \inf\{\omega_c(g,u)(U) : \ E \subset U \subset \Omega, \ \ U \text{ open}\}$$

for all Borel sets $E \subset \Omega$, and

$$\omega_c(g,u)(U) = \sup\{\omega_c(g,u)(K) : \ K \subset U, \ \ K \text{ compact}\}$$

for all open sets $U \subset \Omega$.

Definition 2.5 *Let $u \in C(\Omega)$ and $x_0 \in \Omega$. Then u is called strictly c-convex at x_0 if $\partial_c u(x_0) \neq \emptyset$ and for any $p \in \partial_c u(x_0)$ we have*

$$u(x) > u(x_0) - c(x-p) + c(x_0 - p)$$

for all $x \in \Omega \setminus \{x_0\}$.

Remark 2.10 The definition means precisely that the set of supporting hypersurfaces of u at x_0 is nonempty and any such supporting hypersurface touches the graph of u only at x_0.

For $u \in C(\Omega)$, define $\Gamma_u = \{x \in \Omega : \partial_c u(x) \neq \emptyset\}$. Then Γ_u is a relatively closed set in Ω since $\Gamma_u = \{x \in \Omega : u_*(x) = u(x)\}$, where u_* is the continuous function defined in (4.1). We then have the following result noticing that $\Gamma_u = \Omega$ iff u is c-convex in Ω.

Proposition 2.4 *Suppose that c satisfies $(H2)$ and $c^* \in C^2(\mathbb{R}^n)$. We have*

1. *If $u \in C^2(\Omega)$, then*

$$\omega_c(g,u)(E) = \int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)| \, dx$$

   for all Borel sets $E \subset \Omega$.

2. *If in addition $c \in C^2(\mathbb{R}^n)$ then for any $u \in C^2(\Omega)$,*

$$\omega_c(g,u)(E) = \int_{E\cap\Gamma_u} g(x - Dc^*(-Du)) \det(I + D^2c^*(-Du)D^2u) \, dx$$

   for all Borel sets $E \subset \Omega$.

Proof: (1) Define $s(x) = x - Dc^*(-Du(x))$ for every x in Ω. Since $c^* \in C^2(\mathbb{R}^n)$ and $u \in C^2(\Omega)$, it follows that $s:\ \Omega \to \mathbb{R}^n$ is a C^1 mapping, and by Proposition 2.3 we have $\partial_c u(x) = \{s(x)\}$ for every x in Ω satisfying $\partial_c u(x) \neq \emptyset$. Let $\tilde{A} = \{x \in \Omega : u$ is not strictly c-convex at $x\}$ and let S be defined as in Corollary 2.1. Then $|S| = 0$ and we claim that $\tilde{A} = (\Omega - \Gamma_u) \cup (\Gamma_u \cap s^{-1}(S))$. Indeed, if $x \in \tilde{A}$ and $x \in \Gamma_u$ then $p = s(x) \in \partial_c u(x)$. Since $\partial_c u(x)$ can not contain more than one element, there exists $y \in \Omega$, $y \neq x$ such that $u(y) = u(x) - c(y-p) + c(x-p)$. Hence,

$$u(z) \geq u(x) - c(z-p) + c(x-p) = u(y) - c(z-p) + c(y-p)$$

for every z in Ω. So $p \in \partial_c u(x) \cap \partial_c u(y)$, i.e., $x \in s^{-1}(S)$. This implies the claim as the reverse relation is obvious. We now let $S' \subset \mathbb{R}^n$ be a Borel set such that $S \subset S'$ and $|S'| = 0$. Put $A = (\Omega - \Gamma_u) \cup (\Gamma_u \cap s^{-1}(S'))$, then clearly A is a measurable set, $\tilde{A} \subset A$, and as $\partial_c u(A) = \partial_c u(\Gamma_u \cap s^{-1}(S')) = s(\Gamma_u \cap s^{-1}(S')) \subset S'$ we have $|\partial_c u(A)| = 0$. We now proceed the proof as follows. From the definition of $\tilde{A}$ it is easy to see that s is one-to-one on $\Omega \setminus \tilde{A}$, and hence one-to-one on $\Omega \setminus A$. Therefore for any Borel subset E of Ω, by using the usual change variables formula we obtain

$$\begin{aligned}
&\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\ dx \\
&= \int_{E\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx \geq \int_{E\setminus A} g(s(x))|\det Ds(x)|\ dx \\
&= \int_{s(E\setminus A)} g(y)\ dy = \int_{\partial_c u(E\setminus A)} g(y)\ dy \overset{(*)}{=} \int_{\partial_c u(E)} g(y)\ dy = \omega_c(g,u)(E).
\end{aligned}$$

Note that the equality $(*)$ holds since $|\partial_c u(E) - \partial_c u(E \setminus A)| = |\partial_c u(E - (E \setminus A))| = |\partial_c u(E \cap A)| = 0$. Thus we have proved that

$$\int_{E\cap\Gamma_u} g(x - Dc^*(-Du)) \det(I + D^2c^*(-Du)D^2u)\ dx \geq \omega_c(g,u)(E)$$

for every Borel set $E \subset \Omega$. Hence, (1) will be proved if we can show the reverse inequality. To do that, let $B = \{x \in \Omega : \det Ds(x) = 0\}$ and let E be a Borel set in Ω. Then for any open set U with $E \subset U \subset \Omega$, we can write the open set $U \setminus B$ as $U \setminus B = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ are cubes with disjoint interior and sides parallel to the coordinate axes. We can choose C_i are small enough so that $s : C_i \to s(C_i)$ is a diffeomorphism. We therefore have

$$\begin{aligned}
&\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\ dx \\
&\leq \int_{U\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx = \int_{(U\setminus B)\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx \\
&= \int_{(\cup_{i=1}^{\infty} C_i)\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx = \int_{(\cup_{i=1}^{\infty} \overset{\circ}{C}_i)\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx \\
&= \sum_{i=1}^{\infty} \int_{\overset{\circ}{C}_i\cap\Gamma_u} g(s(x))|\det Ds(x)|\ dx = \sum_{i=1}^{\infty} \int_{s(\overset{\circ}{C}_i\cap\Gamma_u)} g(y)\ dy \\
&= \sum_{i=1}^{\infty} \omega_c(g,u)(\overset{\circ}{C}_i \cap \Gamma_u) = \omega_c(g,u)((\cup_{i=1}^{\infty} \overset{\circ}{C}_i) \cap \Gamma_u) \leq \omega_c(g,u)(U).
\end{aligned}$$

Hence since the measure $\omega_c(g,u)$ is regular, we deduce that

$$\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\ dx \leq \omega_c(g,u)(E).$$

This combined with the previous inequality yield the desired result for (1).

(2) This is a consequence of (1) and Remark 2.6. $\square$

CHAPTER 3

Notions Of Weak Solutions

3.1 Aleksandrov Solutions

The equation (1.2) is highly fully nonlinear and at least when the cost function c is nice enough, it is degenerate elliptic on the set of c-convex functions. Motivated by Proposition 2.4 and by using the previous results we shall define a notion of weak solutions for (1.2) and study the stability property of the solutions.

Definition 3.1 *We say that a c-convex function $u \in C(\Omega)$ is a generalized solution of* (1.2) *in the sense of Aleksandrov, or simply Aleksandrov solution, if*

$$\omega_c(g,u)(E) = \int_E f(x)\ dx$$

for any Borel set $E \subset \Omega$.

Remark 3.1 In (1.2) the function f on the right hand side can be replaced by a locally finite Borel measure μ on Ω. And as above a notion of Aleksandrov solutions can be defined similarly.

Proposition 3.1 *Suppose c satisfies $(H2)$, and that $c, c^* \in C^2(\mathbb{R}^n)$. Let $u \in C(\Omega)$ be a c-convex function. Then u is an Aleksandrov solution of* (1.2) *iff $\omega_c(g,u)$ is absolutely continuous w.r.t. the Lebesgue measure and* (1.2) *is satisfied pointwise a.e. on Ω.*

PROOF: Observing first that by [**GM96**, Corollary C.5], u is locally semi-convex[1] and hence twice differentiable a.e. on Ω in the sense of Aleksandrov. Also, by using Remark 2.6 and an argument similar to [**McC97**, Proposition A.2] we have that whenever u has an Aleksandrov second derivative $D^2u(x_0)$ at $x_0 \in \Omega$ then

$$\lim_{r \to 0^+} \frac{|\partial_c u(B_r(x_0))|}{|B_r(x_0)|} = \det[I + D^2c^*(-Du(x_0))D^2u(x_0)], \tag{3.1}$$

and if in addition $I + D^2c^*(-Du(x_0))D^2u(x_0)$ is invertible, then $\partial_c u(B_r(x_0))$ shrinks nicely to $x_0 - Dc^*(-Du(x_0))$.

Suppose $\omega_c(g, u)$ is absolutely continuous w.r.t. the Lebesgue measure on the σ-algebra of Borel sets in Ω and with density $F(x)$. The proposition will be proved if we can show that

$$F(x) = g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)] \text{ a.e. } x \text{ in } \Omega. \tag{3.2}$$

Since $g > 0$ a.e. on $\mathbb{R}^n$ we get $\omega_c(u)$ is also absolutely continuous w.r.t. the Lebesgue measure. Combining this with (3.1) we see that

$$\det[I + D^2c^*(-Du(x))D^2u(x)]$$

is the density of $\omega_c(u)$. Now let M be the set of points $x \in \Omega$ satisfying u has Aleksandrov second derivative at x and $\det[I + D^2c^*(-Du(x))D^2u(x)] > 0$, and let H be a Borel set in $\mathbb{R}^n$ with Lebesgue measure zero such that every point in $\mathbb{R}^n \setminus H$ is a Lebesgue point of g. Define $E = \{x \in M : x - Dc^*(-Du(x)) \in H\}$. Then it is clear from Remark 2.7 that $E = \partial_h(u^*, \mathbb{R}^n)(H) \cap M$ and hence E is Lebesgue measurable by Lemma 2.4 as $u^* \in C(\mathbb{R}^n)$. We claim that $|E| = 0$. Indeed, let $K \subset E$ be a compact set then we have $\partial_c u(K) \subset \partial_c u(E) \subset H$. Hence, $\int_K \det[I + D^2c^*(-Du(x))D^2u(x)]\, dx = \omega_c(u)(K) = 0$. This implies that $|K| = 0$ as the integrand is positive on K. Since E is Lebesgue measurable, the claim follows because $|E| = \sup\{|K| : K \subset E, \text{ K is compact}\}$.

[1]This means that given $x \in \Omega$ there exist a ball $B_r(x)$ and a nonnegative constant λ such that $u(x) + \lambda|x|^2$ is convex on $B_r(x)$ in the standard sense, see [**GM96**, p. 134].

For each $x \in M - E$ since $I + D^2c^*(-Du(x))D^2u(x)$ is positive definite we get $\partial_c u(B_r(x))$ shrinks nicely to $x - Dc^*(-Du(x))$, a Lebesgue point of g. Consequently,

$$\lim_{r\to 0^+} \frac{\omega_c(g,u)(B_r(x))}{|B_r(x)|} = \lim_{r\to 0^+} \frac{|\partial_c u(B_r(x))|}{|B_r(x)|} \frac{1}{|\partial_c u(B_r(x))|} \int_{\partial_c u(B_r(x))} g(y)\ dy$$
$$= g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)]\ \forall x \in M - E.$$

Thus we obtain

$$F(x) = g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)] \text{ a.e. on } M. \quad (3.3)$$

On the other hand, by letting B be a Borel set in Ω such that $M \subset B$ and $|M| = |B|$ we have $|\partial_c u(\Omega - B)| = \int_{\Omega - B} \det[I + D^2c^*(-Du(x))D^2u(x)]\ dx = 0$ since $\det[I + D^2c^*(-Du(x))D^2u(x)]$ is zero a.e. on $\Omega - M$. Therefore,

$$\int_{\Omega - B} F(x)\ dx = \omega_c(g,u)(\Omega - B) = \int_{\partial_c u(\Omega - B)} g(y)\ dy = 0,$$

which gives $F(x) = 0$ a.e. on $\Omega - B$. This implies that

$$F(x) = g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)] \text{ a.e. on } \Omega - M. \quad (3.4)$$

From (3.3) and (3.4) we get (3.2) and the proof is complete. □

A problem considered in this thesis is whether (1.2) has a unique generalized solution in the sense of Aleksandrov satisfying certain boundary condition. There are some obstructions to the existence of a solution, and one needs to impose some conditions on the datum and Ω. An obvious necessary condition of solvability is

$$\int_\Omega f(x)dx \leq \int_{\mathbb{R}^n} g(y)dy =: B(g). \quad (3.5)$$

If $B(g) = +\infty$, then (3.5) is not a restriction on the function $f(x)$. Another condition one may assume is that Ω is strictly convex.

In order to solve the Dirichlet problem for the equation (1.2) we need the following lemma.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u_k \in C(\Omega)$ be a sequence such that $u_k \to u$ uniformly on compact subsets of Ω.*

(i) If $K \subset \Omega$ is compact, then

$$\limsup_{k\to\infty} \partial_c u_k(K) \subset \partial_c u(K),$$

and by Fatou

$$\limsup_{k\to\infty} \omega_c(g, u_k)(K) \leq \omega_c(g, u)(K).$$

(ii) Assume further that $(H2)$ holds, u_k are c-convex on Ω and for every subsequence $\{k_j\}$ and $\{z_{k_j}\} \subset \Omega$ with $z_{k_j} \to z_0 \in \partial\Omega$, we have

$$\liminf_{j\to\infty} u(z_{k_j}) \leq \limsup_{j\to\infty} u_{k_j}(z_{k_j}). \tag{3.6}$$

If K is compact and U is open such that $K \subset U \subset \Omega$, then we get

$$\partial_c u(K) \subset \liminf_{k\to\infty} \partial_c u_k(U)$$

where the inclusion holds for almost every point of the set on the left hand side, and by Fatou

$$\omega_c(g, u)(K) \leq \liminf_{k\to\infty} \omega_c(g, u_k)(U).$$

PROOF: (i). Let $p \in \limsup_{k\to\infty} \partial_c u_k(K)$. Then for each n, there exist k_n and $x_{k_n} \in K$ such that $p \in \partial_c u_{k_n}(x_{k_n})$. By selecting a subsequence $\{x_j\}$ of $\{x_{k_n}\}$ we may assume $x_j \to x_0 \in K$. On the other hand,

$$u_j(x) \geq u_j(x_j) - c(x-p) + c(x_j - p) \quad \forall x \in \Omega$$

and by letting $j \to \infty$, the uniform convergence of u_j on compacts yields

$$u(x) \geq u(x_0) - c(x-p) + c(x_0 - p) \quad \forall x \in \Omega$$

that is, $p \in \partial_c u(x_0)$.

(ii). Without loss of generality we can assume that $\overline{U} \subset \Omega$.

Let $A = \{(x,p)|x \in K \text{ and } p \in \partial_c u(x)\}$ and for every $z \in \mathbb{R}^n$ we define the auxiliary function $v(z) = \sup_{(x,p)\in A} f_{x,p}(z)$, where for each $(x,p) \in A$ we denote $f_{x,p}(z) = -c(z-p) + c(x-p) + u(x)$ for all $z \in \mathbb{R}^n$. We first observe that since $\partial_c u(K)$ is bounded by Lemma 2.3 and c is locally Lipschitz it is easy to see that v is locally Lipschitz on $\mathbb{R}^n$. If $(x,p) \in A$, then $u(z) \geq f_{x,p}(z)$ for $z \in \Omega$ and so $v \leq u$ on Ω. Since by Lemma 2.1 u is c-convex, then by taking $p \in \partial_c u(z)$ we have $v(z) \geq f_{z,p}(z) = u(z)$ for $z \in K$ and so $v = u$ in K. Moreover, $\partial_c(v, \mathbb{R}^n)(x) = \partial_c u(x)$ for every x in K. Now let $S = \{p \in \mathbb{R}^n : \ p \in \partial_c(v,\mathbb{R}^n)(x_1) \cap \partial_c(v,\mathbb{R}^n)(x_2) \text{ for some } \ x_1, x_2 \in \mathbb{R}^n, x_1 \neq x_2\}$. By Corollary 2.1, $|S| = 0$. Therefore (ii) will be proved if we can show that $\partial_c(v,\mathbb{R}^n)(K) \setminus S \subset \liminf_{k\to\infty} \partial_c u_k(U)$. Let $p \in \partial_c(v,\mathbb{R}^n)(K) \setminus S$, then there exists $x_0 \in K$ such that $p \in \partial_c(v,\mathbb{R}^n)(x_0)$ and $p \notin \partial_c(v,\mathbb{R}^n)(x)$ for every x in $\mathbb{R}^n \setminus \{x_0\}$. Hence we have

$$v(x) > v(x_0) - c(x-p) + c(x_0 - p) \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}. \tag{3.7}$$

Now let $\delta_k := \min_{x\in\overline{U}} \{u_k(x) - u_k(x_0) + c(x-p) - c(x_0-p)\}$. Then this minimum is attained at some $x_k \in \overline{U}$. So $\delta_k = u_k(x_k) - u_k(x_0) + c(x_k - p) - c(x_0 - p)$ and $u_k(x) \geq u_k(x_0) - c(x-p) + c(x_0-p) + \delta_k \quad \forall x \in \overline{U}$. Thus we obtain

$$u_k(x) \geq u_k(x_k) - c(x-p) + c(x_k - p) \quad \forall x \in \overline{U}. \tag{3.8}$$

We first claim that $x_k \to x_0$. Indeed, let $\{x_{k_j}\}$ be any convergent subsequence of $\{x_k\}$, say to $\bar{x} \in \overline{U}$. If $\bar{x} \neq x_0$ then since $u_k \to u$ uniformly on $\overline{U}$, passing to the limit in (3.8) and using (3.7) we get

$$\begin{aligned}
u(x) &\geq u(\bar{x}) - c(x-p) + c(\bar{x}-p) \\
&\geq v(\bar{x}) - c(x-p) + c(\bar{x}-p) \\
&> v(x_0) - c(\bar{x}-p) + c(x_0-p) - c(x-p) + c(\bar{x}-p) \\
&= u(x_0) - c(x-p) + c(x_0-p) \quad \forall x \in \overline{U},
\end{aligned}$$

in particular, $u(x_0) > u(x_0)$, a contradiction. So we must have $x_{k_j} \to x_0$ and hence we obtain $x_k \to x_0 \in U$.

We next claim that there exists k_0 such that for all $k \geq k_0$ we have

$$u_k(x) \geq u_k(x_k) - c(x-p) + c(x_k - p) \quad \forall x \in \Omega,$$

in other words, the inequality (3.8) holds true in Ω. Otherwise, we can find a subsequence $\{k_j\}$ and $\{z_{k_j}\} \subset \Omega \setminus \overline{U}$ such that

$$u_{k_j}(z_{k_j}) < u_{k_j}(x_{k_j}) - c(z_{k_j} - p) + c(x_{k_j} - p) \quad \forall j. \tag{3.9}$$

Since Ω is bounded, passing through a subsequence, we can assume that $z_{k_j} \to z_0 \in \overline{\Omega} \setminus U$. If $z_0 \in \Omega \setminus U$, then by letting $j \to \infty$ in (3.9) and using the assumption that $u_{k_j} \to u$ uniformly on compact subsets of Ω, we deduce that

$$v(z_0) \leq u(x_0) - c(z_0 - p) + c(x_0 - p) = v(x_0) - c(z_0 - p) + c(x_0 - p). \tag{3.10}$$

On the other hand, if $z_0 \in \partial\Omega$, then from (3.9) we obtain

$$\limsup_{j\to\infty} u_{k_j}(z_{k_j}) \leq u(x_0) - c(z_0 - p) + c(x_0 - p) = v(x_0) - c(z_0 - p) + c(x_0 - p).$$

But (3.6) and since $u \geq v$ on Ω yield

$$\limsup_{j\to\infty} u_{k_j}(z_{k_j}) \geq \liminf_{j\to\infty} u(z_{k_j}) \geq \liminf_{j\to\infty} v(z_{k_j}) = v(z_0),$$

and therefore (3.10) also holds. This gives a contradiction with (3.7) since $z_0 \neq x_0$. So the claim is proved. But then we get $p \in \partial_c u_k(x_k)$ for all $k \geq k_0$ and hence $p \in \liminf_{k\to\infty} \partial_c u_k(U)$ as $x_k \to x_0 \in U$. This completes the proof.

□

As an immediate consequence we have the following stability property, which will be useful in various contexts later.

Corollary 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that $(H2)$ holds. If $\{u_k\} \subset C(\Omega)$ is a sequence of c-convex functions converging locally uniformly in Ω to a function u and condition* (3.6) *holds, then $\omega_c(g, u_k)$ tend to $\omega_c(g, u)$ weakly, i.e.,*

$$\int_\Omega f(x)\ d\omega_c(g, u_k) \to \int_\Omega f(x)\ d\omega_c(g, u)$$

for any f in $C_0(\Omega)$.

Remark 3.2 We note that by following the proof of Lemma 3.1 we see that if either u_k are in $C^1(\Omega)$ or u_k are convex in the standard sense, then the above results still hold without condition (3.6). The reason is that in these cases we have that (3.8) holds for every x in an open neighborhood of x_k iff it holds for every x in Ω. However, this seems no longer true if u_k are merely c-convex. We also remark that (3.6) is satisfied if either $u_k \to u$ locally uniformly and $u_k \geq u$ on Ω or $u_k \to u$ uniformly on Ω. The verification is obvious in the first case, while for the latter we just use the fact that $\limsup_{j\to\infty}\{a_j + b_j\} \geq \limsup_{j\to\infty} a_j + \liminf_{j\to\infty} b_j$ and apply for $a_j = [u_{k_j}(z_{k_j}) - u(z_{k_j})]$ and $b_j = u(z_{k_j})$.

3.2 Viscosity Solutions

In this section we shall introduce a notion of viscosity solutions for the equation (1.2) when the function f on the right hand side is continuous. We also study the relationship between these solutions with Aleksandrov solutions defined in the previous section. More generally, we consider the Monge-Ampère type equation of the form

$$\det[I + D^2c^*(-Du(x))D^2u(x)] = F(x, u(x), Du(x)) \text{ in } \Omega \tag{3.11}$$

where $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is nonnegative.

Definition 3.2 *A c-convex function $u \in C(\Omega)$ is said to be a viscosity subsolution (or supersolution) of* (3.11) *if for any c-convex C^2 function ψ on Ω such that $u - \psi$ has a strict local maximum (or strict local minimum) at some $x_0 \in \Omega$ we have*

$$\det[I + D^2c^*(-D\psi(x_0))D^2\psi(x_0)] \geq (\leq)\ F(x_0, u(x_0), D\psi(x_0)).$$

And u is said to be a viscosity solution of (3.11) *if it is both a viscosity subsolution and supersolution.*

CHAPTER 4

Maximum Principles

Let Ω be a bounded open set in $\mathbb{R}^n$ and $u \in C(\Omega)$. Consider the classes of functions

$$F(u) := \{v : \ v \text{ is } c\text{-convex in } \Omega \ \text{ and } \ v(x) \le u(x) \ \ \forall x \in \Omega\},$$

$$G(u) := \{w : \ w \text{ is } c\text{-concave in } \Omega \ \text{ and } \ w(x) \ge u(x) \ \ \forall x \in \Omega\},$$

where w is called c-concave if $-w$ is c-convex. Let

$$u_*(x) := \sup_{v \in F(u)} v(x) \qquad \text{and} \qquad u^*(x) := \inf_{w \in G(u)} w(x). \tag{4.1}$$

Then u_* is c-convex and u^* is c-concave on Ω. Moreover, if c satisfies condition $(H3)$, then it follows from Remark 2.5 that u_* and u^* are in $C(\Omega)$. We call these functions the c-convex and c-concave envelopes of u in Ω respectively, and we have the inequalities

$$u_*(x) \le u(x) \le u^*(x) \quad \forall x \in \Omega.$$

We also have that $F(-u) = -G(u)$, and hence

$$\begin{aligned} -(u^*)(x) &= -\inf_{w \in G(u)} w(x) = \sup_{w \in G(u)} -w(x) \\ &= \sup_{-v \in G(u)} v(x) = \sup_{v \in -G(u)} v(x) = \sup_{v \in F(-u)} v(x) = (-u)_*(x). \end{aligned} \tag{4.2}$$

Consider the set of contact points

$$C_*(u) := \{x \in \Omega : \ u_*(x) = u(x)\} \qquad ; \qquad C^*(u) := \{x \in \Omega : \ u^*(x) = u(x)\}$$

which are relative closed in Ω if c satisfies $(H3)$. Then by (4.2) we get

$$C_*(u) = C^*(-u). \tag{4.3}$$

Since $u(x) \geq u_*(x)$ for every x in Ω, it is clear that

$$\partial_c(u_*)(C_*(u)) \subset \partial_c u(C_*(u)). \tag{4.4}$$

It is easy to check that if $x_0 \notin C_*(u)$, then $\partial_c u(x_0) = \emptyset$. Also if A and B are sets, then $\partial_c u(A \cup B) = \partial_c u(A) \cup \partial_c u(B)$. Hence,

$$\partial_c u(\Omega) = \partial_c u(C_*(u)) \cup \partial_c u(\Omega \setminus C_*(u)) = \partial_c u(C_*(u)). \tag{4.5}$$

Let $p \in \partial_c u(C_*(u))$, then $p \in \partial_c u(x_0)$ for some $x_0 \in C_*(u)$. Hence,

$$u(x) \geq u(x_0) - c(x-p) + c(x_0 - p) = u_*(x_0) - c(x-p) + c(x_0-p) \quad \forall x \in \Omega.$$

But then by definition of u_* we get $u_*(x) \geq u_*(x_0) - c(x-p) + c(x_0-p) \quad \forall x \in \Omega$. So $p \in \partial_c u_*(x_0)$. Therefore,

$$\partial_c u(C_*(u)) \subset \partial_c(u_*)(C_*(u)). \tag{4.6}$$

From (4.4), (4.5), (4.6) we obtain

$$\partial_c u(\Omega) = \partial_c u(C_*(u)) = \partial_c(u_*)(C_*(u)). \tag{4.7}$$

Let

$$\partial^c u(x_0) = \{p \in \mathbb{R}^n : \ u(x) \leq u(x_0) + c(x-p) - c(x_0 - p) \quad \forall x \in \Omega\}$$

be the c-superdifferential of u at x_0. Notice then that $\partial^c(-u)(x_0) = \partial_c u(x_0)$.

