SUFFICIENT CONDITIONS FOR LOCAL MINIMIZERS IN CALCULUS OF VARIATIONS

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

---

by
Tadele A. Mengesha
May, 2007

ABSTRACT

SUFFICIENT CONDITIONS FOR LOCAL MINIMIZERS IN CALCULUS OF VARIATIONS

Tadele A. Mengesha
DOCTOR OF PHILOSOPHY

Temple University, May, 2007

Professor Yury Grabovsky, Chair

In this thesis we prove that uniform quasiconvexity and uniform positivity of second variation are sufficient for a continuous Young measure to be a local minimizer of a variational integral. Results corresponding to classical strong local minimizers are also given. The variational problems considered are multiple integrals with Lagrangian behaving as a power function at infinity. Our approach is direct. We evaluate the increment of the variational functional corresponding to a strong variation. The sufficient conditions and growth assumptions on the Lagrangian guarantee that the increment is always positive.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my sincere appreciation and gratitude to my advisor, Professor Yury Grabovsky, for his expert guidance and kind support through out the preparation of thesis. His enthusiasm for the work, constant encouragement and valuable insights were essential for the success of the project.

I would also like to thank members of defense committee Professor Shiferaw Berhanu and Professor Boris Datskovsky for their review of the material and constant support through my stay at Temple. They have also taught me courses and I have benefited greatly from their teaching, guidance and advise.

I would like to thank all department faculty for what they taught me and all the advices they offered. A special thanks to Professor David Zitarelli for his help in many ways in the final years of my study, Professor Omar Hijab, the department chair, for his confidence in me and his inspirational teaching of the courses I studied with him.

I would like to thank the department staff for their administrative assistant and my fellow graduate students in the department for the time we spent either discussing mathematics or playing soccer.

A very special thanks to my wife Kalkidan Amede, with whom we enjoy the presence of the new member of the family our son Ahadu Mengesha, for her love, support and patience throughout the project. Her faith in me is the driving force for many of my accomplishments.

This work was supported in part by Temple University and the National Science Foundation under Grant No. 0094089 (Yury Grabovsky, PI).

DEDICATION

To my parents

W/ro Kassaye Andargie

and

Ato Adamtie Mengesha

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation and background

In this thesis we consider the class of integral functionals of the form

$$E(\boldsymbol{y}) = \int_\Omega W(\boldsymbol{x}, \nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x} \tag{1.1}$$

defined on the set of admissible maps

$$\mathcal{A} = \{\boldsymbol{y} \in W^{1,\infty}(\Omega; \mathbb{R}^m) : \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{x} \in \partial\Omega_1\}$$

where $\boldsymbol{g} \in C^1(\partial\Omega; \mathbb{R}^m)$ and Ω is a smooth (i.e. C^1) bounded domain in $\mathbb{R}^d$ with $\partial\Omega_1$ and $\partial\Omega_2$ smooth, disjoint and relatively open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$. The Lagragian $W : \overline{\Omega} \times \mathbb{M} \to \mathbb{R}$ is a continuous function, where $\mathbb{M}$ denotes the space of $m \times d$ real matrices.

The problem of sufficient conditions for a smooth extremal of a variational functional to be a strong local minimizer is an old one. It has been solved completely by Weierstrass for the case of one independent variable by the methods of field theory. His method was applied by Morrey [32] to address the problem of multiple integrals for the case of one dependent variable. Non-field theory approaches were also developed by Levi [28] (see also [37]) and Hestenes [22] (see also [41]) to treat variational problems.

The aim of this thesis is to present a sufficiency result for variational problems with multiple integrals, where the unknown is a vector field. It generalizes the result proved in [19]. We will extend sufficient conditions of local minimizers to problems with no classical minimizers, where Young measures are used to account the oscillatory properties of minimizing sequences. Applications include models of shape memory materials that are characterized by nonexistence of solutions and where Young measure describes the observed fine-scale microstructure, [7], [4].

For classical variational problems with one independent variable the sufficient conditions consist, beyond Euler's equation, of the positivity of second variation and positivity of the Weierstrass excess function. It has been understood (see for example [3]) that the vectorial analog of the Weierstrass condition is the quasiconvexity condition [31]. The vectorial analog of the classical sufficiency theorem, the one, where the Weierstrass condition is replaced with the quasiconvexity condition, was conjectured by Ball [3, Section 6.2].

The method in the thesis is close in spirit to the expansion method of Levi [28] and the directional convergence method of Hestense [22]. Applying these methods one estimates the normalized functional increment corresponding to a given strong variation. Taheri [41] used the method of Hestenes to treat problems of L^r-local minima and remarked that the results hold in the vectorial case as well to yield a sufficiency result that is based on convexity, rather than quasiconvexity. Zhang [45] has also succeeded in proving the "local" sufficiency theorem (i.e. a sufficiency theorem that holds for domains that are contained in a sufficiently small ball). In this work we present sufficient conditions without restrictions on the size of the domain. The approach undertaken is the result of the insights achieved in [20], where the necessary conditions for strong local minima are examined in greater generality.

Our method allows us to study the effect of all possible strong variations on a given Lagrangian. Using Young's idea of duality [43] we fix a strong variation and consider its action on a functional space of all admissible Lagrangians. The key technical tool in our analysis is a new version of the decomposition lemma

[15, 26] that permits us to split a given variation into its strong and weak part while simultaneously controlling their growth. Using analytical techniques from [15], we show that the actions of the two parts of the variation on the Lagrangian are independent. We then show that each part contributes a non-negative increment to the variational functional, the weak part—because of the positivity of second variation, the strong part—because of the quasiconvexity conditions.

The same strategy has been used in [19] to obtain sufficiency result for strong local minimizers. As an extension of the work to Young measure local minimizers we follow [19] in allowing all strong variations and show how Stone-Čech compactification method of DiPerna and Majda [10] enables us to handle variations with unbounded gradients, when the Lagrangian has power growth at infinity.

The classical minimizers analyzed in the thesis are of class C^1 and the smoothness assumption is very important. For the $W^{1,p}$ non-smooth extremals, our sufficiency theorems are false, as shown in [27, Corollary 7.3] for the quasiconvex integrands and in [39] for the polyconvex ones. In fact, if the gradient of the extremal vector field has a jump discontinuity, then there are additional necessary conditions (see e.g. [21, Section 4]).

The thesis is organized in the following way. In the remaining part of this chapter we define some notations and discuss Young measures to make the thesis self-contained. In Chapter 2 we will define the different notions of local minimizers and extend the notion to problems that may have no classical minimizers but minimizing sequences. We will also formulate necessary conditions for local minimizers. In Chapter 3 we present a sufficiency result for Young measure local minimizers and show that the result can be specialized to strong local minimizers. The last chapter, Chapter 4 is devoted to the proof of the main result.

1.2 Weak convergence and compactness

We define some terminologies that will be used throughout the thesis. For $1 \leq p < \infty$, we say that $\{u_n\}$ converges weakly to u in $L^p(\Omega)$ and we write $u_n \rightharpoonup u$ in $L^p(\Omega)$ if

$$\int_\Omega u_n(x)g(x)dx \to \int_\Omega u(x)g(x)dx, \quad \forall g \in L^q(\Omega), \quad 1/p + 1/q = 1.$$

For $p = \infty$, $\{u_n\}$ converges weakly * to u in $L^\infty(\Omega)$ written $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ if

$$\int_\Omega u_n(x)g(x)dx \to \int_\Omega u(x)g(x)dx, \quad \forall g \in L^1(\Omega).$$

We use the usual notation $W^{1,p}(\Omega;\mathbb{R}^m)$ for Sobolev spaces and we say a sequence $\boldsymbol{u}_n \in W^{1,p}(\Omega;\mathbb{R}^m)$ converges weakly to $\boldsymbol{u}$, $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ if $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ and $\nabla \boldsymbol{u}_n \rightharpoonup \nabla \boldsymbol{u}$ in L^p. For a weakly convergent sequence $\boldsymbol{u}_n$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ to $\boldsymbol{u}$, $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $L^p(\Omega;\mathbb{R}^m)$, and $\nabla \boldsymbol{u}_n \rightharpoonup \nabla \boldsymbol{u}$ in $L^p(\Omega;\mathbb{M})$.

The following proposition gives a criterion for weak relative compactness in L^p for $1 \leq p \leq \infty$.

Proposition 1 *For $1 < p \leq \infty$, a sequence $\{u_n\}$ is weakly relatively compact in L^p (weak * relatively compact if $p = \infty$) if and only if the sequence is L^p- bounded. i.e $\sup_n \|u_n\|_{L^p} < \infty$. For $p = 1$, the sequence $\{u_n\}$ is weakly relatively compact in L^1 if and only if it is L^1-bounded and equiintegrable.*

Definition 1 *We say that $\{u_n\}$ is equiintegrable if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that for every measurable set E, $|E| < \delta$, we have*

$$\sup_n \int_E |u_n(x)|dx < \epsilon.$$

A failure of equiintegrability implies a concentration phenomenon in the limiting behavior of the sequence, as can be seen on any Dirac delta sequence, for example $u_n(x) = n\chi_{[0,1/n]}(x)$.

By $C_0(\mathbb{R}^n)$ we denote the closure of continuous functions on $\mathbb{R}^n$ with compact support. The dual of $C_0(\mathbb{R}^n)$ can be identified with the space $\mathcal{M}(\mathbb{R}^n)$ of signed Radon measures with finite mass via the dual pairing

$$\langle \nu, f\rangle = \int_{\mathbb{R}^n} f d\nu.$$

Given a sequence $\{\nu_n\} \subset \mathcal{M}(\mathbb{R}^n)$, we say that $\nu_n \overset{*}{\rightharpoonup} \nu$ in the sense of measures if

$$\langle \nu_n, f\rangle \to \langle \nu, f\rangle$$

for all $f \in C_0(\mathbb{R}^n)$. Applying the theorem of Banach-Alaoglue we have the following compactness result for measures, [11].

Proposition 2 *Suppose that the sequence $\{\nu_n\}$ is bounded in $\mathcal{M}(\mathbb{R}^n)$. Then there exists a subsequence $\{\nu_{n_k}\}$ and a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ with $\nu_{n_k} \overset{*}{\rightharpoonup} \nu$ in the sense of measures.*

1.3 Young measures

Young measures (also known as parameterized measures or generalized curves) were first introduced by L. C. Young in his study of the optimal control theory. A discussion is found in his book [43]. Tartar([44]) also introduced Young measures to study the oscillation effects as well as compactness and existence questions in nonlinear partial differential equations. One of the many application of Young measures is finding weak limits of continuous nonlinear composition with sequences. Assume that $u_n \rightharpoonup u$ in L^1, and let ϕ be a continuous function. It is definitely false that $\phi(u_n)$ converges weakly to $\phi(u)$, as simple examples illustrate. Indeed if we take the sequence $u_n = n\chi_{[0,1/n^2]}(x)$, then $u_n \rightharpoonup 0$ in L^1. However, for $\phi(x) = x^2$, $\phi(u_n) = n^2\chi_{[0,1/n^2]}$ does not converge weakly in L^1. Young measures will enable us to have a measure-theoretic characterization of the incompatibility of weak convergence and nonlinear composition.

We follow [33] in defining Young measures. More about Young measures can be found in [43], [44] and [35].

A map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^n)$ is called weak* measurable if the functions

$$\boldsymbol{x} \mapsto \langle \nu(\boldsymbol{x}), f \rangle$$

are measurable for all $f \in C_0(\mathbb{R}^n)$. We say that $\nu(\boldsymbol{x}) \geq 0$ if $\langle \nu(\boldsymbol{x}), f \rangle \geq 0$, for all $f \geq 0$, and $f \in C_0^\infty$. We write $\nu_{\boldsymbol{x}}$ instead of $\nu(\boldsymbol{x})$.

Theorem 1 (Fundamental theorem on Young measures) *Let $\boldsymbol{z}_n : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions. Then there exists a subsequence $\boldsymbol{z}_{n_k}$ and a weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ such that the following holds.*

(i) $\nu_{\boldsymbol{x}} \geq 0, \|\nu_{\boldsymbol{x}}\|_{\mathcal{M}(\mathbb{R}^m)} = \int_{\mathbb{R}^m} d\nu_{\boldsymbol{x}} \leq 1,$ *for a.e.* $\boldsymbol{x} \in \Omega$

(ii) For all $f \in C_0(\mathbb{R}^m)$

$$f(\boldsymbol{z}_{n_k}) \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty(\Omega)$$

where

$$\overline{f}(\boldsymbol{x}) = \langle \nu_{\boldsymbol{x}}, f \rangle = \int_{\mathbb{R}^m} f d\nu_{\boldsymbol{x}}.$$

(iii) Let $K \subset \mathbb{R}^m$ *be compact. Then supp* $\nu_x \subset K$ *for a.e.* $\boldsymbol{x} \in \Omega$ *if dist*$(\boldsymbol{z}_{n_k}, K) \to 0$ *in measure.*

(iv) $\nu_{\boldsymbol{x}}$ *is a probability measure for a.e.* $\boldsymbol{x} \in \Omega$ *if*

$$\lim_{M\to\infty} \sup_k |\{|\boldsymbol{z}_{n_k}| \geq M\}| = 0 \tag{1.2}$$

(v) If (1.2) holds and $\Omega_0 \subset \Omega$ *measurable then whenever the sequence* $\{f(\boldsymbol{x}, \boldsymbol{z}_{n_k}(\boldsymbol{x}))\}$ *is weakly convergent in* $L^1(\Omega_0)$ *for any continuous function* $f(\boldsymbol{x}, \boldsymbol{z}) : \Omega_0 \times \mathbb{R}^m \to \mathbb{R}$*, then*

$$f(\boldsymbol{x}, \boldsymbol{z}_{n_k}) \rightharpoonup \overline{f} \quad \text{in } L^1(\Omega_0), \quad \overline{f}(\boldsymbol{x}) = \langle \nu_{\boldsymbol{x}}, f(\boldsymbol{x}, \cdot) \rangle.$$

See [35] or [33] for the proof.