Lemma 4.1 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in C(\bar{\Omega})$ is such that $u \leq 0$ on $\partial\Omega$. Then for any $x_0 \in \Omega$ with $u(x_0) > 0$, we have*

$$\Omega(x_0, u(x_0)) \subset \partial^c(u^*)(C^*(u))$$

where $\Omega(x,t) = \{y \in \mathbb{R}^n : c(z-y) - c(x-y) + t > 0 \quad \forall z \in \bar{\Omega}\}$.

PROOF: Let $y \in \Omega(x_0, u(x_0))$, then

$$c(z-y) - c(x_0 - y) + u(x_0) > 0 \quad \forall z \in \bar{\Omega}. \tag{4.8}$$

Let

$$\lambda_0 := \inf\{\lambda : \ \lambda + c(z-y) - c(x_0-y) \geq u(z) \quad \forall z \in \bar{\Omega}\}.$$

By continuity we have

$$\lambda_0 + c(z-y) - c(x_0-y) \geq u(z) \quad \forall z \in \bar{\Omega}. \tag{4.9}$$

Consider the minimum

$$\min_{z \in \bar{\Omega}} [\lambda_0 + c(z-y) - c(x_0-y) - u(z)]$$

which is nonnegative by (4.9). This minimum is attained at some point $\bar{z} \in \bar{\Omega}$, and we have

$$\lambda_0 + c(\bar{z}-y) - c(x_0-y) - u(\bar{z}) = 0, \tag{4.10}$$

because on the contrary

$$\lambda_0 + c(z-y) - c(x_0-y) - u(z) \geq \epsilon > 0 \quad \forall z \in \bar{\Omega}$$

and λ_0 would not be the minimum. We now claim that $\bar{z} \in \Omega$. Indeed, since $u \leq 0$ on $\partial\Omega$, the claim will be proved if we show that $u(\bar{z}) > 0$. By taking $z = x_0$ in (4.9) we get $\lambda_0 \geq u(x_0)$, and consequently by (4.8)

$$c(z-y) - c(x_0-y) + \lambda_0 > 0 \quad \forall z \in \bar{\Omega}.$$

Combining with (4.10) yields $u(\bar{z}) = c(\bar{z}-y) - c(x_0-y) + \lambda_0 > 0$. Thus we must have $\bar{z} \in \Omega$. Therefore we have proved that if $y \in \Omega(x_0, u(x_0))$, then there exists $\bar{z} \in \Omega$ such that

$$u(\bar{z}) = c(\bar{z}-y) - c(x_0-y) + \lambda_0,$$

and since the above minimum is zero we also have

$$u(z) \leq \lambda_0 + c(z-y) - c(x_0-y) \qquad \forall z \in \bar{\Omega}.$$

Therefore by definition of u^* we obtain

$$u(z) \le u^*(z) \le \lambda_0 + c(z-y) - c(x_0-y) \qquad \forall z \in \Omega.$$

In particular,

$$u(\bar z) \le u^*(\bar z) \le \lambda_0 + c(\bar z-y) - c(x_0-y) = u(\bar z).$$

So $u^*(\bar z) = \lambda_0 + c(\bar z - y) - c(x_0 - y) = u(\bar z)$ and hence $\bar z \in C^*(u)$. Moreover by combining with the above inequality we get

$$u^*(z) \le \lambda_0 + c(z-y) - c(x_0-y) = u^*(\bar z) + c(z-y) - c(\bar z - y) \quad \forall z \in \Omega.$$

So $y \in \partial^c(u^*)(\bar z) \subset \partial^c(u^*)(C^*(u))$ and this completes the proof. □

We notice that from Lemma 4.1, (4.2), (4.3) and (4.7) we have

$$\begin{aligned} &\Omega(x_0, u(x_0)) \subset \partial^c(u^*)(C^*(u)) = \partial_c(-(u^*))(C^*(u)) \\ &= \partial_c((-u)_*)(C^*(u)) = \partial_c((-u)_*)(C_*(-u)) = \partial_c(-u)(C_*(-u)). \end{aligned} \tag{4.11}$$

Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous cost function satisfying $c(0) = \min_{x\in\mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For each x in $\overline{\Omega}$ and each $t \ge 0$, let $\Omega(x,t)$ be as in Lemma 4.1, i.e.,

$$\Omega(x,t) = \{y \in \mathbb{R}^n : -t - c(z-y) + c(x-y) < 0, \forall z \in \bar\Omega\},$$

and define

$$\tilde\Omega(x,t) = \{y \in \mathbb{R}^n : -t - c(z-y) + c(x-y) \le 0, \forall z \in \bar\Omega\}.$$

First observe that $\Omega(x,0) = \emptyset$, $x \in \Omega(x,t)$ for any $t > 0$ and $\tilde\Omega(x,t)$ is closed. Since c is uniformly continuous on any bounded set of $\mathbb{R}^n$, it is easy to see that $\Omega(x,t)$ is an open set for every x in $\overline{\Omega}$ and every $t \ge 0$. Also if $0 \le t_1 < t_2$ then $\overline{\Omega(x,t_1)} \subset \tilde\Omega(x,t_1) \subset \Omega(x,t_2)$, and $\cup_{t\ge 0}\Omega(x,t) = \mathbb{R}^n$. Particularly, we have $\overline{\Omega(x,t_1) \cap B(x,t_1)} \subset \Omega(x,t_2) \cap B(x,t_2)$ where the first set is compact and the later is a nonempty open set. Therefore, for any $x \in \overline{\Omega}$ and any $0 \le t_1 < t_2$, we have that

$$|\overline{\Omega(x,t_1) \cap B(x,t_1)}| < |\Omega(x,t_2) \cap B(x,t_2)|. \tag{4.12}$$

The following lemma will be needed later.

Lemma 4.2 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{x\in\mathbb{R}^n} c(x)$. Then $|\tilde{\Omega}(x,t) - \Omega(x,t)| = 0$ for all $x \in \overline{\Omega}$ and all $t > 0$.*

Proof: Let $x_0 \in \overline{\Omega}$ and $t_0 > 0$. Define

$$\mathcal{F} = \{c\text{-convex function } v \in C(\overline{\Omega}) : v \leq 0 \text{ on } \overline{\Omega} \text{ and } v(x_0) \leq -t_0\}$$

and

$$w(x) = \sup_{v\in\mathcal{F}} v(x) \;\forall x \in \overline{\Omega}.$$

If we let $v(x) = -t_0 - c(x-x_0) + c(0)$ then it is clear that $v \in \mathcal{F}$. So $\mathcal{F} \neq \emptyset$ and moreover we have w is bounded from below on $\overline{\Omega}$, $w \leq 0$ on $\overline{\Omega}$ and $w(x_0) = -t_0$. By using $w(x_0) = -t_0$ it is easy to see that $\tilde{\Omega}(x_0,t_0) = \partial_c(w,\overline{\Omega})(x_0)$[1]. Now let

$$\tilde{S} = \{p \in \mathbb{R}^n : \; p \in \partial_c(w,\overline{\Omega})(x_1) \cap \partial_c(w,\overline{\Omega})(x_2) \text{ for some } x_1, x_2 \in \overline{\Omega},\; x_1 \neq x_2\}.$$

Then $|\tilde{S}| = 0$ by Lemma 2.2. We shall complete the proof by showing that $\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0) \subset \tilde{S}$. Indeed, if $y \in \tilde{\Omega}(x_0,t_0)$ is such that $y \notin \tilde{S}$ then as $y \in \partial_c(w,\overline{\Omega})(x_0)$ we get $-t_0 - c(z-y) + c(x_0-y) < w(z)$ for all z in $\overline{\Omega}\setminus\{x_0\}$. Particularly, since $w \leq 0$ on $\overline{\Omega}$ and $-t_0 < 0$ we obtain $-t_0 - c(z-y) + c(x_0-y) < 0$ for all z in $\overline{\Omega}$. That is, $y \in \Omega(x_0,t_0)$ and hence $\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0) \subset \tilde{S}$ as desired. □

Now suppose $g \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. on $\mathbb{R}^n$. Let $B(g) = \int_{\mathbb{R}^n} g(y)\, dy$ and for each $t \geq 0$, define

$$h(t) = \inf_{x\in\overline{\Omega}} \int_{\Omega(x,t)\cap B(x,t)} g(y)\, dy. \tag{4.13}$$

Then clearly $h : [0,+\infty) \to [0, B(g))$ with $h(0) = 0$. We remark that by using the Dominated Convergence Theorem and Lemma 4.2 it can be shown easily that the function $f(x) := \int_{\Omega(x,t)\cap B(x,t)} g(y)\, dy$ is continuous on $\overline{\Omega}$ (a similar argument will be employed in the proof of Lemma 4.3 below). Therefore, in the definition of h the infimum is achieved, i.e.,

$$h(t) = \min_{x\in\overline{\Omega}} \int_{\Omega(x,t)\cap B(x,t)} g(y)\, dy.$$

By this fact and (4.12), we also have h is strictly increasing.

[1] Notice that if c satisfies $(H3)$, then from Lemma 2.3 the set $\tilde{\Omega}(x,t)$ is bounded whenever $x \in \Omega$.

Lemma 4.3 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{\mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then the map $h : [0, +\infty) \to [0, B(g))$ is continuous, strictly increasing, and onto with $h(0) = 0$. Consequently, it is invertible and $h^{-1} : [0, B(g)) \to [0, +\infty)$ is also continuous, strictly increasing with $h^{-1}(0) = 0$.*

PROOF: It remains to prove h is continuous and onto. Firstly, let $t_0 \in [0, +\infty)$ and we want to show h is continuous at t_0. For this it suffices to prove that for any sequence $\{t_n\} \subset (0, +\infty)$ with $t_n \to t_0$, there exists a subsequence $\{n_j\}$ such that $h(t_{n_j}) \to h(t_0)$. If $\{t_n\}$ is such a sequence, then by the remark before this lemma there exists $\{x_n\} \subset \overline{\Omega}$ satisfying

$$h(t_n) = \int_{\Omega(x_n,t_n)\cap B(x_n,t_n)} g(y)\ dy \qquad \forall n.$$

As Ω is bounded we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x_0 \in \overline{\Omega}$ so that $x_{n_j} \to x_0$ as $j \to \infty$. We consider the following two cases.

Case 1: $t_0 = 0$. We have

$$\begin{aligned} h(t_{n_j}) &= \int_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})} g(y)\ dy \\ &= \int_{B(x_0,1)} g(y)\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y)\ dy \longrightarrow 0 = h(0) \end{aligned}$$

by the Dominated Convergence Theorem since $\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y) \longrightarrow 0$ for all $y \neq x_0$.

Case 2: $t_0 > 0$. We have

$$\begin{aligned} &h(t_{n_j}) = \int_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})} g(y)\ dy = \int_{B(x_0,2t_0)} g(y)\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y)\ dy \\ &\longrightarrow \int_{B(x_0,2t_0)} g(y)\chi_{\Omega(x_0,t_0)\cap B(x_0,t_0)}(y)\ dy = \int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\ dy \end{aligned}$$

by the Dominated Convergence Theorem since

$$\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y) \longrightarrow \chi_{\Omega(x_0,t_0)\cap B(x_0,t_0)}(y)$$

for all $y \neq E := [(\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0)) \cap \overline{B}(x_0,t_0)] \cup [\tilde{\Omega}(x_0,t_0) \cap \partial B(x_0,t_0)]$, which has Lebesgue measure zero by Lemma 4.2. This can be easily verified by noticing the fact that $\mathbb{R}^n = [\Omega(x_0,t_0)\cap B]\cup\tilde{\Omega}(x_0,t_0)^c\cup\overline{B}(x_0,t_0)^c\cup E$. Thus we have shown that

$$h(t_{n_j}) \longrightarrow \int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\ dy.$$

Now we claim that $\int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\ dy = h(t_0)$. Indeed, for each $x\in\overline{\Omega}$ we have

$$\begin{aligned}&\int_{\Omega(x,t_0)\cap B(x,t_0)} g(y)\ dy = \lim_{j\to\infty}\int_{\Omega(x,t_{n_j})\cap B(x,t_{n_j})} g(y)\ dy\\ &\geq \liminf_{j\to\infty} h(t_{n_j}) = \int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\ dy\end{aligned}$$

where we have again used Lemma 4.2 in the first equality. Therefore, by taking the infimum on the left hand side we obtain $h(t_0)\geq \int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\ dy$. So the claim is proved since the reverse inequality is obvious. Thus we get $h(t_{n_j})\to h(t_0)$ as desired. This implies that h is continuous at t_0.

Secondly, we want to show that h is onto. We know that $h(0)=0$. Now if we let $a\in(0,B(g))$ then since

$$\int_{\Omega(0,t)\cap B(0,t)} g(y)\ dy \longrightarrow B(g) \text{ as } t\to+\infty$$

we can find a $t_0>0$ such that $a<\int_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\ dy < B(g)$. For any $x\in\overline{\Omega}$ and any $y\in\Omega(0,t_0)\cap B(0,t_0)$ we have

$$\begin{aligned}&-c(z-y)+c(x-y) = -c(z-y)+c(-y)+[c(x-y)-c(-y)]\\ &< t_0 + \sup_{w_1\in\Omega_{t_0};w_2\in B(0,t_0)} |c(w_1)-c(w_2)| =: t_1 < +\infty\ \forall z\in\overline{\Omega}\end{aligned}$$

where $\Omega_{t_0}=\{y\in\mathbb{R}^n : \operatorname{dist}(y,\overline{\Omega})<t_0\}$. Consequently, $\Omega(0,t_0)\cap B(0,t_0)\subset \Omega(x,t_1)\cap B(0,t_0)$ for all x in $\overline{\Omega}$. Hence, by picking t_1 sufficiently large if necessary we can assume that $\Omega(0,t_0)\cap B(0,t_0)\subset\Omega(x,t_1)\cap B(x,t_1)$ for all x in $\overline{\Omega}$. This implies that $\int_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\ dy\leq h(t_1)$. Therefore, we obtain

$$h(0)=0<a<\int_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\ dy \leq h(t_1) < B(g).$$

But then since h is continuous on $[0,+\infty)$, there must exist a $t_2 \in (0,+\infty)$ so that $h(t_2) = a$, which means that h is onto. $\square$

From now on for convenience we will consider h^{-1} as a one-to-one function from $[0, B(g)]$ onto $[0,+\infty]$ with $h^{-1}(B(g)) = +\infty$. Therefore, $h^{-1}(a) < +\infty$ only if $0 \leq a < B(g)$. By combining the previous results we obtain the following maximum principle which holds for any continuous function on $\bar{\Omega}$.

Theorem 4.1 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{\mathbb{R}^n} c(x)$. Let $g \in L^1_{loc}(\mathbb{R}^n)$ be positive a.e. and Ω be a bounded open set in $\mathbb{R}^n$. If $u \in C(\bar{\Omega})$ then*

$$\max_{\Omega} u(x) \leq \max_{\partial\Omega} u(x) + h^{-1}(\omega_c(g,-u)(\Omega)).$$

PROOF: Let $M = \max_{\partial\Omega} u(x)$ and let $x_0 \in \Omega$ be such that $u(x_0) > M$. By Lemma 4.1 and (4.11) we have $\Omega(x_0, u(x_0)-M) \subset \partial_c(-u+M)(C_*(-u+M)) = \partial_c(-u)(\Omega)$. This gives

$$h(u(x_0)-M) \leq \int_{\Omega(x_0,u(x_0)-M)} g(y)\ dy \leq \int_{\partial_c(-u)(\Omega)} g(y)\ dy = \omega_c(g,-u)(\Omega).$$

Hence by taking the inverse we obtain $u(x_0) \leq M + h^{-1}(\omega_c(g,-u)(\Omega))$ and the proof is complete. $\square$

We end this section noticing that if the cost function c is convex, C^1 and satisfies that there exist positive constants A, α such that $|Dc(x)| \leq A|x|^\alpha$ for all x in $\mathbb{R}^n$ then Theorem 4.1 also holds with the function h defined in a simpler way, namely $h(t) = \inf_{x\in\bar{\Omega}} \int_{B(x,t)} g(y)\ dy = \min_{x\in\bar{\Omega}} \int_{B(x,t)} g(y)\ dy$. The advantage of this definition is that it is independent of c and in many cases when the function g is simple enough we can calculate h and h^{-1} exactly. For example, when $g \equiv 1$ we have the following result which is an extension of the well known Aleksandrov-Bakelman-Pucci maximum principle.

Theorem 4.2 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a C^1 and convex function satisfying there exist positive constants A, α such that $|Dc(x)| \leq A|x|^\alpha$ for all x in $\mathbb{R}^n$. Let Ω be a bounded open set in $\mathbb{R}^n$. We have* *(a) If* $u \in C(\bar{\Omega})$ *then*

$$\max_{\Omega} u(x) \leq \max_{\partial\Omega} u(x) + A\ \omega_n^{-\frac{\alpha}{n}}\, diam(\Omega)\, |\partial_c(-u)(C_*(-u))|^{\frac{\alpha}{n}}.$$

(b) If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *and in addition* c *satisfies* $(H2)$ *with* $c^* \in C^2(\mathbb{R}^n)$*, then*

$$\max_{\Omega} u(x) \leq \max_{\partial\Omega} u(x) + A\ \omega_n^{-\frac{\alpha}{n}}\, diam(\Omega) \left(\int_{C_*(-u)} |\det(I - D^2c^*(Du)D^2u)|dx \right)^{\frac{\alpha}{n}}.$$

Proof: (a) Let $M = \max_{\partial\Omega} u(x)$ and $x_0 \in \Omega$ be such that $u(x_0) > M$. For any $z \in \bar{\Omega}$ and $y \in \mathbb{R}^n$, we have from the convexity of c and the assumptions that

$$\begin{aligned} &c(z-y) - c(x_0-y) + u(x_0) - M \geq Dc(x_0-y)\cdot(z-x_0) + u(x_0) - M \\ &\geq -A|x_0-y|^{\alpha}|z-x_0| + u(x_0) - M \geq -A\ \mathrm{diam}(\Omega)|x_0-y|^{\alpha} + u(x_0) - M. \end{aligned}$$

So if $y \in B(x_0, R)$ where $R = \left(\dfrac{u(x_0)-M}{A\ \mathrm{diam}(\Omega)} \right)^{\frac{1}{\alpha}}$, then we get

$$c(z-y) - c(x_0-y) + u(x_0) - M > 0 \qquad \forall z \in \bar{\Omega}.$$

That is, $B(x_0, R) \subset \Omega(x_0, u(x_0) - M)$. Therefore, by Lemma 4.1 and (4.11) we obtain

$$\omega_n R^n = |B(x_0,R)| \leq |\Omega(x_0, u(x_0)-M)| \leq |\partial_c(-u)(C_*(-u))|$$

or $u(x_0) - M \leq A\ w_n^{-\frac{\alpha}{n}} \mathrm{diam}(\Omega)|\partial_c(-u)(C_*(-u))|^{\frac{\alpha}{n}}$. This completes the proof of part (a).

(b) This follows from (a) and the first part of Proposition 2.4. □

CHAPTER 5

Comparison principles

We begin with the following basic lemma.

Lemma 5.1 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u, v \in C(\bar{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ in Ω, then*

$$\partial_c v(\Omega) \subset \partial_c u(\Omega).$$

PROOF: The proof is the same as in [**Gut01**, Lemma 1.4.1] for the standard subdifferential but we include it here for convenience.

Let $p \in \partial_c v(\Omega)$. There exists $x_0 \in \Omega$ such that $v(x) \geq v(x_0) - c(x-p) + c(x_0 - p) \quad \forall x \in \Omega$. Define

$$a = \sup_{x \in \bar{\Omega}} \{v(x_0) - c(x-p) + c(x_0 - p) - u(x)\}.$$

Since $v(x_0) \geq u(x_0)$ we have $a \geq 0$. Also, there exists $x_1 \in \bar{\Omega}$ such that $a = v(x_0) - c(x_1 - p) + c(x_0 - p) - u(x_1)$ and so

$$u(x) \geq v(x_0) - c(x-p) + c(x_0-p) - a = u(x_1) - c(x-p) + c(x_1-p) \qquad \forall x \in \Omega.$$

Moreover,

$$v(x_1) \geq v(x_0) - c(x_1 - p) + c(x_0 - p) = u(x_1) + a.$$

Hence, if $a > 0$, then $x_1 \in \Omega$ and we get $p \in \partial_c u(x_1) \subset \partial_c u(\Omega)$. If $a = 0$ then

$$u(x) \geq v(x_0) - c(x-p) + c(x_0-p) \geq u(x_0) - c(x-p) + c(x_0-p) \qquad \forall x \in \Omega,$$

and consequently we obtain $p \in \partial_c u(x_0) \subset \partial_c u(\Omega)$. This completes the proof. $\square$

We next have the following result which gives a stronger conclusion than Lemma 5.1.

Lemma 5.2 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u, v \in C(\Omega)$. Suppose that the set $G := \{x \in \Omega : v(x) > u(x)\}$ satisfies $\overline{G} \subset \Omega$. Then $\partial_c v(G) \subset \mathrm{Int}(\partial_c u(G))$.*

PROOF: If $p \in \partial_c v(G)$, then there exists $x_0 \in G$ such that $v(x) \geq v(x_0) - c(x-p) + c(x_0 - p)$ for all $x \in \Omega$. Let $\epsilon = v(x_0) - u(x_0) > 0$ and consider the hypersurface of the form $v(x_0) - c(x-q) + c(x_0 - q) - \dfrac{\epsilon}{2}$, where q will be chosen in a moment. Fix a ball B sufficiently large such that $x - z \in B$ for all $(x, z) \in \Omega \times B(p, 1)$. Choose $M_\epsilon = \dfrac{\epsilon}{4\|c\|_{Lip(B)} + \epsilon}$. Then for any $q \in B(p, M_\epsilon)$ we have

$$\begin{aligned} & v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} \qquad (5.1)\\ &= v(x_0) - c(x-p) + c(x_0-p) + [c(x-p) - c(x-q)] \\ &\qquad + [c(x_0-q) - c(x_0-p)] - \frac{\epsilon}{2} \\ &\leq v(x) + 2\|c\|_{Lip(B)}|p-q| - \frac{\epsilon}{2} \leq v(x) \quad \forall x \in \Omega. \end{aligned}$$

We shall show that $B(p, M_\epsilon) \subset \partial_c u(G)$. Indeed, for any $q \in B(p, M_\epsilon)$ let

$$a = \sup_{x \in \bar{G}} \{v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x)\}.$$

Since $v(x_0) - u(x_0) = \epsilon$, we get $a > 0$. Also observe that if x is in $\Omega \setminus \bar{G}$, then $v(x) \leq u(x)$ and so by combining with (5.1) we get

$$\begin{aligned} & v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x) \\ &\leq v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - v(x) \leq 0 < a. \end{aligned}$$

So in fact $a = \sup_{x\in\Omega} \{v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x)\}$. Now by the definition of a there exists $x_1 \in \bar{G}$ such that $a = v(x_0) - c(x_1 - q) + c(x_0 -$

$q) - \frac{\epsilon}{2} - u(x_1)$ and hence by the above observation we have

$$v(x_0)-c(x_1-q)+c(x_0-q)-\frac{\epsilon}{2}-u(x_1) \geq v(x_0)-c(x-q)+c(x_0-q)-\frac{\epsilon}{2}-u(x)$$

for all $x \in \Omega$, or equivalently,

$$u(x) \geq u(x_1) - c(x-q) + c(x_1 - q) \quad \forall x \in \Omega. \tag{5.2}$$

On the other hand, applying (5.1) at $x = x_1$ yields

$$v(x_1) \geq v(x_0) - c(x_1 - q) + c(x_0 - q) - \frac{\epsilon}{2} = u(x_1) + a > u(x_1).$$

Therefore, $x_1 \in G$ and hence from (5.2) we get $q \in \partial_c u(x_1) \subset \partial_c u(G)$, i.e., $B(p, M_\epsilon) \subset \partial_c u(G)$. This completes the proof of the lemma. □

We recall a lemma from [**GM96**] adapted to the case of the c-subdifferential.

Lemma 5.3 (Lemma 4.3 from [GM96]) *Suppose c satisfies $(H3)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u, v \in C(\Omega)$. Assume that $G = \{x \in \Omega : v(x) > u(x)\}$ is bounded, and $X = \{x \in \Omega : \partial_c u(x) \cap \partial_c v(G) \neq \emptyset\}$ is nonempty. If $p \in \Omega$, $u(p) = v(p)$, $\partial_c u(p) \cap \partial_c v(p) = \emptyset$, then dist$(p, X) > 0$.*

Proof: Suppose $\text{dist}(p, X) = 0$. Then there exist $x_n \in X$ such that $x_n \to p$. Then there exist $z_n \in G$ and y_n such that $y_n \in \partial_c u(x_n) \cap \partial_c v(z_n)$. Since $p \in \Omega$, it follows from Lemma 2.3 that $\partial_c u(\cup_n \{x_n\})$ is bounded. So passing through a subsequence, we may assume that $z_n \to z_0$ and $y_n \to y_0$. Since $y_n \in \partial_c u(x_n)$, we have $u(z) \geq u(x_n) - c(z - y_n) + c(x_n - y_n)$ for all $z \in \Omega$. Letting $n \to \infty$ yields $y_0 \in \partial_c u(p)$, and from the hypotheses $y_0 \notin \partial_c v(p)$. So there exists $t \in \Omega$ such that

$$v(t) < v(p) - c(t - y_0) + c(p - y_0). \tag{5.3}$$

On the other hand, since $y_n \in \partial_c v(z_n)$, we have

$$\begin{aligned} v(t) &\geq v(z_n) - c(t - y_n) + c(z_n - y_n) \\ &\geq u(z_n) - c(t - y_n) + c(z_n - y_n), \text{ since } z_n \in G \\ &\geq u(p) - c(z_n - y_0) + c(p - y_0) - c(t - y_n) + c(z_n - y_n), \text{ since } y_0 \in \partial_c u(p). \end{aligned}$$

Letting $n \to \infty$ we get $v(t) \geq u(p) - c(t - y_0) + c(p - y_0) = v(p) - c(t - y_0) + c(p - y_0)$ contradicting (5.3). □

We now consider conditions which force $\omega_c(g,u)(G) > \omega_c(g,v)(G)$. This will be used to prove the comparison principle Theorem 5.1.

Lemma 5.4 *Suppose c satisfies $(H2)$ and g is positive a.e. and locally integrable in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $G = \{x \in \Omega : v(x) > u(x)\}$ where $u, v \in C(\Omega)$. Suppose that $G \neq \emptyset$, $\overline{G} \subset \Omega$ and $\partial_c u(x_0) \cap \partial_c v(x_0) = \emptyset$ for some $x_0 \in \partial G$. Assume further that $x_0 \in Int(spt(\omega_c(g,u)))$. Then we have*

$$\omega_c(g,u)(G) > \omega_c(g,v)(G).$$

Proof: Let $X = \{x \in \Omega : \partial_c u(x) \cap \partial_c v(G) \neq \emptyset\}$. Then if $X \neq \emptyset$ we have from Lemma 5.3 that $\operatorname{dist}(x_0, X) > 0$. Therefore, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and $\partial_c u(B(x_0,r)) \cap \partial_c v(G) = \emptyset$, in particular, $\partial_c u(G \cap B(x_0,r)) \cap \partial_c v(G) = \emptyset$. This obviously holds if $X = \emptyset$. From Lemma 5.2, $\partial_c v(G) \subset \partial_c u(G)$. Thus we must have $\partial_c v(G) \subset \partial_c u(G \setminus B(x_0,r))$. Since $x_0 \in \operatorname{Int}(\operatorname{spt}(\omega_c(g,u)))$, there exists $r > 0$ small enough such that $B(x_0,r) \subset \operatorname{spt}(\omega_c(g,u))$. As $x_0 \in \partial G$, we then get that $\emptyset \neq G \cap B(x_0,r) \subset \operatorname{spt}(\omega_c(g,u))$. We therefore obtain

$$\begin{aligned} \omega_c(g,u)(G) &= \omega_c(g,u)(G \setminus B(x_0,r)) + \omega_c(g,u)(G \cap B(x_0,r)) \\ &\geq \omega_c(g,v)(G) + \omega_c(g,u)(G \cap B(x_0,r)) > \omega_c(g,v)(G). \end{aligned}$$

This completes the proof. □

By using Lemma 5.4 we are able to prove the following comparison principle, which in particular gives the uniqueness of solutions for the Dirichlet problems considered in the next section. In the following theorem we denote $S := \operatorname{spt}(\omega_c(u)) \setminus \overline{\operatorname{Int}(\operatorname{spt}(\omega_c(u)))}$.

Theorem 5.1 *Suppose c satisfies $(H2)$. Let g be positive a.e. and locally integrable in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u, v \in C(\bar{\Omega})$ be c-convex in Ω, and*

$$\omega_c(g,u)(E) \leq \omega_c(g,v)(E) \text{ for all Borel sets } E \subset \Omega. \tag{5.4}$$

Assume that

$$\text{for every open set } D \Subset \Omega \text{ with } |\partial_c v(D \setminus spt(\omega_c(u)))| = 0, \text{ there exists} \quad (5.5)$$
$$\text{a closed set } F \subset \partial_c v(S \cap D) \text{ such that } |\partial_c v(S \cap D) \setminus F| = 0.$$

Then we have

$$\min_{\bar{\Omega}} \{u(x) - v(x)\} = \min_{\partial\Omega} \{u(x) - v(x)\}.$$

Proof: For simplicity we shall present the proof when $g \equiv 1$, i.e., when $\omega_c(g, u)$ and $\omega_c(g, v)$ are replaced by $\omega_c(u)$ and $\omega_c(v)$ respectively. However, it can be readily checked that the same argument works for general g as well. For this general case, one only need to note that by our assumptions on the function g we have $\text{spt}(\omega_c(g, u)) = \text{spt}(\omega_c(u))$ and if $E, F \subset \mathbb{R}^n$ are two measurable sets with $E \subset F$ and $\int_E g < \infty$, then $\int_E g = \int_F g$ if and only if $|E| = |F|$.

By adding a constant to v if necessary, we can assume without loss of generality that $\min_{\partial\Omega} \{u(x) - v(x)\} = 0$. We shall prove that $u(x) \geq v(x) \quad \forall x \in \bar{\Omega}$. Indeed, suppose not, then there exists $\bar{x} \in \Omega$ such that $v(\bar{x}) - u(\bar{x}) = \max_{\bar{\Omega}} [v(x) - u(x)] > 0$. Let $\bar{\delta} = [v(\bar{x}) - u(\bar{x})] > 0$. For every $0 < \delta < \bar{\delta}$, define $w_\delta(x) := v(x) - \delta$ in $\overline{\Omega}$ and

$$D_\delta := \{x \in \overline{\Omega} : w_\delta(x) > u(x)\} = \{x \in \Omega : w_\delta(x) > u(x)\}.^1$$

We have $\bar{x} \in D_\delta$, and $w_\delta(x) = v(x) - \delta \leq u(x) - \delta < u(x)$ for $x \in \partial\Omega$. Hence $\overline{D_\delta} \subset \Omega$ and $\partial D_\delta = \{x \in \Omega : w_\delta(x) = u(x)\}$. Applying Lemma 5.2 we obtain $\partial_c v(D_\delta) = \partial_c w_\delta(D_\delta) \subset \text{Int}(\partial_c u(D_\delta))$. It follows that $\omega_c(u)(D_\delta) > 0$ and therefore

$$\text{spt}(\omega_c(u)) \cap D_\delta \neq \emptyset, \qquad \text{for } 0 < \delta < \bar{\delta}. \tag{5.6}$$

Denote $V = \text{Int}(\text{spt}(\omega_c(u)))$ and fix a $\delta \in (0, \bar{\delta})$. We then consider the following cases.

Case 1: $\overline{V} \cap D_\delta = \emptyset$.

[1] If $\delta_1 < \delta_2$, then $\overline{D_{\delta_2}} \subset \text{Int}(D_{\delta_1})$.

Since $\operatorname{spt}(\omega_c(u)) = S \cup \bar{V}$, from (5.6) we get that $S \cap D_\delta = \operatorname{spt}(\omega_c(u)) \cap D_\delta \neq \emptyset$, and so

$$\begin{aligned} |\partial_c v(D_\delta)| \leq |\partial_c u(D_\delta)| &= \omega_c(u)(\operatorname{spt}(\omega_c(u)) \cap D_\delta) \\ &= \omega_c(u)(S \cap D_\delta) \leq |\partial_c v(S \cap D_\delta)| \leq |\partial_c v(D_\delta)|. \end{aligned}$$

Therefore $|\partial_c v(D_\delta \setminus \operatorname{spt}(\omega_c(u))| = 0$. Thus, from (5.5) there exists F_δ closed, $F_\delta \subset \partial_c v(S \cap D_\delta)$ with $|\partial_c v(S \cap D_\delta) \setminus F_\delta| = 0$. So

$$|\partial_c v(D_\delta)| = |\partial_c v(S \cap D_\delta)| = |F_\delta|. \tag{5.7}$$

In addition, $F_\delta \subset \partial_c v(\overline{D_\delta})$, so it follows from Lemma 2.3 that F_δ is a compact set. Moreover, $F_\delta \subset \partial_c v(D_\delta) \subset \operatorname{Int}(\partial_c u(D_\delta))$. Therefore, we obtain

$$|F_\delta| < |\partial_c u(D_\delta)|. \tag{5.8}$$

From (5.7) and (5.8) we deduce that $|\partial_c v(D_\delta)| < |\partial_c u(D_\delta)|$ and this gives a contradiction.

Case 2: $\overline{V} \cap D_\delta \neq \emptyset$.

Since D_δ is open, $V \cap D_\delta \neq \emptyset$. We then decompose the nonempty open set V into the union of its disjoint connected open components $V_1 \cup V_2 \cup ... \cup V_k \cup ....$ Note that the number of connected components of V is at most countable.

Case 2 A: there exists a connected component V_j of V such that $\overline{V_j} \cap D_\delta \neq \emptyset$ and $\overline{V_j} \cap D_\delta^c \neq \emptyset$.

This implies that $V_j \cap D_\delta \neq \emptyset$ and $\overline{V_j} \cap D_\delta^c \neq \emptyset$. But as V_j is connected, then we can find a connected open component, say O, of the nonempty open set $V_j \cap D_\delta$ such that $\overline{O} \cap \partial D_\delta \neq \emptyset$. Now let

$$\tilde{O} = \{x \in O \mid u \text{ and } v \text{ are differentiable at } x\}.$$

Since u and v are differentiable a.e. on Ω we get $\tilde{O} = O$ a.e. Now if $\partial_c w_\delta(x) \subset \partial_c u(x)$ for all $x \in \tilde{O}$, then we have $x - \nabla c^*(-\nabla w_\delta(x)) = x - \nabla c^*(-\nabla u(x)) \quad \forall x \in \tilde{O}$. Therefore we obtain $\nabla u(x) = \nabla w_\delta(x)$ for a.e. $x \in O$. It follows that $u - w_\delta$ is constant on O, and hence constant on $\overline{O}$ by the continuity. By using the fact that $u = w_\delta$ on ∂D_δ and $\overline{O} \cap \partial D_\delta \neq \emptyset$ we then get

$u = w_\delta$ on O. Since O is nonempty and $O \subset D_\delta$, this contradicts the definition of D_δ. Thus we must have $\partial_c w_\delta(x_0) \cap \partial_c u(x_0) = \emptyset$ for some $x_0 \in \tilde{O}$. We first claim that $0 < w_\delta(x_0) - u(x_0) < \max_{\overline{\Omega}}[w_\delta(x) - u(x)] = \max_{\overline{D_\delta}}[w_\delta(x) - u(x)]$. Indeed, we only need to show the second inequality. Suppose by contradiction that it is false, then by letting $p = \partial_c w_\delta(x_0)$ we have for every $x \in \Omega$

$$u(x_0) - c(x-p) + c(x_0 - p) = w_\delta(x_0) - c(x-p) + c(x_0-p) - [w_\delta(x_0) - u(x_0)]$$
$$\leq w_\delta(x) - \max_{\overline{\Omega}}[w_\delta(x') - u(x')] \leq w_\delta(x) - [w_\delta(x) - u(x)] = u(x),$$

so $p = \partial_c u(x_0)$, a contradiction. This proves the claim. Now define $w_\delta^*(x) = w_\delta(x) - [w_\delta(x_0) - u(x_0)]$ in $\overline{\Omega}$ and let $G := \{x \in \Omega : w_\delta^*(x) > u(x)\} \subset D_\delta$. Then we get G is nonempty by the claim. Moreover it is clear that G is open, $\overline{G} \subset \Omega$ and $\partial G = \{x \in \Omega : w_\delta^*(x) = u(x)\}$. We also have $w_\delta^*(x_0) = w_\delta(x_0) - [w_\delta(x_0) - u(x_0)] = u(x_0)$. Therefore, $x_0 \in \partial G$ and by applying Lemma 5.4 and noting that $x_0 \in \mathrm{Int}(\mathrm{spt}(\omega_c(u)))$ we obtain that $|\partial_c u(G)| > |\partial_c v(G)|$, a contradiction.

Case 2 B: otherwise we have that each $\overline{V_i}$ is either contained in D_δ or contained in D_δ^c. Define two open sets V_δ^* and V_δ^{**} as follows

$$V_\delta^* := \cup_{\overline{V_i} \subset D_\delta} V_i \subset D_\delta \quad \text{and} \quad V_\delta^{**} := \cup_{\overline{V_i} \subset D_\delta^c} V_i \subset D_\delta^c.$$

Then we have $V = V_\delta^* \cup V_\delta^{**}$, $V_\delta^* \cap V_\delta^{**} = \emptyset$, $\overline{V} = \overline{V_\delta^*} \cup \overline{V_\delta^{**}}$. Also $V_\delta^* \neq \emptyset$, $\overline{V_\delta^*} \subset \overline{D_\delta}$ and $\overline{V_\delta^{**}} \subset D_\delta^c$. Let $\tilde{V}_\delta^* = \{x \in V_\delta^* \mid u \text{ and } v \text{ are differentiable at } x\}$. If we can find a $x_0 \in \tilde{V}_\delta^*$ such that $\partial_c w_\delta(x_0) \not\subset \partial_c u(x_0)$, then by arguing as in Case 2 A we obtain a contradiction. Therefore, we can assume that $\partial_c w_\delta(x) \subset \partial_c u(x)$ for all $x \in \tilde{V}_\delta^*$. But again as in Case 2 A these yield that $v - u$ is constant on each connected open component of V_δ^*. We claim that $\overline{V_\delta^*} \subset D_\delta$. Indeed, since otherwise there exist $\hat{x} \in \partial D_\delta$ and a sequence $\{x_n\} \subset V_\delta^*$ such that $x_n \to \hat{x}$. Therefore, $v(x_n) - u(x_n) \to v(\hat{x}) - u(\hat{x}) = \delta$. Since $0 < \delta < \bar{\delta}$ and $v(x) - u(x) > \delta$ for all $x \in D_\delta$, we can pick a n_0 large enough such that $\delta < v(x_{n_0}) - u(x_{n_0}) < \bar{\delta}$. Define $\hat{\delta} := v(x_{n_0}) - u(x_{n_0})$, then $\hat{\delta} \in (0, \bar{\delta})$. As $x_{n_0} \in V_\delta^*$, x_{n_0} must belong to some connected open component V_j of V_δ^*. But since $v - u$ is constant on V_j, we then deduce that $v(x) - u(x) = \hat{\delta}$ for all

$x \in V_j$. However, this implies that $V_j \subset \partial D_{\hat{\delta}}$, which is impossible because V_j is a nonempty open set. This yields a contradiction and the claim is proved. Now as $\overline{V_\delta^{**}} \subset D_\delta^c$ and by the claim, we have

$$spt(\omega_c(u)) \cap D_\delta = (\overline{V_\delta^*} \cup \overline{V_\delta^{**}} \cup S) \cap D_\delta = \overline{V_\delta^*} \cup (S \cap D_\delta).$$

So we obtain

$$\begin{aligned} |\partial_c v(D_\delta)| &\leq |\partial_c u(D_\delta)| = \omega_c(u)(spt(\omega_c(u)) \cap D_\delta) = \omega_c(u)(\overline{V_\delta^*} \cup (S \cap D_\delta)) \\ &\leq |\partial_c v(\overline{V_\delta^*} \cup (S \cap D_\delta))| = |\partial_c v(\overline{V_\delta^*}) \cup \partial_c v(S \cap D_\delta)| \leq |\partial_c v(D_\delta)|. \end{aligned}$$

Thus

$$|\partial_c v(D_\delta)| = |\partial_c v(\overline{V_\delta^*}) \cup \partial_c v(S \cap D_\delta)| = |\partial_c v(\overline{V_\delta^*}) \cup F_\delta|. \tag{5.9}$$

We have that F_δ is compact since F_δ is closed and $F_\delta \subset \partial_c v(S \cap D_\delta) \subset \partial_c v(\overline{D_\delta})$, where the last set is bounded by Lemma 2.3. Also $\partial_c v(\overline{V_\delta^*})$ is compact. Moreover, $\partial_c v(\overline{V_\delta^*}) \cup F_\delta \subset \partial_c v(D_\delta) \subset Int(\partial_c u(D_\delta))$. Therefore, we get

$$|\partial_c v(\overline{V_\delta^*}) \cup F_\delta| < |\partial_c u(D_\delta)|. \tag{5.10}$$

From (5.9) and (5.10) we deduce $|\partial_c v(D_\delta)| < |\partial_c u(D_\delta)|$ obtaining a contradiction. The proof is completed. □

Remark 5.1 (On condition (5.5)**)** We remark that condition (5.5) is satisfied if any of the following conditions holds.

1. For each $D \Subset \Omega$ open, the set $S \cap D$ is closed.
2. If $\text{stp}(\omega_c(u)) = \bar{V}$ with V open subset of Ω. In this case we have $S = \emptyset$.
3. If $\text{spt}(\omega_c(u))$ is convex with nonempty interior, then we also have $S = \emptyset$.
4. If $\text{spt}(\omega_c(u))$ is a finite set, then it follows from Lemma 2.4(i) that (5.5) holds.

CHAPTER 6

Dirichlet Problems

6.1 Homogeneous Dirichlet Problem

Definition 6.1 *A bounded set $\Omega \subset \mathbb{R}^n$ is called strictly convex if for any $z \in \partial\Omega$, there exists a supporting hyperplane H of $\overline{\Omega}$ such that $H \cap \overline{\Omega} = \{z\}$.*

Definition 6.2 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. A bounded set $\Omega \subset \mathbb{R}^n$ is called strictly c-convex if for any $z \in \partial\Omega$, any $\delta > 0$ and any $a > 0$, there exist $y, y^* \in \mathbb{R}^n$ such that*

$$c(x-y) - c(z-y) \geq 0 \;\forall x \in \partial\Omega, \text{ and } c(x-y) - c(z-y) \geq a \;\forall x \in \bar{\Omega} \setminus B(z,\delta) \tag{6.1}$$

and

$$c(z-y^*) - c(x-y^*) \geq 0 \;\forall x \in \partial\Omega, \text{ and } c(z-y^*) - c(x-y^*) \geq a \;\forall x \in \partial\Omega \setminus B(z,\delta). \tag{6.2}$$

To illustrate this definition we give several remarks.

Remark 6.1 *If $c(x) = \phi(|x|)$ with $\phi : [0,\infty) \to \mathbb{R}$ continuous and strictly increasing,, then (6.1) implies that Ω satisfies the exterior sphere condition, that is, for every $z \in \partial\Omega$, there exists an open ball B satisfying $\bar{B} \cap \bar{\Omega} = \{z\}$.*

Proof: Let $z \in \partial\Omega$. We claim that we can find a $y \in \mathbb{R}^n \setminus \bar{\Omega}$ such that

$$|x - y| \geq |z - y|, \qquad \text{for all } x \in \bar{\Omega}. \tag{6.3}$$

If we assume the claim for the moment, then we see that $B(y, |z-y|) \cap \bar{\Omega} = \emptyset$ and $z \in \overline{B}(y, |z-y|) \cap \bar{\Omega}$. Therefore, if we let $\bar{y}$ be the midpoint of the segment $\overline{yz}$ and B be the open ball centered at $\bar{y}$ and with radius $|z-y|/2$, then it is clear that $\bar{B} \cap \bar{\Omega} = \{z\}$ as desired.

It remains to prove the claim. Let a be such that

$$a > \max_{x, \tilde{y} \in \bar{\Omega}} [c(x - \tilde{y}) - c(z - \tilde{y})]$$

and let $\delta = \text{diam}(\Omega)/2$. Then by (6.1) there exists $y \in \mathbb{R}^n$ such that

$$c(x - y) - c(z - y) \geq 0, \qquad \text{for all } x \in \partial\Omega, \tag{6.4}$$

and

$$c(x - y) - c(z - y) \geq a, \qquad \text{for all } x \in \bar{\Omega} \setminus B\left(z, \frac{\text{diam}(\Omega)}{2}\right). \tag{6.5}$$

From (6.5) and the choice of a, we must have $y \notin \bar{\Omega}$. Then if $x \in \Omega$, let $\bar{x}$ be a point on the segment $\overline{xy}$ and on $\partial\Omega$. From the form of c and (6.4) we have, $c(x - y) \geq c(\bar{x} - y) \geq c(z - y)$ and we are done. □

Remark 6.2 *If $c(x) = \phi(|x|)$ is convex with $\phi : [0, \infty) \to \mathbb{R}$ continuous and strictly increasing, and the open set Ω verifies the first inequality in* (6.2)*, then Ω satisfies the enclosing sphere condition, that is, for each $z \in \partial\Omega$ there exists a ball $B_R \supset \Omega$ with $z \in \partial B_R$,.*

Proof: We first see that the first inequality in (6.2) holds in Ω. Because if $x \in \Omega$, then there exist two points $x_1, x_2 \in \partial\Omega$ such that $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$. Hence

$$\begin{aligned} c(x - y^*) &= c(t(x_1 - y^*) + (1-t)(x_2 - y^*)) \leq tc(x_1 - y^*) + (1-t)c(x_2 - y^*) \\ &\leq tc(z - y^*) + (1-t)c(z - y^*) = c(z - y^*). \end{aligned}$$

Therefore

$$\Omega \subset \{x : \phi(|x-y^*|) \le \phi(|z-y^*|)\} = \{x : |x-y^*| \le |z-y^*|\} = B_{|z-y^*|}(y^*),$$

that is, Ω satisfies the enclosing sphere condition and in particular, Ω is strictly convex. □

Remark 6.3 *If $c : \mathbb{R}^n \to \mathbb{R}$ is convex and $\lim_{|x|\to\infty} \dfrac{c(x)}{|x|} = +\infty$, and Ω is strictly convex, then* (6.1) *holds.*

Proof: Let $z \in \partial\Omega$, $\delta > 0$, $a > 0$, and $P(x) = 0$ be the equation of the supporting hyperplane to Ω at z. We can assume $\bar{\Omega} \subset \{x : P(x) \ge 0\}$. Since Ω is strictly convex, there exists $\eta > 0$ such that $\{x \in \bar{\Omega} : P(x) \le \eta\} \subset B(z, \delta)$. That is, $P(x) \ge \eta$ for all $x \in \bar{\Omega} \setminus B(x, \delta)$. We can write $P(x) = A \cdot (x - z)$ with $A \in \mathbb{R}^n$. Since $\partial c(\mathbb{R}^n) = \mathbb{R}^n$ (∂c means the standard subdifferential of c), we get that $\frac{a}{\eta} A \in \partial c(w)$ for some $w \in \mathbb{R}^n$. If $y = z - w$, then $\frac{a}{\eta} A \in \partial c(z - y)$ and hence

$$c(x-y) - c(z-y) \ge \frac{a}{\eta} A \cdot (x-z) = \frac{a}{\eta} P(x) \ge 0$$

for all $x \in \bar{\Omega}$, and

$$c(x-y) - c(z-y) \ge \frac{a}{\eta} A \cdot (x-z) = \frac{a}{\eta} P(x) \ge \frac{a}{\eta}\eta = a$$

for all $x \in \bar{\Omega} \setminus B(z, \delta)$. □

Remark 6.4 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $c(x) = \phi(|x|)$ for some nondecreasing function $\phi : [0, \infty) \to \mathbb{R}$ satisfying $\phi \in C^1(m, \infty)$ for some $m \ge 0$ and $\lim_{t\to+\infty} \phi'(t) = +\infty$. If $\Omega \subset \mathbb{R}^n$ is a bounded open set satisfying the enclosing sphere condition, then Ω is strictly c-convex.*

Proof: In view of Remark 6.3, we only need to verify (6.2). We may assume $m = 0$ because we will see that we can pick y^* below as far as we want from $\overline{\Omega}$. Let $z \in \partial\Omega$, $\delta > 0$, $a > 0$, and $B_R(z_0)$ be an enclosing ball for z. If H is the supporting hyperplane to the ball $B_R(z_0)$ at z, then $H \cap \overline{\Omega} = \{z\}$. Let H^- denote the halfspace containing Ω and let H^+ denote the complementary halfspace. Consider the line passing through z and orthogonal to H. Let L and $L^{'}$ be the rays starting from z and lying in H^+ and H^- respectively. We have that $z_0 \in L'$.