Definition 2 *We call* $\nu = \{\nu_x\}$ *the family of Young measures generated by the sequence* $\boldsymbol{z}_{n_k}$.

Remark 1 Condition (1.2) is satisfied if there exists a continuous, nondecreasing function $g : [0, \infty) \to [0, \infty]$, $\lim_{t\to\infty} g(t) = \infty$ such that

$$\sup_k \int_\Omega g(|\boldsymbol{z}_{n_k}|)d\boldsymbol{x} < \infty.$$

Indeed, because g is nondecreasing,

$$g(M) \sup_k |\{|\boldsymbol{z}_{n_k}\}| \geq M\}| \leq \sup_k \int_\Omega g(|\boldsymbol{z}_{n_k}|)d\boldsymbol{x} < \infty$$

and because $\lim_{M\to\infty} g(M) = \infty$ we must have (1.2). By taking $g(t) = t^p$, for any $p > 0$, we see that every bounded sequence in $L^p(\Omega)$ contains a subsequence that generates a family of probability measures.

Remark 2 As a dual of the set continuous functions vanishing at infinity, Young measures are identified by their action on $C_0(\mathbb{R}^n)$. Thus property *(ii)* of Theorem 1 defines ν. This observation can be used to prove the following corollary which will be useful later.

Corollary 1 *[35] Suppose that the sequences* $\boldsymbol{z}_n$ *and* $\boldsymbol{w}_n$ *are bounded in* $L^p(\Omega; \mathbb{R}^m)$. *Then if* $|\{\boldsymbol{z}_n \neq \boldsymbol{w}_n\}| \to 0$, *then the Young measures generated by the sequences are the same.*

PROOF: By the above observation we only need to show that for all $f \in C_0(\mathbb{R}^n)$, the sequences $f(\boldsymbol{z}_n)$ and $f(\boldsymbol{w}_n)$ weak* converge to the same value in $L^\infty(\Omega)$. Thus it suffices to show that if $\xi(\boldsymbol{x}) \in L^1(\Omega)$, then

$$|\int_\Omega (\xi(\boldsymbol{x})(f(\boldsymbol{z}_n(\boldsymbol{x})) - f(\boldsymbol{w}_n)))\, |dx \to 0.$$

But this follows from the assumption and the absolute continuity of the integral since

$$|\int_\Omega (\xi(\boldsymbol{x})(f(\boldsymbol{z}_n(\boldsymbol{x})) - f(\boldsymbol{w}_n)))\, |dx \leq \int_{\{\boldsymbol{z}_n \neq \boldsymbol{w}_n\}} 2\|f\|_\infty |\xi(\boldsymbol{x})| d\boldsymbol{x} \to 0$$

as $n \to \infty$. ∎

From the corollary it follows that Young measures completely miss concentration effects of sequences. Indeed, two sequences one exhibiting concentration and the other one with the concentration cut off may generate the same Young measure.

Example 1 Consider the periodic extension of the sawtooth function

$$s(x) = \begin{cases} x & \text{on } [0, 1/4) \\ 1/2 - x & \text{on } [1/4, 3/4) \\ x - 1 & \text{on } [3/4, 1] \end{cases}$$

and consider the sequence $z_n(x) = n^{-1}s(nx)$. Obviously for any $f \in C_0(\mathbb{R})$, we have $f(z_n) \to f(0)$ uniformly, and so the associated Young measure is $\nu = \{\delta_0\}$. However if we consider the sequences of gradients $\{\nabla z_n = z_n'\}$, then for any $f \in C_0(\mathbb{R})$ we have $f(z_n'(x))$ is a periodic function of period 1. By Riemann-Lebesgue Lemma,

$$f(z_n'(x)) \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty([0, 1])$$

where $\overline{f}(x)$ is the average $\frac{1}{2}f(1) + \frac{1}{2}f(-1)$. We may write $\overline{f}$ as $\overline{f} = \langle \nu, f \rangle$, where $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ is the Young measure associated to the sequence of derivatives.

Example 2 Following the above example, let $\Omega = [0, 1] \times [0, 1]$. Then the Young measure associated to the gradient of the sequence

$$z_n(x_1, x_2) = n^{-1}s(nx_2)$$

is $\nu = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$ where $A = (0, 1)$, and $B = (0, -1)$.

Motivated by non-linear elasticity where the free energy associated to a particular deformation depends on the deformation gradient, we are interested in Young measures generated by sequences of gradients. A characterization of gradient Young measures is given by Kinderlehrer and Pedregal and can be found in [35]. The following lemma(Lemma 8.3, pg 138 [35]) helps us to incorporate boundary values for a new gradient sequence generating a Young measure.

Lemma 1 *Let* $\boldsymbol{v}_n$ *be a bounded sequence in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *such that the sequence* $\{\nabla \boldsymbol{v}_n\}$ *generates the Young measure* $\nu = \{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Omega}$*, and* $\boldsymbol{v}_n \rightharpoonup \boldsymbol{u}$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$*. Then there exists a new sequence* $\{\boldsymbol{u}_k\}$*, bounded in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *such that* $\{\nabla \boldsymbol{u}_k\}$ *generates the same Young measure* ν *and* $\boldsymbol{u}_k - \boldsymbol{u} \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ *for all* k*. If for* $p<\infty$ $\{|\nabla \boldsymbol{v}_n|^p\}$ *is equiintegrable, then so is* $\{|\nabla \boldsymbol{u}_k|^p\}$*.*

The representation of weak limits of sequences of type $\{f(\nabla \boldsymbol{z}_n\}$ for f a continuous function in terms of Young measures is only valid if one can rule out concentration effects. There are several tools to account possible development of concentrations. They can be considered as generalizations of Young measures as presented by DiPerna and Majda, [10]. There is also the approach of Fonseca, [14] where a measure, Λ on $\Omega\times\mathbb{S}$ is associated to the sequence $\{\nabla \boldsymbol{z}_i\}$to capture concentration effects. Here $\mathbb{S}$ is the unit sphere in $\mathbb{R}^m$. First observe that given π a finite nonnegative measure on Ω and $\boldsymbol{\alpha}:\Omega\to\mathbb{S}$ π-measurable, one can define a finite, nonnegative Radon measure $\Lambda = \delta_{\boldsymbol{\alpha}(\boldsymbol{x})}\otimes\pi$ on $\Omega\times\mathbb{R}^m$. Then $supp(\Lambda)\subset \quad supp(\pi)\times\mathbb{S}$ and

$$\int_{\Omega\times\mathbb{R}^m} f(\boldsymbol{x},\boldsymbol{y})d\Lambda(\boldsymbol{x},\boldsymbol{y}) = \int_{\Omega} f(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{x}))d\pi(\boldsymbol{x}),$$

for any Λ integrable function f.