Let $L'_{z_0} = \{y \in L' : |y - z| \geq R\}$. If $y \in L'_{z_0}$, then $\Omega \subset B(y, |y - z|)$ and hence $c(z - y) - c(x - y) \geq 0$ for all $x \in \partial\Omega$. Therefore, (6.2) will be proved if we can show that there exists $y^* \in L^{'}_{z_0}$ such that $c(z - y^*) - c(x - y^*) \geq a$ for all $x \in \partial\Omega \setminus B(z, \delta)$. From the strict convexity of Ω, there exists $\beta > 0$ such that

$$\frac{x-y}{|x-y|} \cdot \frac{z-x}{|z-x|} \geq \beta, \qquad \forall x \in \partial\Omega \backslash B(z,\delta), \quad \text{and } \forall y \in L'_{z_0} \text{ with } |y-z| \text{ large.}$$

Again from the convexity of c we then get

$$\begin{aligned} c(z-y) - c(x-y) &\geq Dc(x-y) \cdot (z-x) = \phi'(|x-y|)\,|z-x|\,\frac{x-y}{|x-y|} \cdot \frac{z-x}{|z-x|} \\ &\geq \phi'(|x-y|)\,\beta\,\delta, \end{aligned}$$

for all $x \in \partial\Omega \backslash B(z, \delta)$ and all $y \in L'_{z_0}$ with $|y - z|$ large. Since $\lim_{t\to+\infty} \phi'(t) = +\infty$, (6.2) follows picking $y = y^* \in L'_{z_0}$ sufficiently far from z. This completes the proof. ☐

Under the assumption that the domain is strictly c-convex we solve the homogeneous Dirichlet problem as follows.

Theorem 6.1 *Suppose that c satisfies condition $(H1)$ and $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set and $\psi : \partial\Omega \to \mathbb{R}$ be a continuous function. Then there exists a unique c-convex function $u \in C(\bar{\Omega})$ Aleksandrov generalized solution to the problem*

$$\begin{aligned} \det[I + D^2c^*(-Du(x))D^2u(x)] = 0 \quad &\text{in } \Omega, \\ u = \psi \quad &\text{on } \partial\Omega. \end{aligned}$$

PROOF: Define

$$\mathcal{F} := \{f(x) = -c(x-y) - \lambda : \; y \in \mathbb{R}^n, \lambda \in \mathbb{R} \text{ and } f(x) \leq \psi(x) \text{ on } \partial\Omega\}.$$

Then since ψ is continuous on $\partial\Omega$ we have that $\mathcal{F}$ is nonempty. Let

$$u(x) = \sup\{f(x) : f \in \mathcal{F}\}. \tag{6.6}$$

Claim 1: $u = \psi$ on $\partial\Omega$.

It is clear from the definition of u that $u \leq \psi$ on $\partial\Omega$. Now let $z \in \partial\Omega$, and $\epsilon > 0$. Then we can find $\delta > 0$ such that $|\psi(x) - \psi(z)| < \epsilon$ for all $x \in B(z,\delta)\cap\partial\Omega$. Choose $a = \psi(z)-\epsilon-m$ where $m = \min\{\psi(x)|x \in \partial\Omega \setminus B(z,\delta)\}$. Since Ω is c-strictly convex, there exists $y \in \mathbb{R}^n$ such that $c(x-y)-c(z-y) \geq 0 \quad \forall x \in \partial\Omega$ and $c(x-y) - c(z-y) \geq a \quad \forall x \in \partial\Omega \setminus B(z,\delta)$.

Let $f(x) := -[c(x-y) - c(z-y)] + \psi(z) - \epsilon$. We claim that $f \leq \psi$ on $\partial\Omega$. Indeed, if $x \in B(z,\delta) \cap \partial\Omega$, then $f(x) = -[c(x-y) - c(z-y)] + \psi(z) - \epsilon \leq \psi(z) - \epsilon \leq \psi(x)$. On the other hand, if $x \in \partial\Omega \setminus B(z,\delta)$, then we have $c(x-y) - c(z-y) \geq a$ and hence

$$f(x) = -[c(x-y)-c(z-y)]+\psi(z)-\epsilon \leq -[\psi(z)-\epsilon-m]+\psi(z)-\epsilon = m \leq \psi(x).$$

Therefore $f \in \mathcal{F}$. Thus, $u(z) \geq f(z) = \psi(z) - \epsilon$ for all $\epsilon > 0$. Hence, $u(z) \geq \psi(z)$ and this proves Claim 1.

Claim 2: u is c-convex and $u \in C(\Omega)$.

From the definition it is clear that u is uniformly bounded from below on $\bar{\Omega}$. Now let $g(x) := -c(x) + \max_{\partial\Omega}[\psi + c]$. It is clear that $g \geq \psi$ on $\partial\Omega$, g is c-convex and as $c \in C^1(\mathbb{R}^n)$ we have $\partial_c g(\Omega) = \{0\}$ and so $|\partial_c g(\Omega)| = 0$. Hence for each $f(x) = -c(x-y) - \lambda \in \mathcal{F}$, it follows from the comparison principle Theorem 5.1 that $f \leq g$ in $\overline{\Omega}$ and therefore u is uniformly bounded from above on $\bar{\Omega}$. Thus, we get u is uniformly bounded on $\bar{\Omega}$. Particularly, this implies that u is c-convex and moreover from Remark 2.5 we obtain $u \in C(\Omega)$.

Claim 3: u is continuous up to the boundary.

Let $z \in \partial\Omega$ and $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to z$. For any $\epsilon > 0$, we can find $\delta > 0$ such that $|\psi(x) - \psi(z)| < \epsilon$ for all $x \in B(z,\delta) \cap \partial\Omega$. Choose $b = M - \psi(z) - \epsilon$ where $M = \max\{\psi(x)|x \in \partial\Omega \setminus B(z,\delta)\}$. Since Ω is c-strictly convex, there exists $y^* \in \mathbb{R}^n$ such that $c(z-y^*) - c(x-y^*) \geq 0 \quad \forall x \in \partial\Omega$ and $c(z-y^*) - c(x-y^*) \geq b \quad \forall x \in \partial\Omega \setminus B(z,\delta)$.

Let $h(x) := c(z - y^*) - c(x - y^*) + \psi(z) + \epsilon$. For $x \in B(z,\delta) \cap \partial\Omega$ we have $h(x) \geq \psi(z) + \epsilon > \psi(x)$, and for $x \in \partial\Omega \setminus B(z,\delta)$ we have $h(x) \geq b + \psi(z) + \epsilon \geq \psi(x)$. Therefore, we get $h(x) \geq \psi(x)$ on $\partial\Omega$. Moreover, h is c-convex, $\partial_c h(\Omega) = \{y^*\}$, so $|\partial_c h(\Omega)| = 0$. Thus for any $f(x) = -c(x - y) - \lambda \in \mathcal{F}$, as in Claim 2 we obtain $f \leq h$ in $\overline{\Omega}$. Hence $u(x) \leq h(x) = c(z - y^*) - c(x - y^*) + \psi(z) + \epsilon$ in $\overline{\Omega}$. This yields $\limsup u(x_n) \leq \psi(z) + \epsilon$ and hence $\limsup u(x_n) \leq \psi(z)$ since $\epsilon > 0$ was chosen arbitrary. On the other hand, for any $\epsilon > 0$, by constructing a function $f \in \mathcal{F}$ as in Claim 1 we get $u(x_n) \geq f(x_n) = -[c(x_n - y) - c(z - y)] + \psi(z) - \epsilon$. So $\liminf u(x_n) \geq \psi(z) - \epsilon$ for any $\epsilon > 0$ and hence $\liminf u(x_n) \geq \psi(z)$. Thus $\lim_{n\to+\infty} u(x_n) = \psi(z) = u(z)$ and we obtain $u \in C(\bar{\Omega})$.

Claim 4: $|\partial_c u(\Omega)| = 0$.

Let $p \in \partial_c u(\Omega)$. Then there exists $x_0 \in \Omega$ such that

$$u(x) \geq u(x_0) - c(x - p) + c(x_0 - p) \quad \forall x \in \Omega.$$

Therefore if we let $f(x) := u(x_0) - c(x - p) + c(x_0 - p)$, then since $u(x) = \psi(x)$ on $\partial\Omega$ we get $f(x) \leq \psi(x)$ on $\partial\Omega$. We now claim that in fact there is $\zeta \in \partial\Omega$ satisfying $f(\zeta) = \psi(\zeta)$. Indeed, since otherwise there exists $\epsilon > 0$ such that $f + \epsilon \leq \psi$ on $\partial\Omega$. Then the function $f(x) + \epsilon \in \mathcal{F}$ and hence $u(x) \geq f(x) + \epsilon$ for all $x \in \Omega$. In particular, $u(x_0) \geq f(x_0) + \epsilon = u(x_0) + \epsilon$. This is a contradiction. So $f(\zeta) = \psi(\zeta) = u(\zeta)$ for some $\zeta \in \partial\Omega$. But then we get

$$\begin{aligned} u(x) &\geq u(x_0) - c(x - p) + c(x_0 - p) \\ &= u(x_0) - c(\zeta - p) + c(x_0 - p) - c(x - p) + c(\zeta - p) \\ &= f(\zeta) - c(x - p) + c(\zeta - p) = u(\zeta) - c(x - p) + c(\zeta - p) \quad \forall x \in \Omega \end{aligned}$$

So $p \in \partial_c(u, \overline{\Omega})(x_0) \cap \partial_c(u, \overline{\Omega})(\zeta)$, i.e., $p \in \tilde{S}$ where $\tilde{S}$ is defined as in Lemma 2.2. That is $\partial_c u(\Omega) \subset \tilde{S}$ and the claim follows from Lemma 2.2.

Thus we have shown the existence of a generalized c-convex solution. The uniqueness follows from the comparison principle Theorem 5.1 and this completes the proof. □

Remark 6.5 *In case* $c(x) = \frac{1}{p}|x|^p$*, with* $1 < p < \infty$*, then the conclusions of Theorem 6.1 and Lemma 6.1 hold when* Ω *satisfies only condition* (6.1)*, in particular, this is satisfied when* Ω *is strictly convex by Remark 6.3. And consequently, for power cost functions Theorem 6.2, Theorem 6.3 and Corollary 6.1 below are also true when the domain* Ω *satisfies only condition* (6.1)*.*

PROOF: We notice that condition (6.2) is only used in Claim 3 to prove that for any $z \in \partial\Omega$ and any $\epsilon > 0$ we have $\limsup_{x\to z, x\in\Omega} u(x) \le \psi(z) + \epsilon$. Therefore the remark will be proved if we establish this using only condition (6.1). In fact, from Remark 6.1, Ω satisfies the exterior sphere condition and thus Ω is q-regular with q the conjugate of p. From [**BR02**, Theorem 4.7] there exists $w \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$ weak solution to the q-Laplacian

$$-\text{div}\ \left(|Dw(x)|^{q-2}Dw(x)\right) + n = 0, \quad \text{in } \Omega \text{ and } w = -\psi \text{ on } \partial\Omega.$$

Notice that div $(|Dw(x)|^{q-2}Dw(x)) =$ div $(Dc^*(Dw(x)))$. For each $f(x) = -c(x-y)-\lambda \in \mathcal{F}$, we have $-f(x) \ge -\psi(x)$ on $\partial\Omega$, and $-$div $(Dc^*(-Df(x)))+ n = 0$. Hence by the comparison principle [**BR02**, Theorem 3.1] for the q-Laplacian we get that $-f \ge w$ in $\bar{\Omega}$, and therefore $u(x) = \sup_{f\in\mathcal{F}} f(x) \le -w(x)$ for all $x \in \bar{\Omega}$. □

Remark 6.6 When ψ is a constant function, the proof above shows that Theorem 6.1 holds when Ω satisfies a condition weaker than c-strictly convex, namely: for any $z \in \partial\Omega$, there exist $y, y^* \in \mathbb{R}^n$ such that $c(x-y)-c(z-y) \ge 0$ and $c(z-y^*) - c(x-y^*) \ge 0$ for all x on $\partial\Omega$.

Remark 6.7 Suppose c satisfies condition $(H1)$. Then the argument in the proof of Claim 2 in fact shows that for any c-convex function $u \in C(\overline{\Omega})$ we have $\max_{\overline{\Omega}}[u+c] = \max_{\partial\Omega}[u+c]$.

6.2 Nonhomogeneous Dirichlet Problem

Throughout this section we assume that $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and is positive a.e. We begin with the following lemma.

Lemma 6.1 *Suppose that c satisfies condition $(H1)$, and $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi : \partial\Omega \to \mathbb{R}$ be a continuous function. Then for any set $E \Subset \Omega$ and for any number $\alpha \in \mathbb{R}$, there exists a c-convex function $u \in C(\overline{\Omega})$ satisfying $u = \psi$ on $\partial\Omega$ and $u \leq \alpha$ on E.*

PROOF: Define

$$\mathcal{F} = \{f(x) = -c(x-y) - \lambda : \ y \in \mathbb{R}^n, \lambda \in \mathbb{R}, \ f \leq \psi \text{ on } \partial\Omega \text{ and } f \leq \alpha \text{ on } E\}.$$

Since ψ is continuous on $\partial\Omega$ we have that $\mathcal{F}$ is nonempty. Let

$$u(x) = \sup_{f \in \mathcal{F}} f(x) \text{ for all } x \in \overline{\Omega}.$$

The proof now follows as in Theorem 6.1 with some obvious modifications in Claim 1. □

Theorem 6.2 *Suppose that c satisfies condition (H1), $\lim_{|x|\to\infty} \dfrac{c(x)}{|x|} = +\infty$, and $c(0) = \min_{x\in\mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, $\psi \in C(\partial\Omega)$, distinct points $x_1, \cdots, x_N \in \Omega$, and $a_1, \cdots, a_N$ positive numbers. If*

$$a_1 + \cdots + a_N < \int_{\mathbb{R}^n} g(y)\ dy, \tag{6.7}$$

then there exists a unique function $u \in C(\overline{\Omega})$, c-convex solution to the problem

$$\omega_c(g,u) = \sum_{i=1}^{N} a_i\,\delta_{x_i} \ \ in\ \Omega, \tag{6.8}$$

$$u = \psi \ \ on\ \partial\Omega.$$

PROOF: Let

$$\mathcal{H} = \{v \in C(\overline{\Omega}) : v \text{ is } c\text{-convex in } \Omega,\ v|_{\partial\Omega} = \psi,$$
$$\omega_c(g,v)(\Omega) = \sum_{i=1}^{N} \omega_c(g,v)(x_i), \text{ and } \int_{\partial_c v(x_i)} g(y)\ dy \leq a_i \text{ for } i = 1, \cdots, N\}.$$

From Theorem 6.1, let W be the solution to $\omega_c(g,W) = 0$ and $W = \psi$ on $\partial\Omega$. We have $W \in \mathcal{H}$, and it follows from definition (6.6) that

$$v \leq W, \text{ for each } v \in \mathcal{H}. \tag{6.9}$$

For each $v \in \mathcal{H}$ define

$$V[v] = \int_\Omega [W(x) - v(x)]\, dx \geq 0,$$

and let

$$\beta = \sup_{v \in \mathcal{H}} V[v].$$

We shall prove that there exists $u \in \mathcal{H}$ such that $\beta = V[u]$ and u is the desired solution to the problem (6.8).

Claim 1. *There exists $L \in \mathbb{R}$ such that*

$$L \leq v \text{ in } \Omega \text{ and for all } v \in \mathcal{H}. \tag{6.10}$$

Proof of claim 1. Applying Theorem 4.1 to $-v$ we get

$$\max_\Omega(-v) \leq \max_{\partial\Omega}(-v) + h^{-1}(\omega_c(g,v)(\Omega)),$$

where the function h is given in (4.13). Since $\omega_c(g,v)(\Omega) \leq a_1 + \cdots + a_N < B(g)$, and h^{-1} is increasing, we obtain that

$$\min_\Omega v \geq L := \min_{\partial\Omega} \psi - h^{-1}(a_1 + \cdots + a_N) > -\infty.$$

Claim 2. *There exists a c-convex function $w \in C(\bar{\Omega})$ with $w = \psi$ on $\partial\Omega$ and*

$$w(x) \leq v(x), \qquad \text{in } \bar{\Omega} \text{ and for all } v \in \mathcal{H}.$$

Proof of Claim 2.

Using Lemma 6.1 we can construct a c-convex function $w \in C(\bar{\Omega})$ such that $w = \psi$ on $\partial\Omega$, and $w \leq L$ on $\{x_1, \cdots, x_N\}$. Next let $v \in \mathcal{H}$ and define

$$\mathcal{G}_v = \{h \in C(\overline{\Omega}) : h \text{ is } c\text{-convex in } \Omega,\ h|_{\partial\Omega} \leq \psi,\ h(x_i) \leq v(x_i) \text{ for } i = 1, \cdots, N\},$$

and

$$\tilde{v}(x) = \sup_{h \in \mathcal{G}_v} h(x), \qquad x \in \bar{\Omega}.$$

Notice that $v \in \mathcal{G}_v$, and so $v \leq \tilde{v}$. Also $v(x_i) = \tilde{v}(x_i)$ for $1 \leq i \leq N$. We claim that $v = \tilde{v}$. We have $\tilde{v} = v$ on $\partial\Omega$. So if we prove that $\omega_c(g,v) \leq \omega_c(g,\tilde{v})$, the claim will follow from the comparison principle Theorem 5.1. In order to show that $\omega_c(g, v) \leq \omega_c(g, \tilde{v})$, and since the measure $\omega_c(g, v)$ is concentrated on $\{x_1, \cdots, x_N\}$, it is enough to prove that

$$\partial_c v(x_i) \subset \partial_c \tilde{v}(x_i), \quad i = 1, \cdots, N.$$

If $p \in \partial_c v(x_i)$, then $v(x) \geq v(x_i) - c(x-p) + c(x_i - p) =: h(x)$ for all $x \in \bar{\Omega}$. But $h \in \mathcal{G}_v$ so $h \leq \tilde{v}$ in $\bar{\Omega}$. Since $h(x_i) = v(x_i) = \tilde{v}(x_i)$, we then get $p \in \partial_c \tilde{v}(x_i)$ as desired. We now notice that from Claim 1, we have that $w(x_i) \leq L \leq v(x_i)$ for $1 \leq i \leq N$ and so $w \in \mathcal{G}_v$, and consequently $w \leq v$. This completes the proof of Claim 2.

Recall that

$$\beta = \sup_{v \in \mathcal{H}} V[v],$$

and from Claim 2, $\beta \leq V[w] < \infty$. Then there exists a sequence $\{u_n\} \subset \mathcal{H}$ such that $V[u_n] \uparrow \beta$ as $n \to \infty$. From (6.9) and Claim 2 we have that

$$w(x) \leq u_n(x) \leq W(x), \qquad \forall x \in \bar{\Omega}. \tag{6.11}$$

Claim 2A. *There is a subsequence $\{u_{n_k}\}$ and $u \in C(\bar{\Omega})$ with $u = \psi$ on $\partial\Omega$ such that $u_{n_k} \to u$ locally uniformly in Ω as $k \to \infty$. We denote this subsequence u_k.*

From (6.11), $\{u_n\}$ is uniformly bounded in $\bar{\Omega}$. Since u_n is c-convex in Ω, from Remark 2.2 we know that given $K \subset \Omega$ compact, u_n is Lipschitz in K, say with constant $C(K, n)$. We claim that $C(K, n)$ is uniformly bounded in n. Indeed, from the hypotheses on c we have that c satisfies condition $(H3)$ and therefore by Lemma 2.3 there exists $R > 0$ such that $\partial_c u_n(K) \subset B(0, R)$ for all $n = 1, 2, ...$ Choose the ball B which is large enough such that $z - p \in B$ for all $z \in K$ and $p \in B(0, R)$. Then for any $x, y \in K$, by choosing $p \in \partial_c u_n(y)$ and since c is convex on $\mathbb{R}^n$ we have

$$u_n(x) - u_n(y) \geq -c(x-p) + c(y-p) \geq -\|c\|_{Lip(B)}|x - y|.$$

Similarly, we also have $u_n(y) - u_n(x) \geq -\|c\|_{Lip(B)}|x-y|$. Thus $|u_n(x) - u_n(y)| \leq \|c\|_{Lip(B)}|x-y|$ for all $x, y \in K$, that is, $C(K,n) \leq \|c\|_{Lip(B)}$ for all n. This proves the claim. Therefore $\{u_n\}$ are equicontinuous on K and uniformly bounded in Ω. From (6.11) and since $w = W = \psi$ on $\partial\Omega$, by Arzela-Ascoli's lemma there exists a subsequence $\{u_{n_k}\}$ converging uniformly on compact subsets of Ω to a function $u \in C(\overline{\Omega})$ satisfying $u = \psi$ on $\partial\Omega$. Also u is c-convex by Lemma 2.1. This completes the proof of Claim 2A.

Claim 3. *$u \in \mathcal{H}$, and $V[u] = \sup_{v\in\mathcal{H}} V[v]$.*

It is enough to show $u \in \mathcal{H}$. We first see that $w_c(g, u_k) \to w_c(g,u)$ weakly in Ω. To prove this we use Lemma 3.1 and so we only have to check that condition (3.6) holds. Indeed, if $z_{k_j} \to z_0 \in \partial\Omega$, then from (6.11) we have

$$\begin{aligned}\limsup u_{k_j}(z_{k_j}) &\geq \limsup w(z_{k_j}) = w(z_0) = \psi(z_0) \\ &= \lim W(z_{k_j}) = \liminf W(z_{k_j}) \geq \liminf u(z_{k_j}).\end{aligned}$$

Second, since the measures $w_c(g, u_k)$ are concentrated on $\{x_1, \cdots, x_N\}$, it follows from Lemma 3.1 that

$$\int_{\partial_c u_k(x_i)} g(y)\, dy = w_c(g,u_k)(\{x_i\}) \to w_c(g,u)(\{x_i\}) = \int_{\partial_c u(x_i)} g(y)\, dy,$$

as $k \to \infty$ for $1 \leq i \leq N$. This implies that $\int_{\partial_c u(x_i)} g(y)\, dy \leq a_i$ for $1 \leq i \leq N$. Also the measure $\omega_c(g,u)$ is concentrated on $\{x_1, \cdots, x_N\}$ since it is the limit of measures concentrated on that set. Thus, we get Claim 3.