Definition 3 *A* varifold *is a nonnegative measure* Λ *on* $\Omega\times\mathbb{S}$*.*

Fonseca's Varifold Theorem says that every bounded sequence of varifold of the form $\Lambda_n = \delta_{\boldsymbol{\alpha}_n(\boldsymbol{x})}\otimes\pi_n$ has a varifold limit.

Theorem 2 *Suppose* $\Lambda_n = \delta_{\boldsymbol{\alpha}_n(\boldsymbol{x})}\otimes\pi_n$ *is a bounded sequence of varifolds, where* π_n *is a measure on* $\overline{\Omega}$ *and* $\boldsymbol{\alpha}_n:\overline{\Omega}\to\mathbb{S}$ *is* π_n *measurable. Then there exists a subsequence,* Λ_n*, not relabeled, a positive measure* π *on* $\overline{\Omega}$ *and a family of probability measures* $\lambda_{\boldsymbol{x}}$ *on* $\mathbb{S}$ *such that for every* $f\in\mathcal{C}_0(\overline{\Omega}\times\mathbb{R}^m)$

$$\begin{aligned}\lim_{n\to\infty}\int_{\overline{\Omega}\times\mathbb{R}^m} f(\boldsymbol{x},\boldsymbol{y})d\Lambda_n(\boldsymbol{x},\boldsymbol{y}) &= \lim_{n\to\infty}\int_{\Omega} f(\boldsymbol{x},\boldsymbol{\alpha}_n(\boldsymbol{x}))d\pi_n(\boldsymbol{x})\\ &= \int_{\overline{\Omega}}\left(\int_{\mathbb{S}} f(\boldsymbol{x},\boldsymbol{y})d\lambda_{\boldsymbol{x}}(\boldsymbol{y})\right)d\pi(\boldsymbol{x}).\end{aligned}$$

Remark 3 The support of $\Lambda = \lambda_x \otimes \pi$ is contained in $\overline{\Omega} \times \mathbb{S}$

Remark 4 Given a bounded sequence $\boldsymbol{u}_n \in W^{1,p}(\Omega;\mathbb{R}^m)$ we can apply Theorem 2 to the varifold $\Lambda_n = \delta_{\frac{\nabla \boldsymbol{u}_n(\boldsymbol{x})}{|\nabla \boldsymbol{u}_n(\boldsymbol{x})|}} \otimes |\nabla \boldsymbol{u}_n|^p$ to obtain a measure $\Lambda = \lambda_{\boldsymbol{x}} \otimes \pi$ such that

$$\int_\Omega f(\boldsymbol{x}, \frac{\nabla \boldsymbol{u}_n(\boldsymbol{x})}{|\nabla \boldsymbol{u}_n(\boldsymbol{x})|})|\nabla \boldsymbol{u}_n(\boldsymbol{x})|^p d\boldsymbol{x} \to \int_{\overline{\Omega}} \int_{\mathbb{S}^{md-1}} f(\boldsymbol{x}, \boldsymbol{y}) d\lambda_x(\boldsymbol{y}) d\pi(\boldsymbol{x}).$$

where $\mathbb{S}^{md-1}$ is the unit sphere in $\mathbb{M}$. In particular, if $f(\boldsymbol{x}, \boldsymbol{y}) \in C(\overline{\Omega} \times \mathbb{M})$ is homogeneous of degree p in the second variable, then

$$\lim_{n \to \infty} \int_\Omega f(\boldsymbol{x}, \nabla \boldsymbol{u}_n(\boldsymbol{x})) d\boldsymbol{x} = \int_{\overline{\Omega}} \int_{\mathbb{S}^{md-1}} f(\boldsymbol{x}, \boldsymbol{y}) d\lambda_x(\boldsymbol{y}) d\pi(\boldsymbol{x}).$$

CHAPTER 2

LOCAL MINIMIZERS AND NECESSARY CONDITIONS

2.1 Local minimizers

The notion of local minimizer of the integral function E defined in (1.1), in contrast with the global one, depends in an essential way on the topology on the space $\mathcal{A}$ of functions on which the variational functional is defined. The classical notions of strong and weak local minima correspond to the L^∞, and $W^{1,\infty}$ topologies on $\mathcal{A}$ respectively. We define the two notions below.

Definition 4 *We say that $\boldsymbol{y} \in \mathcal{A}$ is a weak local minimizer of E, if there exists an $\epsilon > 0$ such that $E(\boldsymbol{y}) \leq E(\widetilde{\boldsymbol{y}})$ for all $\widetilde{\boldsymbol{y}} \in \mathcal{A}$ that satisfy $\|\widetilde{\boldsymbol{y}} - \boldsymbol{y}\|_{L^\infty} < \epsilon$, and $\|\nabla\widetilde{\boldsymbol{y}} - \nabla\boldsymbol{y}\|_{L^\infty} < \epsilon$.*

Definition 5 *We say that $\boldsymbol{y} \in \mathcal{A}$ is a strong local minimizer of E, if there exists an $\epsilon > 0$ such that $E(\boldsymbol{y}) \leq E(\widetilde{\boldsymbol{y}})$ for all $\widetilde{\boldsymbol{y}} \in \mathcal{A}$ that satisfy $\|\widetilde{\boldsymbol{y}} - \boldsymbol{y}\|_{L^\infty} < \epsilon$.*

We observe that the notion of strong local minima is stronger than that of the weak one.

Example 3 *[5]* Consider the one dimensional variational problem of locally minimizing

$$I(y) = \int_0^1 (y'^2(x) - y'^4(x))dx,$$

over the set

$$\{y : [0,1] \to \mathbb{R} : y \quad \text{is absolutely continuous on} \quad [0,1], y(0) = y(1) = 0\}$$

Here $y' = \frac{dy}{dx}$. The constant function $y = 0$ is a weak local minimizer. Indeed, for small $\epsilon > 0$, and $\|y'\| < \epsilon$ we have

$$I(y) \geq (1 - \epsilon^2) \int_0^1 y'^2 > 0 = I(0).$$

However $y = 0$ is not a strong local minimizer. To see this we construct a sequence of admissible functions which converges to 0 uniformly but their functional value is negative. Consider the periodic sawtooth function defined in Example 1. Define

$$y_n(x) = 2\frac{1}{n}s(nx).$$

Then $y_n \to 0$ uniformly and

$$I(y_n) = \int_0^1 (4s'(x)^2 - 16s'(x)^4)dx = -12 < 0 = I(0),$$

as desired.

Let us formulate the definition of local minima in terms of sequences. Let

$$\text{Var}(\mathcal{A}) = \{\boldsymbol{\phi} \in W^{1,\infty}(\Omega; \mathbb{R}^m) : \boldsymbol{\phi}|_{\partial\Omega_1} = 0\}.$$

We will call $\text{Var}(\mathcal{A})$ the space of variations because for any $\boldsymbol{\phi} \in \text{Var}(\mathcal{A})$ and for any $\boldsymbol{y} \in \mathcal{A}$ we have $\boldsymbol{y} + \boldsymbol{\phi} \in \mathcal{A}$.

Definition 6 *A strong (weak)variation is a sequence* $\{\boldsymbol{\phi}_n\} \subset Var(\mathcal{A})$ *such that* $\boldsymbol{\phi}_n \to \mathbf{0}$ *in* $L^\infty(\Omega; \mathbb{R}^m)$ $(W^{1,\infty}(\Omega; \mathbb{R}^m))$.

As an example, the sequence $\boldsymbol{\phi}_n = \frac{1}{n}\boldsymbol{\phi}$ where $\boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A})$ is a weak variation where as, for any $\boldsymbol{x}_0 \in \Omega$, and $\boldsymbol{\phi} \in C_0^1(\Omega; \mathbb{R}^m)$, the sequence

$$\boldsymbol{\phi}_n(\boldsymbol{x}) = \frac{1}{n}\boldsymbol{\phi}(n(\boldsymbol{x} - \boldsymbol{x}_0))$$

is a strong but not a weak variation. For our purposes it will be convenient to rephrase the definition of a local minimizer in terms of the variations. The map $\boldsymbol{y} \in \mathcal{A}$ is a *strong (weak) local minimizer* if and only if for each strong (weak) variation $\{\boldsymbol{\phi}_n\}$ there exists $N > 0$ such that for any $n \geq N$

$$E(\boldsymbol{y} + \boldsymbol{\phi}_n) \geq E(\boldsymbol{y}). \tag{2.1}$$

In other words, strong variations can not lower the value of the functional at a strong local minimizer. Since the uniform topology of $L^\infty(\overline{\Omega}; \mathbb{R}^m)$ is meterizable, the sequence-based definition is equivalent to Definition 5.

This sequence-based definition enables us to extend the notion of a local minimizer to variational problems that do not even have classical solutions. Such problems appear in Calculus of Variations and in applications like phase transitions in solids, [7], [4]. Let us assume $\{\boldsymbol{y}_j\} \subset \mathcal{A}$ is bounded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Applying Theorem 1, there exists a family of probability measures $\nu = \{\nu_{\boldsymbol{x}}\}$ on $\mathbb{M}$ such that for any $f(\boldsymbol{x}, \boldsymbol{F})$ continuous function on $\overline{\Omega} \times \mathbb{M}$ and $\xi \in L^1(\Omega)$,

$$\int_\Omega f(\boldsymbol{x}, \nabla \boldsymbol{y}_n(\boldsymbol{x}))\xi(\boldsymbol{x})d\boldsymbol{x} \to \int_\Omega \int_{\mathbb{M}} f(\boldsymbol{x}, \boldsymbol{F})\xi(\boldsymbol{x})d\nu_{\boldsymbol{x}}(\boldsymbol{F})d\boldsymbol{x}, \quad \text{as } n \to \infty$$

Here because the sequence $\boldsymbol{y}_n$ is bounded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, $supp(\nu_{\boldsymbol{x}}) \subset B(\mathbf{0}, R)$, for some $R > 0$, for a.e $\boldsymbol{x} \in \Omega$.

Definition 7 *We say that the Young measure $\nu = \{\nu_{\boldsymbol{x}}\}$ is continuous if $\boldsymbol{x}_n \to \boldsymbol{x}$ in Ω implies that*

$$\int_{\mathbb{M}} f(\boldsymbol{x}, \boldsymbol{F})d\nu_{\boldsymbol{x}_n}(\boldsymbol{F}) \to \int_{\mathbb{M}} f(\boldsymbol{x}, \boldsymbol{F})d\nu_{\boldsymbol{x}}(\boldsymbol{F}) \quad \textit{as } n \to \infty.$$

for all $f \in C(\Omega \times \mathbb{M})$.

Example 4 *Probability measures of type*

$$\nu_{\boldsymbol{x}} = \sum_{i=1}^{n} \alpha_i \delta_{A_i(\boldsymbol{x})}, \quad \sum_{i=1}^{n} \alpha_i = 1$$

where A_i is a continuous function for $i = 1, \ldots n$, constitute a continuous Young measure $\nu = \{\nu_{\boldsymbol{x}}\}$.

From now on we assume that $\nu = \{\nu_{\boldsymbol{x}}\}$ is a continuous Young measure and is defined for all $\boldsymbol{x} \in \Omega$.

Definition 8 *We say that $\nu = \{\nu_{\boldsymbol{x}}\}$ is a Young measure local minimizer if for each strong variation $\{\boldsymbol{\phi}_n\} \subset Var(\mathcal{A})$, there exists N such that*

$$\int_\Omega \int_{\mathbb{M}} W(\boldsymbol{x}, \boldsymbol{F} + \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) d\boldsymbol{x} \geq \int_\Omega \int_{\mathbb{M}} W(\boldsymbol{x}, \boldsymbol{F}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) d\boldsymbol{x}$$

for all $n \geq N$.

We remark that this definition reduces to Definition 5, when the sequence $\boldsymbol{y}_j$ converges to $\boldsymbol{y}$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, and $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^m)$. In this case $\nu = \{\delta_{\nabla \boldsymbol{y}(\boldsymbol{x})}\}$

Example 5 *([33], [6])* Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$. Consider minimizing the variational problem

$$E(y) = \int_\Omega y_{x_1}^2 + (y_{x_2}^2 - 1)^2 dx_1 dx_2$$

over the set of all $y \in W_0^{1,\infty}(\Omega; \mathbb{R})$. Notice that the infimum is 0, but cannot be attained. To see this, first observe that $E(y) \geq 0$ for all $y \in W_0^{1,\infty}(\Omega; \mathbb{R})$. Moreover, modifying slightly the sequence discussed in Example 2, we define y_n as

$$y_n(x_1, x_2) = n^{-1} s(n x_2)) \quad \text{for} \quad \delta < x_2 < 1 - \delta$$

and linearly interpolate to achieve the boundary values on $\partial\Omega$. Considering first the limit $n \to \infty$ and then $\delta \to 0$ one obtains that $\inf E = 0$. On the other hand if there were a y such that $E(y) = 0$, then $\nabla y = (0,1)$ a.e. Then

$y = x_2 + c$ contradicting the the boundary condition. It is not difficult to see that the minimizing sequence generates the homogeneous Young measure

$$\nu = \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(0,-1)}.$$

and is a Young measure local minimizer of E.