Claim 4. *The function u solves the nonhomogeneous Dirichlet problem (6.8).*

It is enough to show that $\int_{\partial_c u(x_i)} g(y)\, dy = a_i$ for $1 \leq i \leq N$. Suppose by contradiction this is not true. Then there exists $1 \leq i_0 \leq N$ such that $\int_{\partial_c u(x_{i_0})} g(y)\, dy < a_{i_0}$. By relabelling the indices we may assume that there exists a number $1 \leq \ell \leq N$ such that

$$\begin{aligned}\int_{\partial_c u(x_i)} g(y)\, dy &< a_i \qquad \text{for } 1 \leq i \leq \ell \text{ and} \\ \int_{\partial_c u(x_i)} g(y)\, dy &= a_i \qquad \text{for } \ell+1 \leq i \leq N.\end{aligned}$$

Given $n \in \mathbb{N}$ define

$$G_n = \{v \in C(\overline{\Omega}) : v \text{ is } c\text{-convex in } \Omega,\ v|_{\partial\Omega} \leq \psi,\ v(x_k) \leq u(x_k) - \frac{1}{n} \text{ for } 1 \leq k \leq \ell$$
$$\text{and } v(x_m) \leq u(x_m) \text{ for } \ell + 1 \leq m \leq N\}.$$

Since $u - \dfrac{1}{n} \in G_n$, we have $G_n \neq \emptyset$. Define $w_n(x) = \sup_{v \in G_n} v(x)$ for $x \in \bar{\Omega}$. We have

$$w_n(x_k) = u(x_k) - \frac{1}{n} \qquad \text{for } 1 \leq k \leq \ell \text{ and}$$
$$w_n(x_m) \leq u(x_m) \qquad \text{for } \ell + 1 \leq m \leq N.$$

We now claim that

$$\exists n_0 \text{ such that } \forall n \geq n_0,\ w_n(x_m) = u(x_m) \text{ for } \ell + 1 \leq m \leq N. \tag{6.12}$$

Indeed, by definition $\int_{\partial_c u(x_m)} g(y)\, dy = a_m > 0$ for $\ell + 1 \leq m \leq N$, and so $|\partial_c u(x_m)| > 0$ for $\ell + 1 \leq m \leq N$. Hence there exists $p_m \in \partial_c u(x_m)$ such that $f_m(x_k) < u(x_k)$ for $1 \leq k \leq \ell$, where $f_m(x) = u(x_m) - c(x - p_m) + c(x_m - p_m)$. Because if on the contrary for each $p \in \partial_c u(x_m)$ there exists x_k for some $1 \leq k \leq \ell$ and with $f_m(x_k) = u(x_k)$, then $p \in \partial_c u(x_k)$ and so

$$\partial_c u(x_m) \subset \{p \in \mathbb{R}^n : p \in \partial_c u(x) \cap \partial_c u(y) \text{ for some } x, y \in \Omega,\ x \neq y\}.$$

Hence by Corollary 2.1, it follows that $|\partial_c u(x_m)| = 0$, a contradiction. Therefore there exists n_0 such that

$$f_m(x_k) \leq u(x_k) - \frac{1}{n_0}$$

for all $\ell + 1 \leq m \leq N$ and $1 \leq k \leq \ell$, and so the last inequality holds for all $n \geq n_0$. Hence $f_m \in G_n$ for $\ell + 1 \leq m \leq N$ and for all $n \geq n_0$, and consequently, $w_n(x_m) \geq f_m(x_m) = u(x_m)$ and (6.12) is proved.

Claim 4A. *$w_n \in \mathcal{H}$ for sufficiently large n.*

Notice that constructing a sufficiently negative c-convex function $w(x)$ whose values are ψ on $\partial\Omega$ as in Claim 2, we have that $w \in G_n$ and so $w_n \in C(\bar{\Omega})$ with $w_n = \psi$ on $\partial\Omega$.

We first show that the measures $\omega_c(g, w_n)$ are concentrated on $\{x_1, \cdots x_N\}$. Let $B \subset \Omega$ be an open ball such that $B \cap \{x_1, \cdots x_N\} = \emptyset$. We claim that $|\partial_c w_n(B)| = 0$. Let $p \in \partial_c w_n(B)$. Then there exists $z \in B$ such that $w_n(x) \geq w_n(z) - c(x - p) + c(z - p) =: f(x)$ for all $x \in \bar{\Omega}$. We claim that there exists $y \in \bar{\Omega} \setminus B$ such that $f(y) = w_n(y)$. Suppose by contradiction this is not true, so $f(x) < w_n(x)$ for all $x \in \bar{\Omega} \setminus B$. Then $f(x) + \epsilon < w_n(x)$ for all $x \in \bar{\Omega} \setminus B$ for some $\epsilon > 0$ sufficiently small. Notice that $f(x) + \epsilon$ is c-convex in Ω and therefore $f + \epsilon \in G_n$. Hence $f(z) + \epsilon \leq w_n(z) = f(z)$, a contradiction. It then follows from the claim that $p \in \partial_c(w_n, \bar{\Omega})(y)$ and then

$$\partial_c w_n(B) \subset \{p \in \mathbb{R}^n : p \in \partial_c(w_n, \bar{\Omega})(x) \cap \partial_c(w_n, \bar{\Omega})(y) \text{ for some } x, y \in \bar{\Omega},\ x \neq y\}.$$

From Lemma 2.2 we then conclude that $|\partial_c w_n(B)| = 0$.

We have

$$u(x) - \frac{1}{n} \leq w_n(x) \leq u(x), \qquad \text{for } x \in \bar{\Omega} \tag{6.13}$$

where the first inequality holds because $u - \frac{1}{n} \in G_n$ and the second holds because $G_n \subset \mathcal{G}_u$. Consequently, $-\frac{1}{n} \leq w_n(x) - u(x) \leq 0$ in $\bar{\Omega}$ so $w_n \to u$ uniformly in $\bar{\Omega}$ and therefore $w_c(g, w_n) \to w_c(g, u)$ weakly, and since $w_c(g, w_n)$ are concentrated on $\{x_1, \cdots, x_N\}$ we get from Lemma 3.1 that

$$\int_{\partial_c w_n(x_k)} g(y)\, dy \to \int_{\partial_c u(x_k)} g(y)\, dy < a_k$$

for $1 \leq k \leq \ell$, and so there exists $n_1 \geq n_0$ such that for all $n \geq n_1$ we get $\int_{\partial_c w_n(x_k)} g(y)\, dy < a_k$ for $1 \leq k \leq \ell$. Now let $n \geq n_1$, we shall show that $w_n \in \mathcal{H}$. Indeed, from (6.12) and (6.13) we have that

$$\partial_c w_n(x_m) \subset \partial_c u(x_m), \qquad \text{for } \ell + 1 \leq m \leq N,$$

and so

$$\int_{\partial_c w_n(x_m)} g(y)\, dy \leq \int_{\partial_c u(x_m)} g(y)\, dy = a_m, \qquad \text{for } \ell + 1 \leq m \leq N.$$

Thus, Claim 4A is proved.

Finally, from (6.13) and since $u(x_k) - w_{n_1}(x_k) = \dfrac{1}{n_1}$ for $1 \leq k \leq \ell$ we then get that $V[w_{n_1}] = \int_\Omega [W(x) - w_{n_1}(x)]\, dx = V[u] + \int_\Omega [u(x) - w_{n_1}(x)]\, dx > V[u]$, a contradiction by Claim 4A. This completes the proof of Claim 4 and hence the theorem is proved since the uniqueness follows from Theorem 5.1. □

6.3 Nonhomogeneous Dirichlet Problem with general right hand side

Theorem 6.3 *Suppose that c satisfies condition (H1), $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$, $c(0) = \min_{x\in\mathbb{R}^n} c(x)$, and $g \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi \in C(\partial\Omega)$. Suppose that μ is a Borel measure in Ω satisfying spt(μ) $\subset \Omega$ and*

$$\mu(\Omega) < \int_{\mathbb{R}^n} g(y)\, dy. \tag{6.14}$$

Then there exists $u \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g, u) = \mu$ in Ω and $u = \psi$ on $\partial\Omega$. Moreover, the solution is unique if in addition μ satisfies

$$\text{for every open set} \quad D \Subset \Omega, S \cap D \ \text{ is a closed set}, \tag{6.15}$$

where $S := spt(\mu) \setminus \overline{\mathrm{Int}(spt(\mu))}$.

Proof: First fix a subdomain Ω' of Ω such that $\mathrm{spt}(\mu) \subset \Omega' \Subset \Omega$. By the assumptions we can select a sequence of measures $\{\mu_j\}$ satisfying $\mu_j \rightharpoonup \mu$ weakly in Ω, each μ_j is a finite combination of delta masses with $\mathrm{spt}(\mu_j) \subset \Omega'$ and $\{\mu_j(\Omega)\}$ is uniformly bounded by a positive constant A which is strictly less than $\int_{\mathbb{R}^n} g(y)\, dy$. Hence for each j, by Theorem 6.2 there exists a unique c-convex weak solution $u_j \in C(\overline{\Omega})$ to the problem $\omega_c(g, u_j) = \mu_j$ in Ω and

$u_j = \psi$ on $\partial\Omega$. Then by using the maximum principle Theorem 4.1, we have

$$
\begin{aligned}
-\min_{\Omega} u_j &= \max_{\Omega}(-u_j) \le \max_{\partial\Omega}(-u_j) + h^{-1}(\omega_c(g,u_j)(\Omega)) \\
&= -\min_{\partial\Omega}\psi + h^{-1}(\mu_j(\Omega)) \le -\min_{\partial\Omega}\psi + h^{-1}(A) =: C < +\infty.
\end{aligned}
$$

That is,

$$-C \le \min_{\Omega} u_j$$

for all j. Moreover, by Lemma 6.1 we can find a c-convex function $w \in C(\overline{\Omega})$ satisfying $w = \psi$ on $\partial\Omega$ and $w \le -C$ on Ω'. From Theorem 6.1, let $v \in C(\overline{\Omega})$ be the unique c-convex weak solution to the problem $\omega_c(v) = 0$ in Ω and $v = \psi$ on $\partial\Omega$. Then we have

$$w(x) \le u_j(x) \le v(x) \quad \forall x \in \overline{\Omega} \tag{6.16}$$

for every j, where the first inequality is proved using the argument in the proof of Claim 2, Theorem 6.2, and the second follows from Theorem 5.1. It can be showed easily that $\{u_j\}$ has a subsequence converging locally uniformly to some function u in Ω. Then by (6.16) we have $w \le u \le v$. Therefore $u \in C(\overline{\Omega})$ with $u = \psi$ on $\partial\Omega$. Hence by Lemma 2.1 and Corollary 3.1 we obtain that u is a weak solution to the Dirichlet problem. The uniqueness follows from the comparison principle Theorem 5.1. □

In the last theorem, the assumption that the measure μ has compact support in Ω can be substituted by the existence of a c-convex subsolution to the equation with the given boundary condition. Indeed, we have the following.

Corollary 6.1 *Suppose that c satisfies condition (H1),* $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$, $c(0) = \min_{x\in\mathbb{R}^n} c(x)$*, and $g \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi \in C(\partial\Omega)$. Suppose that μ is a finite Borel measure in Ω satisfying* (6.15) *and there exists a c-convex function $w \in C(\overline{\Omega})$ such that $\omega_c(g,w) \ge \mu$ in Ω and $w = \psi$ on $\partial\Omega$. Then there exists a unique $u \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g,u) = \mu$ in Ω and $u = \psi$ on $\partial\Omega$.*

Proof: Let $S = \mathrm{spt}(\mu) \setminus \overline{\mathrm{Int}(\mathrm{spt}(\mu))}$ and for each positive integer j denote $\Omega_j = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 1/j\}$. Define

$$K_j = (\Omega_j \cap S) \cup \overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}$$

which is a subset of $\mathrm{spt}(\mu)$, and $\mu_j(E) = \dfrac{j}{j+1}\mu(E \cap K_j)$ for every Borel set $E \subset \Omega$. Then since μ satisfies condition (6.15) we have K_j is a closed set and hence $\mathrm{spt}(\mu_j) = \mathrm{spt}(\mu|_{K_j}) = K_j$. Moreover, as S does not contain any interior point and the sets $\Omega_j \cap S$ and $\overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}$ are disjoint, we get $\mathrm{Int}(\mathrm{spt}(\mu_j)) = \mathrm{Int}(K_j) = \mathrm{Int}(\overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}) = \Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))$. Therefore, if for each j we let $S_j = \mathrm{spt}(\mu_j) \setminus \overline{\mathrm{Int}(\mathrm{spt}(\mu_j))}$, then we obtain $S_j = \Omega_j \cap S$. This implies that condition (6.15) is satisfied with S replaced by S_j. We note also that $\mu_j \rightharpoonup \mu$ weakly in Ω since for each compact set $K \subset \Omega$ we have $\mu_j(K) \to \mu(K)$ because

$$\begin{aligned}\lim_{j\to\infty} \mu(K \cap K_j) &= \lim_{j\to\infty} \mu((K \cap \Omega_j \cap S) \cup (K \cap \overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}))\\ &= \mu((K \cap S) \cup (K \cap \overline{\mathrm{Int}(\mathrm{spt}(\mu))})) = \mu(K \cap \mathrm{spt}(\mu)) = \mu(K).\end{aligned}$$

Now since $\mu_j(\Omega) \leq \dfrac{j}{j+1}\mu(\Omega)$ and $\mu(\Omega) \leq \omega_c(g, w)(\Omega) \leq \int_{\mathbb{R}^n} g$, we get $\mu_j(\Omega) < \int_{\mathbb{R}^n} g$ as μ is a finite measure. Thus, by Theorem 6.3 for each j there exists $u_j \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g, u_j) = \mu_j$ in Ω and $u_j = \psi$ on $\partial\Omega$. Hence as $\omega_c(g, u_j) = \mu_j \leq \mu \leq \omega_c(g, w)$ and the measures μ_j satisfy condition (6.15), by applying the comparison principle Theorem 5.1 we obtain

$$w(x) \leq u_j(x) \leq v(x) \quad \forall x \in \overline{\Omega} \tag{6.17}$$

for every j, where $v \in C(\bar{\Omega})$ is the c-convex weak solution to the homogeneous Dirichlet problem. Then by following the argument in the proof of Theorem 6.3 we get the desired result. □

In general to find a c-convex subsolution is a nontrivial task. However, in the following by using Bakelman's result on the existence of a convex weak solution to the R-curvature problem we shall show that c-convex subsolutions do exist in a number of important cases. In order to do that we need to recall some concepts introduced by Bakelman. Suppose $R \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set. For each convex function u on Ω, define the measure

$$\omega(R,u,E) = \int_{\partial u(E)} R(y)dy \quad \text{for all Borel sets } E \subset \Omega,$$

where ∂u denotes the standard subdifferential. We then have the following relation which in particular says that when g is bounded from below by a positive constant c_0 then a convex subsolution to the R-curvature problem with $R(y) = c_0 \det D^2 c^*(-y)$ is indeed a c-convex subsolution to our problem.

Lemma 6.2 *Suppose c satisfies condition $(H2)$ and $c^* \in C^2(\mathbb{R}^n \setminus \{z_0\})$ for some z_0 in $\mathbb{R}^n$, and $g(y) \geq c_0 > 0$ for a.e. y. Let $R(y) = c_0 \det D^2 c^*(-y)$ a.e. on $\mathbb{R}^n$ and assume that R is positive a.e., and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for any convex function $u \in C(\Omega)$, we have*

$$\omega_c(g,u)(E) \geq c_0|E| + \omega(R,u,E) \quad \textit{for all Borel sets } E \subset \Omega.$$

Proof: Observing that since c^* is convex, we have $R \in L^1_{loc}(\mathbb{R}^n)$ (see for example [**McC97**, Corollary 4.3]. Also, as $\omega_c(g,u) \geq c_0\omega_c(u)$, it is enough to show the above estimate with the left hand side is $\omega_c(u)$ and with $c_0 = 1$. Assume first that u is a convex function satisfying $u \in C^2(\Omega)$ and $D^2u(x) > 0$ in Ω. Let $s(x) = x - Dc^*(-Du(x))$ for x in Ω. Define $K_u = \{x \in \Omega : Du(x) = -z_0\}$, which is relatively closed in Ω and let $B = \{x \in \Omega \setminus K_u : \det Ds(x) = 0\}$. Then by the assumptions we have $s \in C^1(\Omega \setminus K_u)$ and B is relatively closed in the open set $\Omega \setminus K_u$. Now let E be an arbitrary Borel set in Ω. We claim that

$$\omega_c(u)(E \setminus K_u) \geq \int_{E \setminus K_u} |\det(I + D^2c^*(-Du)D^2u)|\, dx. \tag{6.18}$$

Indeed, for any open set U with $E \setminus K_u \subset U \subset \Omega$, let $V = U \setminus K_u$. Then V is open and $E \setminus K_u \subset V \subset \Omega$. Also $V \setminus B$ is open since $V \setminus B = V \cap ((\Omega \setminus K_u) \setminus B)$. We can write the open set $V \setminus B$ as $V \setminus B = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ are cubes with disjoint interior and sides parallel to the coordinate axes. We can choose

C_i are small enough so that $s : C_i \to s(C_i)$ is a diffeomorphism. We therefore have

$$\int_{E\setminus K_u} |\det(I + D^2c^*(-Du)D^2u)|\ dx \le \int_V |\det Ds(x)|\ dx$$
$$= \int_{V\setminus B} |\det Ds(x)|\ dx = \int_{\cup_{i=1}^{\infty} C_i} |\det Ds(x)|\ dx = \int_{\cup_{i=1}^{\infty} \mathring{C}_i} |\det Ds(x)|\ dx$$
$$= \sum_{i=1}^{\infty} \int_{\mathring{C}_i} |\det Ds(x)|\ dx = \sum_{i=1}^{\infty} \int_{s(\mathring{C}_i)} dy = \sum_{i=1}^{\infty} |\partial_c u(\mathring{C}_i)| \text{ by Proposition 2.3(2)}$$
$$= \omega_c(u)(\cup_{i=1}^{\infty} \mathring{C}_i) \le \omega_c(u)(V \setminus B) \le \omega_c(u)(U).$$

Since c satisfies condition $(H2)$, the measure $\omega_c(u)$ is regular. Hence, we deduce from the last inequality that (6.18) holds. Next, as $\partial u(E \cap K_u) = \{Du(x) : x \in E \cap K_u\} \subset \{-z_0\}$ we have $|\partial u(E \cap K_u)| = 0$. Hence,

$$\omega(R, u, E \cap K_u) = 0. \tag{6.19}$$

We also note that since u is convex and $u \in C^2(\Omega)$ we get from Proposition 2.3(2) that $\partial_c u(E \cap K_u) = \{x - Dc^*(-Du(x)) : x \in E \cap K_u\} = \{x - Dc^*(z_0) : x \in E \cap K_u\}$. Thus,

$$\omega_c(u)(E \cap K_u) = |E \cap K_u|. \tag{6.20}$$

Since $D^2c^*(y)$ is symmetric nonnegative definite for every y in $\mathbb{R}^n - \{z_0\}$ and $D^2u(x)$ is symmetric positive definite for all x in Ω, we have $D^2c^*(y)D^2u(x)$ is diagonalizable with nonnegative eigenvalues for every such x and y. But then by diagonalizing, it is easy to see that $\det(I + D^2c^*(-Du(x))D^2u(x)) \ge 1 + \det(D^2c^*(-Du(x))D^2u(x))$ for all x in $\Omega \setminus K_u$. Combining this with (6.18), (6.19) and (6.20) we obtain

$$\begin{aligned}
\omega_c(u)(E) &= \omega_c(u)(E \cap K_u) + \omega_c(u)(E \setminus K_u) \\
&\geq |E \cap K_u| + \int_{E\setminus K_u} |\det(I + D^2c^*(-Du)D^2u)|dx \\
&\geq |E \cap K_u| + \int_{E\setminus K_u} [1 + \det D^2c^*(-Du)\ \det D^2u]\ dx \\
&= |E| + \int_{E\setminus K_u} R(Du(x))\ \det D^2u(x)\ dx = |E| + \omega(R, u, E \setminus K_u) \\
&= |E| + \omega(R, u, E).
\end{aligned}$$

So the lemma holds for any convex function $u \in C^2(\Omega)$ satisfying $D^2u > 0$ in Ω. For the general case, choose a sequence of convex functions $\{v_m\} \subset C^2(\Omega)$ such that $v_m \to u$ locally uniformly in Ω. Define $u_m(x) = v_m(x) + \frac{1}{m}|x|^2$. Then $D^2u_m > 0$ in Ω for every m and $\{u_m\}$ still share the above properties of $\{v_m\}$. Hence, by applying the previous result we get $\omega_c(u_m) \geq |\cdot| + \omega(R, u_m, .)$ in Ω for all m. But as $\{u_m\}$ are convex and $u_m \to u$ locally uniformly we have $\omega_c(u_m)$, $\omega(R, u_m, .)$ converge weakly to $\omega_c(u)$ and $\omega(R, u, .)$ respectively (see Remark 3.2). Therefore, by passing to the limit we get the desired result. □

We will need the following proposition which is a simple extension of Aleksandrov maximum principle (see [**Gut01**, Theorem 1.4.2]). Note that here we only need $u \geq 0$ on $\partial\Omega$ instead of $u = 0$ on $\partial\Omega$ as in Aleksandrov's result.

Proposition 6.1 *Suppose $R(y) \geq c_1|y|^{-2k}$ for a.e. y in $\mathbb{R}^n$ with $k < \frac{1}{2}$. Let Ω be a bounded convex open set in $\mathbb{R}^n$, and $u \in C(\overline{\Omega})$ a convex function with $u \geq 0$ on $\partial\Omega$. We have the following:*

(i) If $k \leq 0$, then

$$u(x) \geq -\left[\frac{1}{c_1 c(n,k)} diam(\Omega)^{n-2k-1} dist(x, \partial\Omega)\, \omega(R, u, \Omega)\right]^{1/(n-2k)} \quad \forall x \in \Omega.$$

(ii) If $0 \leq k < \frac{1}{2}$, then

$$u(x) \geq -\left[\frac{1}{c_1 c(n)} diam(\Omega)^{n-1} dist(x, \partial\Omega)^{1-2k}\, \omega(R, u, \Omega)\right]^{1/(n-2k)} \quad \forall x \in \Omega.$$

Proof: Let $x_0 \in \Omega$ be such that $u(x_0) < 0$ and let $\mathcal{F} = \{v \in C(\bar{\Omega}) : v \text{ is convex}, v \leq u \text{ on } \partial\Omega, \text{and } v(x_0) \leq u(x_0)\}$. Then $\mathcal{F} \neq \emptyset$ as $u \in \mathcal{F}$. Define

$$w(x) = \sup_{v \in \mathcal{F}} v(x) \qquad \text{for } x \in \bar{\Omega}.$$

Since Ω is convex, there exists $h \in C(\bar{\Omega})$ which is harmonic in Ω and $h = u$ on $\partial\Omega$. Then by writing w as a supremum of affine functions it is easy to see that $u(x) \leq w(x) \leq h(x)$ in $\bar{\Omega}$, and hence the convex function w is in $C(\bar{\Omega})$ with $w = u$ on $\partial\Omega$ and $w(x_0) = u(x_0)$. This implies that $\partial w(\Omega) \subset \partial u(\Omega)$ from [**Gut01**, Lemma 1.4.1]. On the other hand, by the definitions of subdifferential and the function w we have

$$\begin{aligned} \partial w(x_0) &= \{p \in \mathbb{R}^n : w(x_0) + p \cdot (x - x_0) \leq u(x) \text{ on } \partial\Omega\} \\ &\supset \{p \in \mathbb{R}^n : u(x_0) + p \cdot (x - x_0) \leq 0 \text{ on } \partial\Omega\} = \partial v(x_0), \end{aligned}$$

where v is the convex function whose graph is the upside down cone with vertex $(x_0, u(x_0))$ and base Ω, with $v = 0$ on $\partial\Omega$. Therefore, $\partial v(x_0) \subset \partial u(\Omega)$. Moreover, by the proof in [**Gut01**, Theorem 1.4.2] there exists $p_0 \in \mathbb{R}^n$ with $|p_0| = \dfrac{-u(x_0)}{\operatorname{dist}(x_0, \partial\Omega)}$ such that the convex hull K of the n-dimensional ball $B\left(0, \frac{-u(x_0)}{\operatorname{diam}(\Omega)}\right)$ and p_0 is contained in $\partial v(x_0)$. Consequently,

$$\omega(R, u, \Omega) = \int_{\partial u(\Omega)} R(y)\, dy \geq c_1 \int_K |y|^{-2k}\, dy \geq c_1 \int_{H(t,\alpha,\beta)} |y|^{-2k}\, dy, \quad (6.21)$$

where $t = -u(x_0) > 0$, $\alpha = \operatorname{dist}(x_0, \partial\Omega)$, $\beta = \operatorname{diam}(\Omega)$, and $H(t, \alpha, \beta)$ denotes the convex cone in $\mathbb{R}^n$ with vertex at $(0, ..., 0, t/\alpha)$ and with base the ball $B(0, t/\beta) \subset \mathbb{R}^{n-1}$, the orthogonal hyperplane to the x_n-axis at the origin. Noticing that in the last inequality above we have used the fact that the function $|y|^{-2k}$ is radial. If $0 \leq k < \frac{1}{2}$, then we have

$$\begin{aligned} &\int_{H(t,\alpha,\beta)} |y|^{-2k}\, dy \geq \left(\frac{t}{\alpha}\right)^{-2k} |H(t, \alpha, \beta)| \qquad (6.22) \\ &= \omega_{n-2} \left(\frac{t}{\alpha}\right)^{-2k} \int_0^{\frac{t}{\alpha}} \left[\int_0^{\frac{t-\alpha z}{\beta}} h^{n-2}\, dh\right] dz \\ &= \frac{\omega_{n-2}}{n-1} \left(\frac{t}{\alpha}\right)^{-2k} \int_0^{\frac{t}{\alpha}} \left(\frac{t - \alpha z}{\beta}\right)^{n-1} dz = \frac{\omega_{n-2}}{n(n-1)} \frac{t^{n-2k}}{\alpha^{1-2k} \beta^{n-1}}. \end{aligned}$$