2.2 Necessary conditions

In this section we derive necessary conditions for a continuous Young measure $\nu = \{\nu_{\boldsymbol{x}}\}$ to be a local minimizer of the functional integral (1.1). These conditions are appropriate generalizations of the classical necessary conditions for an element $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^m)$ to be a local minimizer discussed in [19]. It is customary in Calculus of Variations to assume that $W(\boldsymbol{x}, \boldsymbol{F})$ is of class C^2 in the variable $\boldsymbol{F}$. Here, we assume a little less. We will assume that W is of class C^2 on some open set $\mathcal{O}$ containing the support of $\nu_{\boldsymbol{x}}$ for all $\boldsymbol{x} \in \Omega$. In case $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^m)$ is a strong local minimizer, then the set $\mathcal{O}$ is an open set containing the range of $\nabla \boldsymbol{y}(\boldsymbol{x})$. We denote $\boldsymbol{F}(\boldsymbol{x}) = \nabla \boldsymbol{y}(\boldsymbol{x})$, and the range of $\nabla \boldsymbol{y}(\boldsymbol{x})$ by $\mathcal{R}$.

Suppose that ν is a Young measure local minimizer of (1.1). For $\boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A})$, we denote the functional increment by $\Delta E(\boldsymbol{\phi})$,

$$\Delta E(\boldsymbol{\phi}) = \int_\Omega \int_{\mathbb{M}} \{W(\boldsymbol{x}, \boldsymbol{F} + \nabla \boldsymbol{\phi}) - W(\boldsymbol{x}, \boldsymbol{F})\} d\nu_x(\boldsymbol{F}) d\boldsymbol{x}$$

One may write the increment in a different way using the function $\overline{W}(\boldsymbol{x}, \boldsymbol{G})$ defined as

$$\overline{W}(\boldsymbol{x}, \boldsymbol{G}) = \int_M W(\boldsymbol{x}, \boldsymbol{F} + \boldsymbol{G}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}). \tag{2.2}$$

The function $\overline{W}(\boldsymbol{x}, \boldsymbol{G})$ is continuous in $\boldsymbol{x}$ and C^2 in $\boldsymbol{G}$ in a small neighborhood of $\boldsymbol{0}$, since the probability measure $\nu_{\boldsymbol{x}}$ is continuous and is supported in a compact set. Moreover the derivatives are given by

$$\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{G}) = \int_{\mathbb{M}} W_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{F}+\boldsymbol{G}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}),\ \ \overline{W}_{\boldsymbol{FF}}(\boldsymbol{x}, \boldsymbol{G}) = \int_{\mathbb{M}} W_{\boldsymbol{FF}}(\boldsymbol{x}, \boldsymbol{F}+\boldsymbol{G}) d\nu_{\boldsymbol{x}}(\boldsymbol{F})$$

Then the functional increment corresponding to a variation $\boldsymbol{\phi}$ is given by

$$\Delta E(\boldsymbol{\phi}) = \int_\Omega \{\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}(\boldsymbol{x})) - \overline{W}(\boldsymbol{x}, \mathbf{0})\} d\boldsymbol{x}$$

Perturbing by the weak variations

$$\boldsymbol{\phi}_\epsilon(\boldsymbol{x}) = \epsilon\boldsymbol{\phi}(\boldsymbol{x})$$

for smooth $\boldsymbol{\phi} \in C^1(\overline{\Omega}; \mathbb{R}^m) \cap \mathrm{Var}(\mathcal{A})$, we get the inequality

$$0 \leq \Delta E(\epsilon\boldsymbol{\phi}) = \int_\Omega \overline{W}(\boldsymbol{x}, \epsilon\nabla\boldsymbol{\phi}(\boldsymbol{x})) - \overline{W}(\boldsymbol{x}, \mathbf{0}) d\boldsymbol{x}. \tag{2.3}$$

Using Taylor expansion of $\overline{W}$ around $G = \mathbf{0}$, for small ϵ we have

$$\begin{aligned} \Delta E(\epsilon\boldsymbol{\phi}) =& \epsilon \int_\Omega (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} \\ &+ \frac{\epsilon^2}{2} \int_\Omega (\overline{W}_{\boldsymbol{FF}}(\boldsymbol{x}, \mathbf{0})\nabla\boldsymbol{\phi}(\boldsymbol{x}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} + o(\epsilon^2). \end{aligned} \tag{2.4}$$

Dividing (2.4) by ϵ and letting $\epsilon \to 0$ we get that

$$\int_\Omega (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} \geq 0$$

if $\epsilon > 0$ and

$$\int_\Omega (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} \leq 0$$

if $\epsilon < 0$. Thus we obtain the weak form of the equilibrium equation

$$\int_\Omega (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} = 0, \tag{2.5}$$

for any $\boldsymbol{\phi} \in C^1(\overline{\Omega}; \mathbb{R}^m) \cap \mathrm{Var}(\mathcal{A})$. Applying the Divergence Theorem in the sense of distributions we get,

$$\int_\Omega (\nabla \cdot \overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} + \int_{\partial\Omega_2} (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0})\boldsymbol{n}(\boldsymbol{x}), \boldsymbol{\phi}(\boldsymbol{x})) dS = 0. \tag{2.6}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outer unit normal at $\boldsymbol{x} \in \partial\Omega$.

Applying (2.5) to the Taylor expansion (2.4) we obtain,

$$0 \leq \Delta E(\epsilon\boldsymbol{\phi}) = \frac{\epsilon^2}{2} \int_\Omega (\overline{W}_{\boldsymbol{FF}}(\boldsymbol{x}, \mathbf{0})\nabla\boldsymbol{\phi}(\boldsymbol{x}), \nabla\boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} + o(\epsilon^2).$$

Then dividing both sides of the inequality by ϵ^2, and letting $\epsilon \to 0$, we get another necessary condition

$$\int_\Omega (\overline{W}_{\boldsymbol{FF}}(\boldsymbol{x},\mathbf{0})\nabla\boldsymbol{\phi}(\boldsymbol{x}),\nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq 0 \tag{2.7}$$

for any $\boldsymbol{\phi} \in C^1(\overline{\Omega};\mathbb{R}^m) \cap \mathrm{Var}(\mathcal{A})$. By approximation argument we can show that (2.7) holds for any $\boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A})$.

Definition 9 *The quantity*

$$\delta^2 E = \int_\Omega (\overline{W}_{\boldsymbol{FF}}(\boldsymbol{x},\mathbf{0})\nabla\boldsymbol{\phi}(\boldsymbol{x}),\nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \tag{2.8}$$

is called the second variation of the variational problem (1.1).

Remark 5 We observe that $\boldsymbol{\phi}_\epsilon = \epsilon\boldsymbol{\phi}$ is a weak variation. Therefore by taking $\nu_{\boldsymbol{x}} = \delta_{\nabla \boldsymbol{y}(\boldsymbol{x})}$ (2.6) and (2.7) are also necessary conditions for $\boldsymbol{y} \in C^1$ to be a weak local minimizer of (1.1).

To obtain other necessary conditions we perturb by the strong variation

$$\boldsymbol{\phi}_\epsilon(\boldsymbol{x}) = \epsilon\boldsymbol{\phi}(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\epsilon}) \tag{2.9}$$

where $\boldsymbol{\phi} \in C_0^\infty(B(\mathbf{0},1);\mathbb{R}^m)$ and $\boldsymbol{x}_0 \in \Omega$. Then

$$\Delta E(\boldsymbol{\phi}_\epsilon) = \int_\Omega (\overline{W}(\boldsymbol{x},\nabla\boldsymbol{\phi}(\frac{\boldsymbol{x}-\boldsymbol{x}_0}{\epsilon})) - \overline{W}(\boldsymbol{x},\mathbf{0}))d\boldsymbol{x} \geq 0$$

The above integration is actually on $B(\boldsymbol{x}_0,\epsilon)$ since $\boldsymbol{\phi}$ is supported on $B(\mathbf{0},1)$. Changing variables,

$$\boldsymbol{z} = \frac{\boldsymbol{x}-\boldsymbol{x}_0}{\epsilon}, \quad \boldsymbol{x} = \boldsymbol{x}_0 + \epsilon\boldsymbol{z}, \quad d\boldsymbol{x} = \epsilon^d d\boldsymbol{z},$$

we have

$$\Delta E(\boldsymbol{\phi}_\epsilon) = \epsilon^d \int_{B(\mathbf{0},1)} \{\overline{W}(\boldsymbol{x}_0+\epsilon\boldsymbol{z},\nabla\boldsymbol{\phi}(\boldsymbol{z})) - \overline{W}(\boldsymbol{x}_0+\epsilon\boldsymbol{z},\mathbf{0})\}d\boldsymbol{z}$$

Dividing by ϵ^d, and taking the limit as $\epsilon \to 0^+$, we get

$$\lim_{\epsilon\to 0^+} \int_{B(\mathbf{0},1)} \{\overline{W}(\boldsymbol{x}_0+\epsilon\boldsymbol{z},\nabla\boldsymbol{\phi}(\boldsymbol{z})) - \overline{W}(\boldsymbol{x}_0+\epsilon\boldsymbol{z},\mathbf{0})\}d\boldsymbol{z} \geq 0.$$

By continuity of $\overline{W}$ and applying Bounded Convergence Theorem we obtain

$$\int_{B(\mathbf{0},1)} \overline{W}(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{z}))d\boldsymbol{z} \geq |B(\mathbf{0},1)|\overline{W}(\boldsymbol{x}_0, \mathbf{0}). \tag{2.10}$$

By an approximation argument (2.10) holds for any $\boldsymbol{\phi}(\boldsymbol{z}) \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$.

Definition 10 (Quasiconvexity) *We say that a continuous function* $W : \mathbb{M} \to \mathbb{R}$ *is* quasiconvex at $\boldsymbol{F}$, if

$$\int_{B(\mathbf{0},1)} W(\boldsymbol{F} + \nabla\boldsymbol{\phi}(\boldsymbol{z}))d\boldsymbol{z} \geq |B(\mathbf{0},1)|W(\boldsymbol{F}) \tag{2.11}$$

for any $\boldsymbol{\phi}(\boldsymbol{z}) \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$. W is quasiconvex if it is quasiconvex at each $\boldsymbol{F} \in \mathbb{M}$

The above calculation shows that if ν is a Young measure local minimizer, then $\overline{W}(\boldsymbol{x}_0, \cdot)$ is quasiconvex at $\boldsymbol{G} = \mathbf{0}$ for all $\boldsymbol{x}_0 \in \Omega$.

Similarly for $\boldsymbol{x}_0 \in \partial\Omega_2$, using the strong variations

$$\boldsymbol{\phi}_\epsilon = \epsilon\boldsymbol{\phi}(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\epsilon}) \in Var(\mathcal{A}),$$

and after change of variables, we get

$$\Delta E(\boldsymbol{\phi}_\epsilon) = \epsilon^d \int_{B(\mathbf{0},1)\cap\frac{\Omega-\boldsymbol{x}_0}{\epsilon}} \overline{W}(\boldsymbol{x}_0 + \epsilon\boldsymbol{z}, \nabla\boldsymbol{\phi}(\boldsymbol{z})) - \overline{W}(\boldsymbol{x}_0 + \epsilon\boldsymbol{z}, \mathbf{0})d\boldsymbol{z}$$

Dividing by ϵ^d and letting $\epsilon \to 0$, we obtain

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)} \overline{W}(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{z}))d\boldsymbol{z} \geq |B(\mathbf{0},1)|\overline{W}(\boldsymbol{x}_0, \mathbf{0}) \tag{2.12}$$

where

$$B^-_{\boldsymbol{n}}(\mathbf{0},1) = \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \leq 1, (\boldsymbol{x}, \boldsymbol{n}) < 0\},$$

is the half ball whose outer unit normal at the flat part of its boundary is equal to $\boldsymbol{n}$. If (2.12) holds we say that W is *quasiconvex at the free boundary point* $\boldsymbol{x}_0$.

The following theorem summarizes the above discussion:

Theorem 3 *Let $\nu = \nu_{\boldsymbol{x}}$ be a Young measure local minimizer of (1.1). Then*

i) The equilibrium equations are satisfied weakly

$$\begin{cases} \nabla \cdot \overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0} & \textit{on } \Omega, \\ \overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0})\boldsymbol{n}(\boldsymbol{x}) = \mathbf{0} & \textit{on } \partial\Omega_2 \end{cases} \tag{2.13}$$

(ii) The second variation (2.7) is nonnegative for all $\boldsymbol{\phi} \in Var(\mathcal{A})$.

(iii) Quasiconvexity inequalities (2.10) and (2.12) hold for all $\boldsymbol{\phi} \in W_0^{1,\infty}(B(\mathbf{0}, 1); \mathbb{R}^m)$.

Remark 6 In case the Young measure $\nu_{\boldsymbol{x}} = \delta_{\nabla \boldsymbol{y}(\boldsymbol{x})}$ for $\boldsymbol{y} \in C^1(\overline{\Omega}; \mathbb{R}^m)$, i.e if $\boldsymbol{y}$ is a strong local minimizer, then we recover the necessary conditions discussed in [19] .

Remark 7 (2.12) is stronger than (2.10). Indeed (2.12) depends nontrivially on the fact that $\boldsymbol{\phi}(\boldsymbol{z}) \in C_0^\infty(B(\mathbf{0}, 1); \mathbb{R}^m)$ does not have to vanish on the 'flat side ' of $B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0}, 1)$, while the continuity of ν implies that (2.10) holds for any $\boldsymbol{x} \in \partial\Omega$. There are examples for which (2.10) holds for all for any $\boldsymbol{x} \in \partial\Omega$, but (2.12) fails for at least one point on $\partial\Omega_2$, [5].