If on the other hand $k \leq 0$, then

$$\int_{H(t,\alpha,\beta)} |y|^{-2k}\, dy = \omega_{n-2} \int_0^{\frac{t}{\alpha}} \left[\int_0^{\frac{t-\alpha z}{\beta}} (h^2+z^2)^{-k} h^{n-2}\, dh \right] dz \tag{6.23}$$

$$\geq \omega_{n-2} \int_0^{\frac{t}{\alpha}} \left[\int_{\frac{t-\alpha z}{2\beta}}^{\frac{t-\alpha z}{\beta}} (h^2+z^2)^{-k} h^{n-3} h\, dh \right] dz$$

$$\geq \omega_{n-2} \min\{2^{n-3}, 1\} \int_0^{\frac{t}{\alpha}} \left(\frac{t-\alpha z}{2\beta} \right)^{n-3} \left[\int_{\frac{t-\alpha z}{2\beta}}^{\frac{t-\alpha z}{\beta}} (h^2+z^2)^{-k} h\, dh \right] dz$$

$$= \frac{c(n)}{2} \int_0^{\frac{t}{\alpha}} \left(\frac{t-\alpha z}{2\beta} \right)^{n-3} \left[\int_{z^2+(\frac{t-\alpha z}{2\beta})^2}^{z^2+(\frac{t-\alpha z}{\beta})^2} s^{-k}\, ds \right] dz$$

$$= \frac{c(n)}{2(1-k)} \int_0^{\frac{t}{\alpha}} \left(\frac{t-\alpha z}{2\beta} \right)^{n-3} \left\{ \left[z^2 + \left(\frac{t-\alpha z}{\beta} \right)^2 \right]^{1-k} - \left[z^2 + \left(\frac{t-\alpha z}{2\beta} \right)^2 \right]^{1-k} \right\} dz$$

$$\geq \frac{c(n)3^{1-k}}{2(1-k)} \int_0^{\frac{t}{\alpha}} \left(\frac{t-\alpha z}{2\beta} \right)^{n-2k-1} dz = \frac{c(n)\, 3^{1-k}}{2^{n-2k}(1-k)(n-2k)} \frac{t^{n-2k}}{\alpha\, \beta^{n-2k-1}}.$$

From (6.21), (6.22), (6.23) and the definitions of t, α, and β we derive the desired results. □

We next recall the notion of local parabolic support due to Bakelman which describes more precisely the geometry of domains which are between being strictly convex and satisfying the enclosing sphere condition. For a bounded open convex set $\Omega \subset \mathbb{R}^n$, let $z \in \partial\Omega$. Then there exist a supporting hyperplane α to $\bar{\Omega}$ at z and an open ball $B_R(z)$ such that the convex $(n-1)$-surface $\partial\Omega \cap B_R(z)$ has a one-to-one orthogonal projection $\Pi_\alpha : \partial\Omega \cap B_R(z) \longrightarrow \alpha$. Moreover, the unit normal $\vec{n}$ to α in the direction of the halfspace where $\bar{\Omega}$ lies passes through interior points of Ω. Denote by $\mathcal{S}_R(z)$ the set $\Pi_\alpha(\partial\Omega \cap B_R(z))$. Let $\xi_1, ..., \xi_{n-1}, \xi_n$ be the Cartesian coordinates introduced in the following way: z is the origin, the axes $\xi_1, ..., \xi_{n-1}$ lie in the plane α, and the axis ξ_n is directed along the interior normal $\vec{n}$ to $\partial\Omega$ at the point z. Clearly, the convex surface $\partial\Omega \cap B_R(z)$ is the graph of some convex function $\varphi(\xi_1, ..., \xi_{n-1})$ defined on $\mathcal{S}_R(z)$. Obviously,

$$\varphi(0, ..., 0) = 0 \text{ and } \varphi(\xi_1, ..., \xi_{n-1}) \geq 0 \qquad \forall (\xi_1, ..., \xi_{n-1}) \in \mathcal{S}_R(z).$$

We will say that $\partial\Omega$ has a parabolic support of order $\tau \geq 0$ at the point $z \in \partial\Omega$, if there exists $r_z \in (0, R)$ and $b(z) > 0$ such that

$$\varphi(\xi_1, ..., \xi_{n-1}) \geq b(z)(\xi_1^2 + ... + \xi_{n-1}^2)^{\frac{\tau+2}{2}} \qquad \forall(\xi_1, ..., \xi_{n-1}) \in \mathcal{S}_{r_z}(z),$$

i.e., the convex $(n-1)$-surface $\partial\Omega \cap B_{r_z}(z)$ can be touched from outside by the $(n-1)$-dimensional paraboloid $\xi_n = b(z)(\xi_1^2 + ... + \xi_{n-1}^2)^{\frac{\tau+2}{2}}$ of order $\frac{\tau+2}{2}$ at z.

Definition 6.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set and $\tau \geq 0$. We say $\partial\Omega$ has a parabolic support of order not more than τ if at every boundary point z of Ω, $\partial\Omega$ has a parabolic support of some order $\tau_z \in [0, \tau]$.*

Note that the larger τ is, the less requirement we put on Ω. Also, $\partial\Omega$ has a parabolic support of order $\tau = 0$ iff Ω satisfies the enclosing sphere condition. We will also consider the following condition for the Borel measure μ:

(G) There exist $a > 0$ and $\epsilon > 0$ such that

$$\mu(E) \leq a|E| \text{ for all Borel sets } E \subset \Omega - \overline{\Omega_\epsilon}, \tag{6.24}$$

where $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$. If μ is absolutely continuous with respect to the Lebesgue measure and with density $f(x)$, then by the Lebesgue differentiation theorem (6.24) is equivalent to

$$f(x) \leq a \text{ for a.e. } x \in \Omega - \overline{\Omega_\epsilon}. \tag{6.25}$$

We can now state the following theorem whose second part is due to Bakelman (see [**Bak94**, Theorem 11.4]).

Theorem 6.4 *Let $c(x) = \dfrac{|x|^p}{p}$ with $1 < p < \infty$, and $R(y) = c_0 \det D^2 c^*(-y)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open and strictly convex set, $\psi \in C(\partial\Omega)$, and μ is a finite Borel measure in Ω. Consider the Dirichlet problem*

$$\omega(R, u, .) = \mu \text{ in } \Omega, \text{ and } u = \psi \text{ on } \partial\Omega. \tag{6.26}$$

We have

(i) If $p < 2 + \dfrac{1}{n-1}$, *then* (6.26) *has a convex weak solution* $u \in C(\bar{\Omega})$.

(ii) If $2 + \dfrac{1}{n-1} \leq p$, *and in addition* μ *satisfies the assumption* (G) *and* $\partial\Omega$ *has a parabolic support of order no more than* τ *for some nonnegative* τ *satisfying* $\dfrac{n}{2}\dfrac{p-2}{p-1} < \dfrac{n+\tau+1}{\tau+2}$, *then* (6.26) *has a convex weak solution* $u \in C(\bar{\Omega})$.

Proof: First observe that $R(y) = c_0(q-1)|y|^{n(q-2)} = c_0(q-1)|y|^{-2\frac{n(2-q)}{2}}$, where $q > 1$ is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Hence,

$$\int_{\mathbb{R}^n} R(y)\, dy = c_0(q-1) \int_{\mathbb{R}^n} |y|^{n(q-2)}\, dy = +\infty.$$

(i) Since μ is a finite Borel measure, there exists a sequence of measures μ_j converging weakly to μ such that each μ_j is a finite combination of delta masses with positive coefficients and $\{\mu_j(\Omega)\}$ is bounded by some positive constant B. For each j, by using [**Bak86**, Theorem 2] we can find $u_j \in C(\bar{\Omega})$ which is the convex weak solution to $\omega(R, u_j, .) = \mu_j$ in Ω and $u_j = \psi$ on $\partial\Omega$. If we let $W \in C(\bar{\Omega})$ be the convex weak solution to $\det D^2 W = 0$ in Ω and $W = \psi$ on $\partial\Omega$, then $u_j \leq W$ on $\bar{\Omega}$. We now prove that $\{u_j\}$ is also uniformly bounded from below in Ω. Indeed, let $\xi \in \partial\Omega$ and $\varepsilon > 0$. There exists $\delta > 0$ such that $|\psi(x) - \psi(\xi)| < \varepsilon$ for $|x - \xi| < \delta$, $x \in \partial\Omega$. Let $A \cdot x + b = 0$ be the equation of the supporting hyperplane to Ω at ξ and assume that $\Omega \subset \{x : l(x) \geq 0\}$, where $l(x) := A \cdot x + b$. Since Ω is strictly convex, there is $\eta > 0$ such that $\{x \in \bar{\Omega} : l(x) \leq \eta\} \subset B_\delta(\xi)$. Let $M = \min\{\psi(x) : x \in \partial\Omega, l(x) \geq \eta\}$ and consider the function $a(x) = \psi(\xi) - \varepsilon - \kappa l(x)$, where κ is a constant satisfying $\kappa \geq \max\{\frac{\psi(\xi)-\varepsilon-M}{\eta}, 0\}$. Hence $a(\xi) = \psi(\xi) - \varepsilon$ and it is easy to see that $a(x) \leq \psi(x)$ on $\partial\Omega$. Set $v_j(x) = u_j(x) - a(x)$. Then $v_j \in C(\bar{\Omega})$ is convex and $v_j \geq 0$ on $\partial\Omega$. By using Proposition 6.1 and noting $2 - q < 1/n$ we see that there exists $\alpha > 0$ depending only on q such that

$$v_j(x) \geq -\left[C(n, q, \Omega)\operatorname{dist}(x, \partial\Omega)^\alpha \omega(R, v_j, \Omega)\right]^{1/n(q-1)} \text{ on } \Omega.$$

But we have by the definition of v_j

$$
\begin{aligned}
\omega(R, v_j, \Omega) &= c_0(q-1)\int_{\partial u_j(\Omega)} |y+\kappa A|^{n(q-2)}\,dy \\
&= c_0(q-1)\int_{\partial u_j(\Omega)\cap\{|y|\geq 2\kappa|A|\}} |y+\kappa A|^{n(q-2)}\,dy \\
&+ c_0(q-1)\int_{\partial u_j(\Omega)\cap\{|y|\leq 2\kappa|A|\}} |y+\kappa A|^{n(q-2)}\,dy \\
&\leq \max\{(\frac{3}{2})^{n(q-2)}, (\frac{1}{2})^{n(q-2)}\} c_0(q-1)\int_{\partial u_j(\Omega)} |y|^{n(q-2)}\,dy \\
&+ c_0(q-1)\int_{B(0,3\kappa|A|)} |z|^{n(q-2)}\,dz \\
&= \max\{(\frac{3}{2})^{n(q-2)}, (\frac{1}{2})^{n(q-2)}\}\omega(R, u_j, \Omega) + C(n,q)(\kappa|A|)^{n(q-1)} \\
&\leq C(n, q, \kappa, |A|, B).
\end{aligned}
$$

Therefore, we obtain

$$W(x) \geq u_j(x) \geq a(x) - C(n, q, \kappa, |A|, B, \Omega)\text{dist}(x, \partial\Omega)^{\alpha/n(q-1)} \text{ on } \Omega. \quad (6.27)$$

This shows that $\{u_j\}$ is uniformly bounded on Ω and hence we can extract a subsequence still denoted by $\{u_j\}$ which converges locally uniformly on Ω to some convex function $u \in C(\Omega)$. This gives $\omega(R, u, .) = \mu$ on Ω. Moreover, (6.27) also implies that for every $\xi \in \partial\Omega$ we have $\lim_{\Omega\ni x\to\xi} u(x) = \psi(\xi)$. Thus the proof of (i) is completed. On the other hand, (ii) follows from [**Bak86**, Theorem 6] or [**Bak94**, Theorem 11.4] with $\lambda = 0$. □

Since convex functions are c-convex, Lemma 6.2 together with Theorem 6.4 provides the c-convex subsolution w needed in Corollary 6.1 when g is bounded from below by some positive constant c_0 and the cost function $c(x) \approx \frac{1}{p}|x|^p$, $1 < p < \infty$. We remark that in these cases in fact we have $\omega_c(g, w)(E) \geq c_0|E| + \mu(E)$. Therefore, we do not really need the technical assumption (6.15) in Corollary 6.1 to ensure the existence of a c-convex weak solution. The reason is that in that proof one can simply take $\mu_j(E) = \mu(E\cap\Omega_j) + \dfrac{c_0}{j}|E|$. We then also have $\mu_j \rightharpoonup \mu$ weakly, $\mu_j \leq \omega_c(g, w)$, $\mu_j(\Omega) < \omega_c(g, w)(\Omega) \leq \int_{\mathbb{R}^n} g$, and $\text{spt}(\mu_j) = \bar{\Omega}$. The last fact allows us to use Theorem 5.1 to obtain (6.17).

CHAPTER 7

Second Boundary Value Problems

7.1 Aleksandrov Solution ⇒ Brenier Solution

Let $c : \mathbb{R}^n \to \mathbb{R}$ be a C^1 strictly convex function satisfying $\lim_{|x|\to+\infty} \frac{c(x)}{|x|} = +\infty$. Particularly, these imply that $Dc : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism with $(Dc)^{-1} = Dc^*$, where c^* denotes the Legendre transform of c. If $\Omega \subset \mathbb{R}^n$ is an open set, $u : \Omega \to \mathbb{R}$ is a function defined on Ω and $x_0 \in \Omega$, then we define

$$\partial_c u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) - c(x-p) + c(x_0 - p) \quad \forall x \in \Omega\}$$

and

$$M_u(x_0) = \{z \in \mathbb{R}^n : u(x) \geq u(x_0) + z \cdot (x - x_0) + o(|x - x_0|) \ \text{ for } x \text{ near } x_0\}.$$

It is easy to see that $M_u(x_0)$ is a convex set. On the other hand, if $p \in \partial_c u(x_0)$ then since $c \in C^1(\mathbb{R}^n)$ we have

$$\begin{aligned} u(x) &\geq u(x_0) - c(x-p) + c(x_0 - p) \\ &= u(x_0) - Dc(x_0 - p) \cdot (x - x_0) + o(|x - x_0|) \ \text{ for } x \text{ near } x_0. \end{aligned}$$

Hence, $-Dc(x_0 - p) \in M_u(x_0)$. That is,

$$\partial_c u(x_0) \subset \{p \in \mathbb{R}^n : -Dc(x_0 - p) \in M_u(x_0)\}.$$

Note that in general the above containment is strict, but if u is differentiable at x_0 then as $M_u(x_0) = \{Du(x_0)\}$ and $\partial_c u(x_0) = \{x_0 - Dc^*(-Du(x_0))\}$, we have

$$\partial_c u(x_0) = \{p \in \mathbb{R}^n : -Dc(x_0 - p) \in M_u(x_0)\}.$$

These observations will be useful later and "would motivate" the definition of c^*-convexity, introduced by Ma, Trudinger and Wang, in conjunction with extending Lemma 1 in [**Caf92**] proved for quadratic cost function to more general cost functions (see Lemma 7.4 below).

Now let Ω_1, Ω_2 be two bounded domains in $\mathbb{R}^n$. Suppose that $f \in L^1(\Omega_1)$ and $g \in L^1(\Omega_2)$ are two functions which are positive a.e. on Ω_1 and Ω_2 respectively, and satisfy the mass balance condition

$$\int_{\Omega_1} f(x)\, dx = \int_{\Omega_2} g(y)\, dy. \tag{7.1}$$

In this section we consider the following second boundary value problem for the Monge-Ampère type operators arising in optimal transportation

$$g(x - Dc^*(-Du(x))) \det[I + D^2c^*(-Du(x))D^2u(x)] = f(x) \text{ in } \Omega_1 \tag{7.2}$$

$$\partial_c u(\Omega_1) = \Omega_2, \tag{7.3}$$

where $u \in C(\Omega_1)$ is a c-convex function on Ω_1. First let us recall the two notions of generalized solutions for equations (7.2)-(7.3), Brenier solutions and Aleksandrov solutions. The first notion was introduced in the theory of optimal transportation by Brenier, Caffarelli, Gangbo and McCann (see [**Bre91**],[**Caf96**],[**GM96**]), and the latter was introduced recently and independently by us in the previous sections of the present thesis and Ma, Trudinger and Wang, [**MTW03**]. One can also introduce a notion of viscosity solutions for these equations as done in section 3.2 but we shall not discuss it here.

Definition 7.1 *A c-convex function $u \in C(\Omega_1)$ is called a Brenier solution of* (7.2)-(7.3) *if*

$$\int_{\Omega_1} h(s(x))f(x)dx = \int_{\Omega_2} h(y)g(y)dy, \qquad \textit{for all } h \in C(\mathbb{R}^n) \tag{7.4}$$

or equivalently,

$$\int_{s^{-1}(E)} f(x)dx = \int_{E\cap\Omega_2} g(y)dy, \qquad \text{for all Borel sets } E \subset \mathbb{R}^n, \tag{7.5}$$

where $s : \Omega_1 \to \mathbb{R}^n$ *is a Borel measurable map defined a.e. on* Ω_1 *by the formula* $s(x) = x - Dc^*(-Du(x))$ *whenever* u *is differentiable at* x.

Definition 7.2 *We say that a c-convex function* $u \in C(\Omega_1)$ *is an Aleksandrov solution of* (7.2)-(7.3) *if*

$$|\partial_c u(\Omega_1) - \Omega_2| = 0 \;\; ; \qquad |\partial_c u(\Omega_1)| = |\Omega_2| \tag{7.6}$$

and

$$\int_{\partial_c u(E)} g(y)\; dy = \int_E f(x)\; dx, \qquad \text{for all Borel sets } E \subset \Omega_1. \tag{7.7}$$

Remark 7.1 Notice that (7.6) is equivalent to $\partial_c u(\Omega_1) \subset \Omega_2$ a.e.

If u is an Aleksandrov solution then the measure $\omega_c(u)$ defined on Ω_1 is absolutely continuous w.r.t. the Lebesgue measure. Indeed, suppose $E \subset \Omega_1$ is such that $|E| = 0$ then by (7.7) we obtain $\int_{\partial_c u(E)} g(y)\; dy = 0$. Hence, $|\partial_c u(E)| = 0$ since $\partial_c u(E) \subset \Omega_2$ a.e. by (7.6) and $g > 0$ a.e. on Ω_2.

Lemma 7.1 *If* u *is a generalized solution of equations* (7.2)-(7.3) *in the sense of Aleksandrov then* u *is also a Brenier solution.*

Proof: Let u be an Aleksandrov solution. Then by the above remark we know that if $E \subset \Omega_1$ satisfying $|E| = 0$ then $|\partial_c u(E)| = 0$. We also observe that if $E \subset \Omega_1$ is such that $|\partial_c u(E)| = 0$ then $|E| = 0$. This follows since if $|\partial_c u(E)| = 0$ then by using (7.7) we obtain

$$0 = \int_{\partial_c u(E)} g(y)\; dy = \int_E f(x)\; dx.$$

Now define

$$u^*(y) = \sup_{x\in\Omega_1} [-c(x-y) - u(x)] = \sup_{x\in\Omega_1} [-h(y-x) - u(x)] \quad \forall y \in \mathbb{R}^n,$$

where $h(z) = c(-z)$. Then $u^* \in C(\mathbb{R}^n)$ and is h-convex on $\mathbb{R}^n$. Since u is an Aleksandrov solution, it follows that $\partial_c u(x) \cap \overline{\Omega_2} \neq \emptyset$ for every x in Ω_1. From this, it can be shown that for $x \in \Omega_1$ and $y \in \mathbb{R}^n$ we have $y \in \partial_c u(x)$ iff $x \in \partial_h(u^*, \mathbb{R}^n)(y)$. Let G be the set of points in Ω_1 where u is differentiable. Let $M = \{y \in s(G) : u^* \text{ is differentiable at } y\}$ and $F = s^{-1}(M) \cap G$. Then we have $|F| = |\Omega_1|$, $|M| = |s(G)| = |\partial_c u(G)| = |\partial_c u(\Omega_1)| = |\Omega_2|$ and $s : F \to s(F) = M$ is a bijection with $t(y) = y - Dh^*(-Du^*(y)) = y + Dc^*(-Du^*(y))$ as its inverse. Now if $E \subset \mathbb{R}^n$ is a Borel set then $s^{-1}(E) \subset \Omega_1$ is a Borel set. Hence, we have

$$\int_{s^{-1}(E)} f(x)\ dx = \int_{s^{-1}(E\cap M)} f(x)\ dx = \int_{\partial_c u(s^{-1}(E\cap M))} g(y)\ dy$$
$$= \int_{s(s^{-1}(E\cap M))} g(y)\ dy = \int_{E\cap M} g(y)\ dy = \int_E g(y)\ dy,$$

where the first equality follows since $|s^{-1}(E) - s^{-1}(E \cap M)| = 0$. Indeed, we have

$$|s^{-1}(E) - s^{-1}(E\cap M)| = |\{x \in G : s(x) \in E - M\}|$$
$$= |\{x \in G : \partial_c u(x) \subset E - M\}| = |H|,$$

here H denotes the set $\{x \in G : \partial_c u(x) \subset E - M\}$. We have $\partial_c u(H) \subset \partial_c u(\Omega_1) - M$, and hence $|\partial_c u(H)| = 0$ which gives $|H| = 0$ by the above observation. This in turn implies that $|s^{-1}(E) - s^{-1}(E \cap M)| = 0$ as desired.

□

It is clear that (7.1) is a necessary condition for the existence of a generalized solution for equations (7.2)-(7.3) either in the sense of Brenier or in the sense of Aleksandrov. Moreover, it is known from the optimal transportation theory that under condition (7.1), there exists a unique Brenier solution up to constants. Unfortunately, (7.1) is not sufficient to ensure the existence of an Aleksandrov solution. In general, Brenier's notion of solutions is strictly weaker than that of Aleksandrov as shown by Caffarelli's example in the case c is a quadratic cost function, i.e., $c(x) = \frac{1}{2}|x|^2$. Aleksandrov's notion of solution detects the singular part of the measure $\det[I + D^2c^*(-Du(x))D^2u(x)]$, while

Brenier's does not. Therefore, a basic question arises: under what conditions is a solution of (7.2)-(7.3) in Brenier's sense also a solution in the sense of Aleksandrov. Understanding this question is important for the study of the regularity of optimal map.

7.2 c^*-convexity and preliminary lemmas

In this section we recall the notion of c^*-convexity for subsets of $\mathbb{R}^n$ which was introduced recently by Ma, Trudinger and Wang (see [**MTW03**]). We assume that $c \in C^1(\mathbb{R}^n)$ and is strictly convex.

Definition 7.3 *If $z_1, z_2 \in \mathbb{R}^n$, $\overline{z_1 z_2}$ is the corresponding line segment, and $x_0 \in \mathbb{R}^n$, then a c^*-segment with respect to x_0 is the set*

$$\{y : -Dc(x_0 - y) \in \overline{z_1 z_2}\}.$$

We notice that $\{y : -Dc(x_0 - y) \in \overline{z_1 z_2}\} = \{x_0 - (Dc)^{-1}(-t) : t \in \overline{z_1 z_2}\}$.

It is clear from the assumptions on c that a c^*-segment is a continuous curve. In fact, for any two points $y_1, y_2 \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, there is a unique c^*-segment γ relative to x_0 connecting y_1 and y_2 and it is given by the formula

$$\gamma = \{x_0 - Dc^*(-z) \mid z \in \overline{z_1 z_2}\},$$

where $z_1 = -Dc(x_0 - y_1)$, $z_2 = -Dc(x_0 - y_2)$ and $\overline{z_1 z_2}$ denotes the line segment connecting z_1 and z_2.

Definition 7.4 *Let E_1 and E_2 be two subsets of $\mathbb{R}^n$. We say E_2 is c^*-convex relative to E_1 if for any two points $y_1, y_2 \in E_2$ and any $x_0 \in E_1$, the c^*-segment relative to x_0 connecting y_1 and y_2 lies in E_2.*

Remark 7.2 If $c(x) = |x|^2/2$, then a set E_2 is c^*-convex relative to E_1 if and only if E_2 is a convex set in the standard sense.

Remark 7.3 If E_2 is c^*-convex relative to E_1 then $\overline{E_2}$ is also c^*-convex relative to E_1. A way to see this is for any two points $y_1, y_2 \in \overline{E_2}$ and any $x_0 \in E_1$, choose two sequences $\{y_1^n\}, \{y_2^n\} \subset E_2$ such that $y_1^n \to y_1$ and $y_2^n \to y_2$. Then for any n, by the assumption we have the c^*-segment relative to x_0 connecting y_1^n and y_2^n lies in E_2. On the other hand, these c^*-segments converge in "some reasonable sense" to the c^*-segment relative to x_0 connecting y_1 and y_2. Hence the desired c^*-segment must lie in $\overline{E_2}$.

Lemma 7.2 *Let $\{u_j\} \subset C(\Omega)$ be a sequence of c-convex functions defined on an open set $\Omega \subset \mathbb{R}^n$. Suppose that u_j converge locally uniformly to a c-convex function $u \in C(\Omega)$. We have*

(i) If $p_j \in \partial_c u_j(x_j)$ and $x_j \to x \in \Omega$, $p_j \to p \in \mathbb{R}^n$, then $p \in \partial_c u(x)$.

(ii) $x - Dc^*(-Du_j(x)) \to x - Dc^*(-Du(x))$ for a.e. $x \in \Omega$, or equivalently, $Du_j(x) \to Du(x)$ for a.e. $x \in \Omega$.

PROOF: (i) We have

$$\begin{aligned} & u_j(z) \geq u_j(x_j) - c(z - p_j) + c(x_j - p_j) \\ & = u(x) + [u_j(x_j) - u(x_j)] + [u(x_j) - u(x)] - c(z - p_j) + c(x_j - p_j) \quad \forall z \in \Omega. \end{aligned}$$

Since $x_j \to x \in \Omega$, by taking $j \geq N$ for some number N large enough we can assume that $\{x_j\}$ remains in fixed compact subset K of Ω. So $[u_j(x_j) - u(x_j)] \to 0$ when $j \to \infty$ as $u_j \to u$ uniformly on K. Hence, by letting $j \to \infty$ in the above inequality we obtain

$$u(z) \geq u(x) - c(z - p) + c(x - p) \quad \forall z \in \Omega.$$

That is, $p \in \partial_c u(x)$ as desired.

(ii) Since $u_j, u \in C(\Omega)$ are c-convex, they are differentiable a.e. on Ω. Now let E denote the set of x in Ω such that all u_j and u are differentiable at x. Then $|E| = |\Omega|$, and we shall show that

$$x - Dc^*(-Du_j(x)) \to x - Dc^*(-Du(x)) \quad \forall x \in E.$$

First note that $\{x - Dc^*(-Du_j(x))\}$ are locally bounded in Ω independent of j. Thus for $x \in E$ we may choose a subsequence $\{j_k\}$ such that $x - Dc^*(-Du_{j_k}(x))$ converges as $k \to \infty$, say to $p \in \mathbb{R}^n$. By part (i) we must have $p \in \partial_c u(x)$, so $p = x - Dc^*(-Du(x))$, since u is differentiable at x. Since this is true for $\{u_j\}$ replaced by any subsequence, we conclude that $x - Dc^*(-Du_j(x)) \to x - Dc^*(-Du(x))$ for all $x \in E$. This completes the proof. □

Lemma 7.3 *Let $u \in C(\Omega)$ be a c-convex functions on Ω. If $x \in \Omega$ such that $Du(x)$ exists, then $s(y) = y - Dc^*(-Du(y))$ is continuous at x, or equivalently, $Du(y)$ is continuous at x.*

PROOF: Let $\{x_j\} \subset \Omega$ be any sequence such that $x_j \to x$ and Du exists at each x_j. We shall show that $s(x_j) \to s(x)$. Indeed, since $x_j \to x \in \Omega$, w.l.g. we can assume that $\{x_j\}$ remains in fixed compact subset K of Ω. Hence $\{s(x_j)\}$ is bounded as they are contained in $\partial_c u(K)$. Thus any subsequence of $\{s(x_j)\}$ has a convergent subsequence, and by Lemma 7.2 the limit of this subsequence must be $s(x)$. This implies that $s(x_j) \to s(x)$. □

7.3 Brenier solution ⇒ Aleksandrov solution

In this section we shall prove that Brenier solution of equations (7.2)-(7.3) is indeed Aleksandrov solution provided that Ω_2 is c^*-convex relative to Ω_1. The main tool in showing this is Lemma 7.4 below, which is proved by Ma, Trudinger and Wang in [**MTW03**].

Lemma 7.4 *Suppose Ω_2 is c^*-convex relative to Ω_1. If $u \in C(\Omega_1)$ is a c-convex function on Ω_1 such that $\partial_c u(x) \subset \overline{\Omega_2}$ for a.e. x in Ω_1, then $\partial_c u(\Omega_1) \subset \overline{\Omega_2}$.*

PROOF: Let G denote the set of points where u is differentiable. Also define $s(x) = x - Dc^*(-Du(x))$ whenever u is differentiable at x. Note that we then have $\partial_c u(x) = \{s(x)\}$ if $x \in G$. We first claim that $\partial_c u(x) \subset \overline{\Omega_2}$ for every x in

G. Indeed, if $x \in G$ then $\partial_c u(x)$ is a singleton. Moreover, by the assumption we can choose a sequence $\{x_j\} \subset G$ such that $\partial_c u(x_j) \subset \overline{\Omega_2}$ for every j and $x_j \to x$. Therefore, by Lemma 7.3 we get $s(x_j) \to x - Dc^*(-Du(x))$. Hence, $x - c^*(-Du(x)) \in \overline{\Omega_2}$, i.e., $\partial_c u(x) \subset \overline{\Omega_2}$. So the claim is proved. Now let x_0 be an arbitrary point in $\Omega_1 - G$. If $\partial_c u(x_0)$ is a single point then we can choose a sequence $\{x_j\} \subset G$ such that $\{s(x_j)\} = \partial_c u(x_j) \subset \overline{\Omega_2}$ for every j and $x_j \to x_0$. By passing to a subsequence we can assume w.l.g. that $s(x_j) \to p$ for some $p \in \overline{\Omega_2}$. But then by Lemma 7.2 we have $p \in \partial_c u(x_0)$. Hence $\partial_c u(x_0) = \{p\}$ since $\partial_c u(x_0)$ is a single point. This implies that $\partial_c u(x_0) \subset \overline{\Omega_2}$. On the other hand, if $\partial_c u(x_0)$ is not singleton then the convex set $M_u(x_0)$ contains more than one point. We now claim that: For any $k \geq 1$, if $z_1, ..., z_k$ are extreme points of $M_u(x_0)$ and $\lambda_1, ..., \lambda_k$ are positive numbers such that $\lambda_1 + ... + \lambda_k = 1$, then the unique solution y of

$$-Dc(x_0 - y) = \lambda_1 z_1 + ... + \lambda_k z_k$$

belongs to $\overline{\Omega_2}$.

We prove the claim by induction. Let $z \in M_u(x_0)$ be an extreme point and let y be the unique solution of $-Dc(x_0 - y) = z$. As $z \in M_u(x_0)$ is an extreme point, by arguing as in [**Caf92**] we can find a sequence $\{x_i\} \subset G$ such that $x_i \to x_0$ and $Du(x_i) \to z$. For each i, let p_i be the single element of $\partial_c u(x_i)$. Then by the above result we know that $\{p_i\} \subset \overline{\Omega_2}$. Also, for every i we have

$$-Dc(x_0 - p_i) = Du(x_i). \tag{7.8}$$

By passing to a subsequence we can assume that $p_i \to p \in \overline{\Omega_2}$ for some p. Then by letting $i \to \infty$ in (7.8) we obtain $-Dc(x_0 - p) = z$. Thus we must have $y = p$, i.e., $y \in \overline{\Omega_2}$. So the claim holds for $k = 1$. Now suppose that it holds up to k for some $k \geq 2$. We shall show that it is also true for $k + 1$. Indeed, let y be the unique solution of $-Dc(x_0 - y) = \lambda_1 z_1 + \lambda_2 ... + \lambda_{k+1} z_{k+1}$, where z_i are extreme points of $M_u(x_0)$ and λ_i are positive numbers satisfying $\lambda_1 + ... + \lambda_{k+1} = 1$. Then we have

$$-Dc(x_0 - y) = \lambda_1 z_1 + (1 - \lambda_1) z^* \tag{7.9}$$

with $z^* := \frac{\lambda_2}{1-\lambda_1}z_2 + ... + \frac{\lambda_{k+1}}{1-\lambda_1}z_{k+1}$. Now let y_1, y_2 be the unique solutions of $-Dc(x_0 - y_1) = z_1$ and $-Dc(x_0 - y_2) = z^*$ respectively. By the induction hypothesis we get y_1 and y_2 belong to $\overline{\Omega_2}$ since z^* is a convex combination of k extreme points of $M_u(x_0)$. But since $\overline{\Omega_2}$ is c^*-convex relative to x_0, the c^*-segment connecting y_1 and y_2, $\gamma = \{p \in \mathbb{R}^n : -Dc(x_0 - p) \in \overline{z_1 z^*}\}$, is contained in $\overline{\Omega_2}$. Hence this combined with (7.9) yield $y \in \overline{\Omega_2}$ as wanted. So the claim is proved. We now show that $\partial_c u(x_0) \subset \overline{\Omega_2}$. Let $p \in \partial_c u(x_0)$, then $-Dc(x_0 - p) \in M_u(x_0)$. Hence we can write $-Dc(x_0 - p)$ as a convex combination of extreme points of $M_u(x_0)$. But then $p \in \overline{\Omega_2}$ by the claim. So $\partial_c u(x_0) \subset \overline{\Omega_2}$ and the proof is completed. □

Theorem 7.1 *Let Ω_1, Ω_2 be bounded domains in $\mathbb{R}^n$ such that Ω_2 is c^*-convex relative to Ω_1. Suppose $u \in C(\Omega_1)$ is a c-convex function on Ω_1 satisfying*

$$\int_{\Omega_1} h(s(x))f(x)\,dx = \int_{\Omega_2} h(y)g(y)\,dy \quad \text{for all } h \in C(\mathbb{R}^n), \tag{7.10}$$

where s is defined a.e. on Ω_1 by the formula $s(x) = x - Dc^*(-Du(x))$ whenever u is differentiable at x.

Then $\partial_c u(\Omega_1) \subset \overline{\Omega_2}$ and $|\partial_c u(\Omega_1)| = |\Omega_2|$. Moreover, we have

$$\int_{\partial_c u(E)} g(y)\,dy = \int_E f(x)\,dx \quad \text{for all Borel sets } E \subset \Omega_1. \tag{7.11}$$

Proof: Let G denote the set of points where u is differentiable. Then $|G| = |\Omega_1|$. We first claim that $\partial_c u(x) \subset \overline{\Omega_2}$ for every x in G. Indeed, since otherwise there exists $x_0 \in G$ such that $\partial_c u(x_0) \not\subset \overline{\Omega_2}$, i.e., $s(x_0) \notin \overline{\Omega_2}$. Then we can find a positive number r satisfying $B(s(x_0), 2r) \cap \overline{\Omega_2} = \emptyset$. Also, by Lemma 7.3 there is a $t > 0$ such that $B(x_0, t) \subset \Omega_1$ and $s(B(x_0, t) \cap G) \subset B(s(x_0), r)$. Now choose a nonnegative function $h \in C(\mathbb{R}^n)$ such that $h = 1$ on $B(s(x_0), r)$ and $supp(h) \subset B(s(x_0), 2r)$. Then by substituting this function h into (7.10) we derive that

$$\begin{aligned} 0 &= \int_{\Omega_1} h(s(x))f(x)\,dx \geq \int_{B(x_0,t)\cap G} h(s(x))f(x)\,dx \\ &= \int_{B(x_0,t)\cap G} f(x)\,dx = \int_{B(x_0,t)} f(x)\,dx > 0. \end{aligned}$$

This gives a contradiction and hence the claim is proved. But then by applying Lemma 7.4 we obtain $\partial_c u(\Omega_1) \subset \overline{\Omega_2}$.

First we show that $\int_{\partial_c u(E)} g(y)\, dy \geq \int_E f(x)\, dx$ for every Borel set $E \subset \Omega_1$. For any compact set $K_1 \subset \Omega_1$, the set $K_2 = \partial_c u(K_1)$ is compact. If now $h \in C(\mathbb{R}^n)$ is any function with $h \geq \chi_{K_2}$, then

$$\int_{\Omega_2} h(y)g(y)\, dy = \int_{\Omega_1} h(s(x))f(x)\, dx \geq \int_{\Omega_1} \chi_{K_2}(s(x))f(x)\, dx$$
$$= \int_G \chi_{K_2}(s(x))f(x)\, dx = \int_{\{x\in G: s(x)\in K_2\}} f(x)\, dx \geq \int_{\{x\in K_1\cap G: s(x)\in K_2\}} f(x)\, dx$$
$$= \int_{\{x\in K_1\cap G: \partial_c u(x)\subset K_2\}} f(x)\, dx = \int_{K_1\cap G} f(x)\, dx = \int_{K_1} f(x)\, dx.$$

By the above claim we have $K_2 = \partial_c u(K_1) \subset \partial_c u(\Omega_1) \subset \overline{\Omega_2}$. Hence letting h decrease to χ_{K_2} in the above inequality and noting that $|\partial\Omega_2| = 0$ we get $\int_{K_2} g(y)\, dy \geq \int_{K_1} f(x)\, dx$, or $\omega_c(g,u)(K_1) \geq \int_{K_1} f(x)\, dx$. The regularity of the measure $f(x)dx$ and of the generalized Monge-Ampère measure $\omega_c(g,u)$ then imply that

$$\omega_c(g,u)(E) \geq \int_E f(x)\, dx \quad \text{for all Borel sets } \ E \subset \Omega_1. \tag{7.12}$$

In particular, if $\omega_c(g,u)(S) = 0$ then $\int_S f(x)\, dx = 0$.

To prove the reverse inequality in (7.12) we note first that for any compact set $K_1 \subset \Omega_1$, by Aleksandrov's Lemma we have

$$|\{p \in \mathbb{R}^n : p \in \partial_c u(K_1) \cap \partial_c u(\Omega_1 - K_1)\}| = 0.$$

Then $(\partial_c u)^{-1}(\partial_c u(K_1)) - K_1$ has Lebesgue measure zero because it is contained in $\{p \in \mathbb{R}^n : p \in \partial_c u(K_1) \cap \partial_c u(\Omega_1 - K_1)\}$. Thus together with (7.12), this implies that $\int_{(\partial_c u)^{-1}(\partial_c u(K_1)) - K_1} f(x)\, dx = 0$, or equivalently as $K_1 \subset (\partial_c u)^{-1}(\partial_c u(K_1))$,

$$\int_{(\partial_c u)^{-1}(\partial_c u(K_1))} f(x)\, dx = \int_{K_1} f(x)\, dx. \tag{7.13}$$

Now for any $h \in C(\mathbb{R}^n)$ such that $h \geq \chi_{\partial_c u(K_1)}$ we have

$$\begin{aligned}\int_{\partial_c u(K_1)} g(y)\ dy &= \int_{\Omega_2} \chi_{\partial_c u(K_1)}(y) g(y)\ dy \\ &\leq \int_{\Omega_2} h(y) g(y)\ dy = \int_{\Omega_1} h(s(x)) f(x)\ dx. \end{aligned} \tag{7.14}$$

If we take $0 \leq h \leq 1$, $h \geq \chi_{\partial_c u(K_1)}$ and h converges uniformly to zero on compact sets outside $\partial_c u(K_1)$, then $h(s(x))$ converges uniformly to zero on compact subsets of $\Omega_1 - (\partial_c u)^{-1}(\partial_c u(K_1))$. Therefore,

$$\int_{\Omega_1} h(s(x)) f(x)\ dx \to \int_{(\partial_c u)^{-1}(\partial_c u(K_1))} f(x)\ dx = \int_{K_1} f(x)\ dx,$$

where we have used (7.13) in the last equality. Hence by combining with (7.14) we obtain

$$\int_{\partial_c u(K_1)} g(y)\ dy \leq \int_{K_1} f(x)\ dx \quad \text{for all compact sets } K_1 \subset \Omega_1.$$

The regularity of the measure $f(x)dx$ and of the generalized Monge-Ampère measure $\omega_c(g, u)$ then imply a similar inequality for K_1 replaced by any Borel subset of Ω_1, as above. From this and (7.12) we get (7.11). As a consequence of (7.11) and by using (7.10) with $h \equiv 1$ we have

$$\int_{\partial_c u(\Omega_1)} g(y)\ dy = \int_{\Omega_1} f(x)\ dx = \int_{\Omega_2} g(y)\ dy,$$

and hence $|\partial_c u(\Omega_1)| = |\Omega_2|$ since $\partial_c u(\Omega_1) \subset \overline{\Omega_2}$ and $|\partial \Omega_2| = 0$. The proof is then completed. □

CHAPTER 8

Aleksandrov Type Estimates

In this chapter we prove a quantitative estimate of Aleksandrov type for c-convex functions which generalizes the well known estimate established by Aleksandrov in 1968 for convex functions (see [**Gut01**, Theorem 1.4.2]). Aleksandrov estimate plays an important role in the theory of Monge-Ampère equation and it is one of the main ingredients in Caffarelli regularity theory for the equation. Aleksandrov proved the estimate by purely geometric method and using the fact that the standard subdifferential at a given point is convex which however is no longer true in our general setting. Here we prove our estimate analytically and we only need $u \geq 0$ on the boundary of Ω instead of requiring $u = 0$ on $\partial\Omega$ as Aleksandrov.

Theorem 8.1 *Suppose $c(x) = \frac{1}{p}|x|^p$, $1 < p < 2 + \frac{1}{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set, and $u \in C(\bar{\Omega})$ be a c-convex function on Ω with $u \geq 0$ on $\partial\Omega$. We have*

(i) If $1 < p \leq 2$ then

$$u(x) \geq -C(n,p)\left[diam(\Omega)^{\frac{n-(p-1)}{p-1}}\ dist(x,\partial\Omega)\ |\partial_c u(\Omega)|\right]^{\frac{p-1}{n}} \quad \forall x \in \Omega.$$

(ii) If $2 < p < 2 + \frac{1}{n-1}$ then

$$u(x) \geq -C(n,p)\left[diam(\Omega)^{n-1}\ dist(x,\partial\Omega)^{\frac{(p-1)-n(p-2)}{p-1}}\ |\partial_c u(\Omega)|\right]^{\frac{p-1}{n}} \quad \forall x \in \Omega,$$

where $C(n,p)$ is a constant depending only on the dimension n and p.

PROOF:

Let $x_0 \in \Omega$ be such that $u(x_0) < 0$, and $G(z) = x_0 - Dc^*(-z)$. In order to prove the theorem we first define a c-convex function, whose graph is the upside-down c-cone surface with vertex at $(x_0, u(x_0))$, as follows

$$w(x) = \sup_{v \in \mathcal{F}} v(x) \quad \forall x \in \bar{\Omega},$$

where

$$\mathcal{F} = \{v \in C(\bar{\Omega}) : v \text{ is c-convex } , v(x_0) \leq u(x_0) \text{ and } v|_{\partial\Omega} \leq u|_{\partial\Omega}\}.$$

Then since $u \in \mathcal{F}$ we have $w \geq u$ on $\bar{\Omega}$, $w(x_0) = u(x_0)$, w is c-convex on Ω and $w|_{\partial\Omega} = u|_{\partial\Omega}$.

Claim 1: $w \in C(\bar{\Omega})$. Indeed, since Ω is convex it is q-regular with q is the conjugate of p. From [**BR02**, Theorem 4.7] there exists $h \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$ weak solution to the q-Laplacian

$$-\text{div } \big(|Dh(x)|^{q-2} Dh(x)\big) + n = 0, \quad \text{in } \Omega \text{ and } h = -u \text{ on } \partial\Omega.$$

Notice that div $(|Dh(x)|^{q-2}Dh(x)) = \text{div } (Dc^*(Dh(x)))$. For each $f(x) = -c(x-y)-\lambda \in \mathcal{F}$, we have $-f(x) \geq -u(x)$ on $\partial\Omega$, and $-\text{div } (Dc^*(-Df(x)))+ n = 0$. Hence by the comparison principle [**BR02**, Theorem 3.1] for the q-Laplacian we get that $-f \geq h$ in $\bar{\Omega}$, and therefore $w(x) = \sup_{v \in \mathcal{F}} v(x) \leq -h(x)$ for all $x \in \bar{\Omega}$. Thus we obtain $u \leq w \leq -h$ in $\bar{\Omega}$ and hence the claim follows since $u, -h \in C(\bar{\Omega})$, $u|_{\partial\Omega} = -h|_{\partial\Omega}$ and w is c-convex on Ω.

From Claim 1 above and by Lemma 5.1 we obtain

$$\begin{aligned}
&\partial_c u(\Omega) \supset \partial_c w(\Omega) \supset \partial_c w(x_0) \\
&= \{y : u(x_0) - c(x-y) + c(x_0 - y) \leq u(x) \; \forall x \in \partial\Omega\} \\
&\supset \{y : u(x_0) - c(x-y) + c(x_0 - y) \leq 0 \; \forall x \in \partial\Omega\} \\
&\supset \{y : u(x_0) + < Dc(x_0 - y), x_0 - x > \leq 0 \; \forall x \in \partial\Omega\} =: E
\end{aligned}$$

We also observe that the set E is c^*-convex with respect to x_0 in the sense that if $y_1, y_2 \in E$, then the curve connecting y_1 and y_2

$$\{x_0 - Dc^*(-z) : z \in \overline{z_1 z_2}\} \subset E,$$

where $z_1 = -Dc(x_0 - y_1)$ and $z_2 = -Dc(x_0 - y_2)$. Indeed, if $z = \lambda z_1 + (1-\lambda) z_2$ for $\lambda \in [0,1]$ and $y = x_0 - Dc^*(-z)$ then we have for any $x \in \partial\Omega$

$$
\begin{aligned}
& u(x_0) + < Dc(x_0 - y), x_0 - x >= u(x_0) + < Dc(Dc^*(-z)), x_0 - x > \\
& = \lambda[u(x_0) + < -z_1, x_0 - x >] + (1-\lambda)[u(x_0) + < -z_2, x_0 - x >] \\
& = \lambda[u(x_0) + < Dc(x_0 - y_1), x_0 - x >] \\
& + (1-\lambda)[u(x_0) + < Dc(x_0 - y_2), x_0 - x >] \leq 0.
\end{aligned}
$$

This means $y = x_0 - Dc^*(-z) \in E$ as desired.

Let x^* be the point on $\partial\Omega$ such that $|x_0 - x^*| = \text{dist}(x_0, \partial\Omega)$, and let $z_0 = \frac{-u(x_0)}{\text{dist}(x_0,\partial\Omega)} \frac{x^* - x_0}{|x^* - x_0|}$. Then since Ω is convex, we easily see by a simple geometric observation that

$$
< x^* - x_0, x - x_0 > \leq |x^* - x_0|^2 \text{ for all } x \in \partial\Omega. \tag{8.1}
$$

Claim 2: We have $G(\bar{B}(0, \frac{-u(x_0)}{\text{diam}(\Omega)})) \subset E$ and $G(z_0) \in E$.

Indeed, suppose $y = G(z) = x_0 - Dc^*(-z)$ for some $z \in \bar{B}(0, \frac{-u(x_0)}{\text{diam}(\Omega)})$. Then for every $x \in \partial\Omega$, we have

$$
\begin{aligned}
& u(x_0) + < Dc(x_0 - y), x_0 - x >= u(x_0) + < -z, x_0 - x > \\
& \leq u(x_0) + \text{diam}(\Omega)|z| \leq 0,
\end{aligned}
$$

which gives $y \in E$ as desired. On the other hand, if we let $y_0 = G(z_0)$ then we have for any $x \in \partial\Omega$

$$
\begin{aligned}
& u(x_0) + < Dc(x_0 - y_0), x_0 - x >= u(x_0) + < -z_0, x_0 - x > \\
& = u(x_0) + \frac{-u(x_0)}{\text{dist}(x_0, \partial\Omega)} \frac{1}{|x^* - x_0|} < x^* - x_0, x - x_0 > \\
& \leq u(x_0) + \frac{-u(x_0)}{\text{dist}(x_0, \partial\Omega)} |x^* - x_0| = 0,
\end{aligned}
$$

where we have used (8.1) in the last inequality. This gives $y_0 \in E$ as wanted.

By claim 2 and since E is c^*-convex with respect to x_0 we obtain $G(H) \subset E$, where H is the closed convex hull of the point $\left[\frac{-u(x_0)}{\text{dist}(x_0,\partial\Omega)}\right] \frac{x^* - x_0}{|x^* - x_0|}$ and the ball

$B(0, \frac{-u(x_0)}{\mathrm{diam}(\Omega)})$. Therefore, in summing up we get $G(H) \subset \partial_c u(\Omega)$. Hence,

$$|\partial_c u(\Omega)| \geq |G(H)| = |Dc^*(-H)| = \int_H \det D^2 c^*(-y)\ dy$$
$$= (q-1) \int_H |y|^{n(q-2)}\ dy.$$

Now the inequalities stated in the theorem follow from the estimates (6.22) and (6.23) in the proof of Proposition 6.1. □

APPENDIX A

Perron Method For The Dirichlet Problems

In this appendix we shall show that the standard Perron method can be carried out to prove Theorem 6.2 provided that we assume in addition a subsolution to the problem exists. Suppose c satisfies $(H1)$ with $\lim_{|x|\to\infty} \dfrac{c(x)}{|x|} = +\infty$ and $g \in L^1_{loc}(\mathbb{R}^n)$ is nonnegative a.e. on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, $\psi \in C(\partial\Omega)$ and $\mu = \sum_{i=1}^N a_i \delta_{x_i}$ where $x_i \in \Omega$ and $a_i > 0$. We consider the following Dirichlet problem

$$g(x - Dc^*(-Du(x)))det[I + D^2c^*(-Du(x))D^2u(x)] = \mu \quad \text{in} \quad \Omega \tag{A.1}$$

$$u = \psi \qquad\qquad \text{on} \quad \partial\Omega \tag{A.2}$$

First define

$$\mathcal{F}(g, \mu, \psi) := \{v \in C(\overline{\Omega}) : v \text{ is c-convex, } \omega_c(g, v) \geq \mu \text{ in } \Omega \text{ and } v = \psi \text{ on } \partial\Omega\}.$$

When $g \equiv 1$ we simply write $\mathcal{F}(\mu, \psi)$ for $\mathcal{F}(1, \mu, \psi)$. Suppose $\mathcal{F}(g, \mu, \psi) \neq \emptyset$, and let W be the unique generalized solution for the associated homogeneous Dirichlet problem. Then we have $v \leq W$ on $\overline{\Omega}$ for every $v \in \mathcal{F}(g, \mu, \psi)$. Therefore, all functions in $\mathcal{F}(g, \mu, \psi)$ are uniformly bounded from above on $\overline{\Omega}$ and we can define

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(g, \mu, \psi)\} \qquad \forall x \in \overline{\Omega}.$$

Pick any $\underline{W} \in \mathcal{F}(g,\mu,\psi)$ and we then get $\underline{W} \leq U \leq W$ on $\overline{\Omega}$. So $U = \psi$ on $\partial\Omega$, U is c-convex on Ω and $U \in C(\overline{\Omega})$. Moreover, by the comparison principle we can see that if our nonhomogeneous Dirichlet problem has a weak solution then it must be U.

Theorem A.1 *Suppose c satisfies condition $(H1)$ with $\lim_{|x|\to\infty} \dfrac{c(x)}{|x|} = +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, $\psi \in C(\partial\Omega)$ and $\mu = \sum_{i=1}^{N} a_i \delta_{x_i}$ where $x_i \in \Omega$ and $a_i > 0$. If $\mathcal{F}(g,\mu,\psi) \neq \emptyset$, then there exists a unique c-convex function $u \in C(\overline{\Omega})$ that is a weak solution to the Dirichlet problem* (A.1)-(A.2)*.*

PROOF: Let U and W be defined as above. We shall show that U is the desired solution. First we claim that

(a) If $u,\ v \in \mathcal{F}(g,\mu,\psi)$ then $\max\{u,v\} \in \mathcal{F}(g,\mu,\psi)$.

(b) $U \in \mathcal{F}(g,\mu,\psi)$.

Step 1: Proof of (a). Let $\phi = \max\{u,v\}$, $\Omega_0 = \{x \in \Omega : u(x) = v(x)\}$, $\Omega_1 = \{x \in \Omega : u(x) > v(x)\}$ and $\Omega_2 = \{x \in \Omega : u(x) < v(x)\}$. If $E \subset \Omega_1$ then $\omega_c(g,\phi)(E) \geq \omega_c(g,u)(E)$ and if $E \subset \Omega_2$ then $\omega_c(g,\phi)(E) \geq \omega_c(g,v)(E)$. Also, if $E \subset \Omega_0$, then $\partial_c u(E) \subset \partial_c\phi(E)$ and $\partial_c v(E) \subset \partial_c\phi(E)$. Given $E \subset \Omega$ a Borel set, write $E = E_0 \cup E_1 \cup E_2$ with $E_i \subset \Omega_i$. We then have

$$\begin{aligned}
&\omega_c(g,\phi)(E) = \omega_c(g,\phi)(E_0) + \omega_c(g,\phi)(E_1) + \omega_c(g,\phi)(E_2) \\
&\geq \omega_c(g,u)(E_0) + \omega_c(g,u)(E_1) + \omega_c(g,v)(E_2) \\
&\geq \mu(E_0) + \mu(E_1) + \mu(E_2) = \mu(E).
\end{aligned}$$

Step 2: For each $y \in \Omega$ there exists a uniformly bounded sequence $v_m \in \mathcal{F}(\mu,\psi)$ converging uniformly on compact subsets of Ω to a function $w \in \mathcal{F}(\mu,\psi)$ so that $w(y) = U(y)$.

Since $\mathcal{F}(g,\mu,\psi) \neq \emptyset$ we can pick $u_0 \in \mathcal{F}(g,\mu,\psi)$. If $u \in \mathcal{F}(g,\mu,\psi)$ then $u \leq W$ in $\overline{\Omega}$. Fix $y \in \Omega$, then by definition of U there exists a sequence $u_m \in \mathcal{F}(g,\mu,\psi)$ such that $u_m(y) \to U(y)$ as $m \to +\infty$. Let $v_m = \max\{u_0,u_m\}$. By step 1, $v_m \in \mathcal{F}(g,\mu,\psi)$ and therefore $u_m(y) \leq v_m(y) \leq U(y)$ and so $v_m(y) \to U(y)$. Note also that $\{v_m\}$ is uniformly bounded in $\overline{\Omega}$ since $u_0 \leq v_m \leq W$

in $\overline{\Omega}$. Now since v_m is c-convex in Ω we know that given $K \subset \Omega$ compact, v_m is Lipschitz in K, say with constant $C(K, m)$. We claim that $C(K, m)$ is uniformly bounded in m. Indeed, by Lemma 2.3 there exists $R > 0$ such that $\partial_c v_m(K) \subset B(0, R)$ for all $m = 1, 2, ...$ Choose the ball B which is large enough such that $z - p \in B$ for all $z \in K$ and $p \in B(0, R)$. Then for any $x, y \in K$, by choosing $p \in \partial_c v_m(y)$ and since c is convex on $\mathbb{R}^n$ we have

$$v_m(x) - v_m(y) \geq -c(x-p) + c(y-p) \geq -||c||_{Lip(B)}|x-y|.$$

Similarly, we also have $v_m(y) - v_m(x) \geq -||c||_{Lip(B)}|x-y|$. Thus $|v_m(x) - v_m(y)| \leq ||c||_{Lip(B)}|x-y|$ for all $x, y \in K$, that is, $C(k, m) \leq ||c||_{Lip(B)}$ for all m. This proves the claim. Therefore, $\{v_m\}$ are equicontinuous on K and uniformly bounded in Ω. Hence since $u_0 \leq v_m \leq W$ and $u_0 = W = \psi$ on $\partial\Omega$, by Arzela-Ascoli's lemma there exists a subsequence still denoted by $\{v_m\}$ converging uniformly on compact subsets of Ω to a function $w \in C(\overline{\Omega})$ satisfying $w = \psi$ on $\partial\Omega$ and so $w(y) = U(y)$. Also w is c-convex by Lemma 2.1. Moreover for each $i = 1, ..., N$, by using the first part of Lemma 3.1 we have $\omega_c(g, w)(\{x_i\}) \geq \limsup_{m\to\infty} \omega_c(g, v_m)(\{x_i\}) \geq \mu(\{x_i\})$. Therefore, we get $w \in \mathcal{F}(g, \mu, \psi)$ and hence $w \leq U$ in Ω.

Step 3: Proof of (b). By the observation before this lemma, to prove (b) it suffices to show that $\omega_c(g, U) \geq \mu$ in Ω. Let $i \in \{1, ..., N\}$. By Step 2 there exists a sequence $\{v_m\} \in \mathcal{F}(g, \mu, \psi)$, uniformly bounded on $\overline{\Omega}$, such that $v_m \to w \in \mathcal{F}(g, \mu, \psi)$ uniformly on compacts of Ω as $m \to \infty$ with $w(x_i) = U(x_i)$. If $p \in \partial_c w(x_i)$ then

$$U(x) \geq w(x) \geq w(x_i) - c(x-p) + c(x_i - p) = U(x_i) - c(x-p) + c(x_i - p)$$

for all $x \in \Omega$. So $p \in \partial_c U(x_i)$ and hence $\partial_c w(x_i) \subset \partial_c U(x_i)$. This yields

$$\omega_c(g, U)(\{x_i\}) \geq \omega_c(g, w)(\{x_i\}) \geq \mu(\{x_i\}) = a_i.$$

Therefore, we obtain $\omega_c(g, U) \geq \mu$ in Ω.

Step 4: $\omega_c(g, U) = \sum_{i=1}^{N} \lambda_i a_i \delta_{x_i}$ for some $\lambda_i \geq 1 \quad \forall i = 1, ..., N$. Let x_0 be an arbitrary element in $\Omega - \{x_1, ..., x_N\}$, and $B = B(x_0, r)$ be any open ball centered at x_0 such that $x_i \notin \overline{B}$ for all $i = 1, ..., N$ and $\overline{B} \subset \Omega$. Define

$$w(x) = \sup\{v(x) : v \in \mathcal{F}^*\} \qquad \forall x \in \overline{\Omega},$$

where $\mathcal{F}^* = \{v \in C(\overline{\Omega}), v \text{ is c-convex, and } v(x) \leq U(x) \text{ on } \overline{\Omega} \setminus B\}$. Then $w \in C(\overline{\Omega})$, w is c-convex, $U \leq w$ in $\overline{\Omega}$ and $w = U$ on $\overline{\Omega} \setminus B$ since $U \in \mathcal{F}^*$. Moreover, for every Borel set $E \subset \Omega$ we have

$$\begin{aligned}\omega_c(g, w)(E) &= \omega_c(g, w)(E \cap B) + \omega_c(g, w)(E \cap B^c) \geq \omega_c(g, w)(E \cap B^c) \\ &\geq \omega_c(g, U)(E \cap B^c) \geq \mu(E \cap B^c) = \mu(E \cap \{x_1, ..., x_N\}) = \mu(E).\end{aligned}$$

Therefore, $w \in \mathcal{F}(g, \mu, \psi)$ and hence we obtain $U = w$ on $\overline{\Omega}$. Now let

$$\tilde{S} = \{p \in \mathbb{R}^n : \ p \in \partial_c(w, \overline{\Omega})(z_1) \cap \partial_c(w, \overline{\Omega})(z_2) \ \text{ for some } \ z_1, z_2 \in \overline{\Omega},\ z_1 \neq z_2\}.$$

We shall show that $\partial_c w(B) \subset \tilde{S}$. Indeed, let $p \in \partial_c w(z_1)$ for some $z_1 \in B$ and define $g(x) = w(z_1) - c(x - p) + c(z_1 - p)$. Then $g(x) \leq w(x)$ for every x in $\overline{\Omega}$. We claim that there must be a $z_2 \in \overline{\Omega} \setminus \{z_1\}$ such that $g(z_2) = w(z_2)$ since otherwise we have $\delta := \min_{x \in \overline{\Omega} \setminus B} [w(x) - g(x)] > 0$. Then by letting $f(x) = g(x) + \delta$ we get $f \in C(\overline{\Omega})$, f is c-convex, and $f(x) = g(x) + \delta \leq g(x) + w(x) - g(x) = w(x) = U(x)$ for every x in $\overline{\Omega} \setminus B$. So $f \in \mathcal{F}^*$ and hence $w(z_1) \geq f(z_1) = g(z_1) + \delta = w(z_1) + \delta > w(z_1)$. This gives a contradiction and hence the claim is proved. Therefore, $\partial_c w(B) \subset \tilde{S}$ and so by Lemma 2.2 we obtain $|\partial_c U(B)| = |\partial_c w(B)| = 0$. Thus, the measure $\omega_c(g, U)$ is concentrated on the set $\{x_1, ..., x_N\}$. Hence since $\omega_c(g, U) \geq \mu$, we have $\omega_c(g, U) = \sum_{i=1}^N \lambda_i a_i \delta_{x_i}$ with $\lambda_i \geq 1$ for every $i = 1, ..., N$.

Step 5: $\omega_c(g, U) = \mu$ in Ω. To prove this by step 4 we only need to show that $\lambda_i = 1$ for every $i = 1, ..., N$. Indeed, suppose by contradiction that $\lambda_{i_0} > 1$ for some $i_0 \in \{1, ..., N\}$, we shall derive a contradiction. First choose an open ball $B = B(x_{i_0}, r)$ so that $x_i \notin \overline{B}$ for all $i \in \{1, ..., N\} \setminus \{i_0\}$ and $\overline{B} \subset \Omega$. Define

$$G = \{v \in C(\overline{\Omega}) : v \ \text{ is c-convex}, v(x_{i_0}) \leq U(x_{i_0}) \ \text{ and } \ v(x) \leq U(x) \ \text{ on } \ \overline{\Omega} \setminus B\}$$

and for each $n \in \mathbb{N}$,

$$G_n = \{v \in C(\overline{\Omega}) : v \text{ is c-convex}, v(x_{i_0}) \le U(x_{i_0}) + \frac{1}{n},\ v(x) \le U(x) \text{ on } \overline{\Omega} \setminus B\}.$$

Note that $U \in G$ and $U \in G_n$ for every n. Now define

$$w(x) = \sup\{v(x) : v \in G\} \qquad \forall x \in \overline{\Omega},$$

$$w_n(x) = \sup\{v(x) : v \in G_n\} \qquad \forall x \in \overline{\Omega}.$$

Then it is clear that w, w_n are c-convex and continuous on $\overline{\Omega}$, and $U \le w, w_n$ on $\overline{\Omega}$. Moreover, $w(x_{i_0}) = U(x_{i_0})$, $w_n(x_{i_0}) \le U(x_{i_0}) + \frac{1}{n}$ and w, w_n are equal to U on $\overline{\Omega} \setminus B$. We further have the followings

Claim 1: $U(x) = w(x)$ on $\overline{\Omega}$.

Since $\partial_c U(x_i) \subset \partial_c w(x_i)$ for every $i = 1, ..., N$, we have by Step 3 and Step 4 that $\omega_c(g, w) \ge \omega_c(g, U) \ge \mu$. Therefore $w \in \mathcal{F}(g, \mu, \psi)$ and the claim follows.

Claim 2: $|\partial_c w_n(B)| = |\partial_c w_n(\{x_{i_0}\})|$. This will follow if we can show that $|\partial_c w(B \setminus \{x_{i_0}\})| = 0$. Let

$$\tilde{S}_n = \{p \in \mathbb{R}^n : \ p \in \partial_c(w_n, \overline{\Omega})(z_1) \cap \partial_c(w_n, \overline{\Omega})(z_2) \text{ for some } z_1, z_2 \in \overline{\Omega},\ z_1 \ne z_2\}.$$

We shall show that $\partial_c w(B \setminus \{x_{i_0}\}) \subset \tilde{S}_n$. Indeed, let $p \in \partial_c w(z_1)$ for some $z_1 \in B \setminus \{x_{i_0}\}$ and define $g(x) = w_n(z_1) - c(x - p) + c(z_1 - p)$. Then $g(x) \le w_n(x)$ for every x in $\overline{\Omega}$. We claim that there must be a $z_2 \in \overline{\Omega} \setminus \{z_1\}$ such that $g(z_2) = w_n(z_2)$ since otherwise we have

$$\alpha := \min_{x \in \overline{\Omega} \setminus B} [w_n(x) - g(x)] > 0 \quad \text{and} \quad w_n(x_{i_0}) - g(x_{i_0}) > 0.$$

Then by letting $f(x) = g(x) + \delta$ where $\delta = \min\{\alpha, w_n(x_{i_0}) - g(x_{i_0})\} > 0$ we get $f \in C(\overline{\Omega})$, f is c-convex, $f(x_{i_0}) \le g(x_{i_0}) + w_n(x_{i_0}) - g(x_{i_0}) = w_n(x_{i_0}) \le U(x_{i_0}) + \frac{1}{n}$ and $f(x) \le g(x) + w(x) - g(x) = w(x) = U(x)$ for every x in $\overline{\Omega} \setminus B$. Therefore $f \in G_n$ and hence we obtain $w_n(z_1) \ge f(z_1) = g(z_1) + \delta = w_n(z_1) + \delta > w_n(z_1)$. This is a contradiction, that is, there exists $z_2 \in \overline{\Omega} \setminus \{z_1\}$ such that $g(z_2) = w_n(z_2)$. But this implies that $p \in \partial_c(w_n, \overline{\Omega})(z_1) \cap \partial_c(w_n, \overline{\Omega})(z_2)$, i.e., $p \in \tilde{S}_n$. Thus $\partial_c w_n(B \setminus \{x_{i_0}\}) \subset \tilde{S}_n$ and $|\partial_c w(B \setminus \{x_{i_0}\})| = 0$ by Lemma 2.2.

Claim 3: There exists $n_0 \in \mathbb{N}$ depending only on U, Ω, B and x_{i_0} such that for all $n \geq n_0$ we have

$$w_n(x_{i_0}) = U(x_{i_0}) + \frac{1}{n}.$$

Indeed, let $\tilde{S} = \{p \in \mathbb{R}^n : p \in \partial_c(U, \overline{\Omega})(y) \cap \partial_c(U, \overline{\Omega})(z) \quad \text{for some} \quad y, z \in \overline{\Omega} \ \text{ with } \ y \neq z\}$. Then $|\tilde{S}| = 0$ by Lemma 2.2. But $|\partial_c U(x_{i_0})| = \lambda_{i_0} a_{i_0} > 0$. Therefore, there exists $p \in \partial_c U(x_{i_0})$ such that $p \notin \tilde{S}$. Hence,

$$U(x) > U(x_{i_0}) - c(x - p) + c(x_{i_0} - p) \quad \forall x \in \overline{\Omega} \setminus \{x_{i_0}\}.$$

Thus

$$\delta := \min_{\overline{\Omega} \setminus B} \{U(x) - U(x_{i_0}) + c(x - p) - c(x_{i_0} - p)\} > 0.$$

So we can pick $n_0 \in \mathbb{N}$ large enough such that $\frac{1}{n_0} \leq \delta$. Now for any $n \geq n_0$, let $f(x) = U(x_{i_0}) - c(x - p) + c(x_{i_0} - p) + \frac{1}{n}$. Then $f \in C(\overline{\Omega})$, f is c-convex on Ω and $f(x_{i_0}) = U(x_{i_0}) + \frac{1}{n}$. Moreover, for every x in $\overline{\Omega} \setminus B$ we have

$$\begin{aligned}
&f(x) \leq U(x_{i_0}) - c(x - p) + c(x_{i_0} - p) + \frac{1}{n_0} \\
&\leq U(x_{i_0}) - c(x - p) + c(x_{i_0} - p) + \delta \\
&\leq U(x_{i_0}) - c(x - p) + c(x_{i_0} - p) + U(x) - U(x_{i_0}) + c(x - p) - c(x_{i_0} - p) \\
&= U(x).
\end{aligned}$$

Therefore, $f \in G_n$ and hence

$$U(x_{i_0}) + \frac{1}{n} \geq w_n(x_{i_0}) \geq f(x_{i_0}) = U(x_{i_0}) + \frac{1}{n}.$$

So $w_n(x_{i_0}) = U(x_{i_0}) + \frac{1}{n}$ for all $n \geq n_0$.

Claim 4: There exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to U$ uniformly on $\overline{\Omega}$.

For every n, we have $U \leq w_n \leq W$ on $\overline{\Omega}$. Therefore, by arguing as in the proof of Step 2 there exist $u \in C(\overline{\Omega})$ with $u = U$ on $\partial\Omega$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to u$ locally uniformly on Ω. On the other hand, we have $U \leq u$ in $\overline{\Omega}$ and $u = U$ in $\overline{\Omega} \setminus B$ since $U \leq w_{n_k}$ on $\overline{\Omega}$ and $w_{n_k} = U$ on $\overline{\Omega} \setminus B$. Also, we have u is c-convex on Ω by Lemma 2.1. Moreover, by claim 3 we get

$$u(x_{i_0}) = \lim_{k\to\infty} w_{n_k}(x_{i_0}) = \lim_{k\to\infty} [U(x_{i_0}) + \frac{1}{n_k}] = U(x_{i_0}).$$

So $u \in G$ and hence by claim 1 we obtain $u \leq U$ in $\overline{\Omega}$. Thus, $u = U$ in $\overline{\Omega}$ and $w_{n_k} \to U$ uniformly on $\overline{\Omega}$ since $w_{n_k} = U$ on $\overline{\Omega} \setminus B$.

By claim 4 and Corollary 3.1 we get $\omega_c(g, w_{n_k}) \to \omega_c(g, U)$ weakly. Hence, by using claim 2 and the fact that $|\partial_c U(B)| = |\partial_c U(x_{i_0})|$ we obtain

$$|\partial_c w_{n_k}(x_{i_0})| \to |\partial_c U(x_{i_0})| = \lambda_{i_0} a_{i_0}.$$

But since $\lambda_{i_0} > 1$ we therefore can choose $n_{k_0} \in \mathbb{N}$, $n_{k_0} \geq n_0$ such that $|\partial_c w_{n_{k_0}}(x_{i_0})| \geq a_{i_0}$. Moreover, for every $i = 1, ..., N$, $i \neq i_0$ since $U(x_i) = w_{n_{k_0}}(x_i)$ and $U \leq w_{n_{k_0}}$ in $\overline{\Omega}$ we get $\partial_c U(x_i) \subset \partial_c w_{n_{k_0}}(x_i)$. So $|\partial_c w_{n_{k_0}}(x_i)| \geq |\partial_c U(x_i)| \geq a_i$. Thus $\omega_c(g, w_{n_{k_0}}) \geq \mu$ and hence $w_{n_{k_0}} \in \mathcal{F}(g, \mu, \psi)$. This implies that $U \geq w_{n_{k_0}}$ in $\overline{\Omega}$. Then by combining with claim 3 we obtain

$$U(x_{i_0}) \geq w_{n_{k_0}}(x_{i_0}) = U(x_{i_0}) + \frac{1}{n_{k_0}}.$$

This yields a contradiction. So we must have $\lambda_i = 1$ for all $i = 1, ..., N$, i.e., $\omega_c(g, U) = \mu$ in Ω and completes the proof of the existence of a weak solution to the problem. The uniqueness follows from Lemma 5.1. □

Remark A.1 *Let $\Omega \subset \mathbb{R}^n$ be a strictly convex open set, $\psi \in C(\partial\Omega)$ and $\mu = \sum_{i=1}^N a_i \delta_{x_i}$ where $x_i \in \Omega$ and $a_i > 0$. Then we claim that if c satisfies conditions as in Lemma 6.2 and if*

$$a_1 + \cdots + a_N < \int_{\mathbb{R}^n} \det D^2 c^*(-y)\, dy \tag{A.3}$$

then $\mathcal{F}(\mu, \psi) \neq \emptyset$. In fact, the following proof will show that there exists a convex function in $\mathcal{F}(\mu, \psi)$. Indeed, let R be defined as in Lemma 6.2 and from now on we only work with this function R in this section. Consider the following equation

$$\begin{aligned} R(Du) \det D^2 u &= \mu \quad \text{in } \Omega \\ u &= \psi \quad \text{on } \partial\Omega \end{aligned}$$

*in the generalized sense. By Theorem 2 in [**Bak86**] (see also Theorem 11.2 in [**Bak94**]), this equation has a unique generalized convex solution, say* u. *But then by using Lemma 6.2 we get* $u \in \mathcal{F}(\mu, \psi)$. *So the claim is proved. We remark that if* $c(x) = \frac{1}{p}|x|^p$, $1 < p < +\infty$, *then* $c^*(y) = \frac{1}{q}|y|^q$, *where* q *is the conjugate of* p, *i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$. *Then we have* $R(y) = \det D^2 c^*(-y) = (q-1)|y|^{n(q-2)}$ *and hence all the above conditions are satisfied without any restrictions on* $a_1, \cdots, a_N$ *because* $\int_{\mathbb{R}^n} R(y)\, dy = +\infty$. *Therefore,* $\mathcal{F}(\mu, \psi) \neq \emptyset$ *for all cost functions* $c(x) = \frac{1}{p}|x|^p$, $1 < p < +\infty$.

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