Furthermore inequalities (2.10) and (2.12) can be rewritten in a different way. To this end notice that the equilibrium equations (2.13) can be completely decoupled from the other necessary conditions for strong local minima. This is done by replacing the functional increment

$$\Delta E(\boldsymbol{\phi}) = \int_\Omega (\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}(\boldsymbol{x})) - \overline{W}(\boldsymbol{x}, \mathbf{0}))d\boldsymbol{x}$$

by

$$\Delta' E(\boldsymbol{\phi}) = \int_\Omega \overline{W}^\circ(\boldsymbol{x}, \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x}, \tag{2.14}$$

where

$$\overline{W}^\circ(\boldsymbol{x}, \boldsymbol{F}) = \overline{W}(\boldsymbol{x}, \boldsymbol{F}) - \overline{W}(\boldsymbol{x}, \mathbf{0}) - (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}, \mathbf{0}), \boldsymbol{F}) \tag{2.15}$$

is related to the Weierstrass excess function. The role of the equilibruim equations (2.13) is, therefore, to establish equivalence between $\Delta' E(\boldsymbol{\phi})$ and

$\Delta E(\boldsymbol{\phi})$. In this case the quasiconvexity inequalities (2.10) and (2.12) can be written in terms of $\overline{W}^{\circ}$. Indeed, (2.10) can be written as

$$\int_{B(\mathbf{0},1)} \overline{W}^{\circ}(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq 0 \tag{2.16}$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$, because

$$\int_{B(\mathbf{0},1)} (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}_0, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} = 0.$$

Similarly the quasiconvexity at the free boundary condition (2.12) can be written as

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)} \overline{W}^{\circ}(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq 0 \tag{2.17}$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$, because

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)} (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}_0, \mathbf{0}), \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} = 0.$$

The vanishing of the last integral occurs because of the boundary condition in (2.13).

The quasiconvexity inequality (2.10) was first introduced by Morrey [31] as a necessary and sufficient condition for sequential weak-* lower semicontinuity of the integral functional (1.1). That (2.10)is a necessary condition for local minimizers was proved in Meyers [30]. The necessity of (2.12) for strong local minimizers via the variation (2.9) is due to Ball [5]. As noted by Ball the quasiconvexity conditions (2.10) and (2.12) are generalizations of the Weierstrass condition of one-dimensional calculus of variations. To see this, we begin by stating the following proposition. The proof can be found in [2].

Proposition 3 *The quasiconvexity inequality (2.10) implies that*

$$\overline{W}(\boldsymbol{x}_0, \mathbf{0}) \leq \lambda\overline{W}(\boldsymbol{x}_0, \boldsymbol{a}\otimes\boldsymbol{b}) + (1-\lambda)\overline{W}(\boldsymbol{x}_0, \frac{\lambda}{\lambda-1}\boldsymbol{a}\otimes\boldsymbol{b}) \tag{2.18}$$

for all $\lambda \in (0,1)$*, and all* $\boldsymbol{a} \in \mathbb{R}^m$*,* $\boldsymbol{b} \in \mathbb{R}^d$*.*

Remark 8 Replacing $\boldsymbol{a}$ by $(1-\lambda)\boldsymbol{a}$, dividing by λ and finally letting $\lambda \to 0$, we obtain from (2.18) that

$$\overline{W}(\boldsymbol{x}_0, \boldsymbol{a}\otimes\boldsymbol{b}) - \overline{W}(\boldsymbol{x}_0, \mathbf{0}) - (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}_0, \mathbf{0}), \boldsymbol{a}\otimes\boldsymbol{b}) \geq 0. \tag{2.19}$$

Definition 11 *We say that W is rank one convex at $\boldsymbol{F}$ if*

$$W(\boldsymbol{x}, \boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{b}) - W(\boldsymbol{x}, \boldsymbol{F}) - (W_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{F}), \boldsymbol{a}\otimes\boldsymbol{b}) \geq 0$$

for all $\boldsymbol{a} \in \mathbb{R}^m$, and $\boldsymbol{b} \in \mathbb{R}^d$. W is rank one convex if it is rank one convex at each $\boldsymbol{F} \in \mathbb{M}$.

Proposition 3 and the remark following it say that quasiconvexity of $\overline{W}(\boldsymbol{x}_0, \cdot)$ at $\boldsymbol{G} = \mathbf{0}$ implies that of rank one convexity at the same point. In the case when $d = 1$ or $m = 1$ the rank one convexity condition (2.19) at $\mathbf{0}$ reduces to the well known Weierstrass convexity condition.

Proposition 4 *When $min(m, d) = 1$, then the quasiconvexity inequality (2.10) is equivalent to the Weierstrass convexity condition:*

$$\overline{W}(\boldsymbol{x}_0, \boldsymbol{f}) - \overline{W}(\boldsymbol{x}_0, \mathbf{0})) - (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x}_0, \mathbf{0}), \boldsymbol{f}) \geq 0. \tag{2.20}$$

for all $\boldsymbol{f} \in \mathbb{R}^{m\times d}$; in this case the two notions quasiconvexity and rank one convexity coincide with the usual notion of convexity.

For $d, m > 1$, the three notions of convexity are different, see [35] for more discussion.

CHAPTER 3

SUFFICIENT CONDITIONS FOR LOCAL MINIMIZERS

In this chapter we present a set of sufficient conditions that are slight strengthening of the necessary conditions for local minimizers discussed in the previous chapter. Given a solution of the equilibrium equations, these conditions make sure that perturbation by strong variations will not lower the value of the integral functional, E.

As discussed in [19] the most salient feature of strong variations is a complete absence of any control on the behavior of $\nabla\boldsymbol{\phi}_n$. It is easy to produce a strong variation whose gradients form an unbounded sequence in any of the L^p spaces. In prior work in [18], we avoided this problem simply by restricting our attention to strong variations with uniformly bounded gradients. In this thesis we gain some control on the gradients of the variation by imposing the super-quadratic coercivity condition on the Lagrangian

$$W(\boldsymbol{x},\boldsymbol{F}) \geq c(|\boldsymbol{F}|^p - 1), \tag{3.1}$$

for all $\boldsymbol{F} \in \mathbb{M}$ and some $c > 0$ and $p \geq 2$. The coercivity condition (3.1), however, is insufficient to eliminate any need to consider variations with unbounded gradients and our method demands certain regularity of W at infinity. In order to formulate this regularity condition, we consider the set X_p of continuous functions $W(\boldsymbol{x}, \boldsymbol{F})$ satisfying

$$|W(\boldsymbol{x}, \boldsymbol{F})| \leq c(1 + |\boldsymbol{F}|^p) \tag{3.2}$$

for all $\boldsymbol{x} \in \overline{\Omega}$, $\boldsymbol{F} \in \mathbb{M}$ and some $c > 0$. The set X_p is a Banach space with respect to the norm

$$\|W\|_{X_p} = \sup_{\boldsymbol{F} \in \mathbb{M}} \max_{\boldsymbol{x} \in \overline{\Omega}} \frac{|W(\boldsymbol{x}, \boldsymbol{F})|}{1 + |\boldsymbol{F}|^p}.$$

Let

$$\mathcal{L} = \{V \in X_p : |V(\boldsymbol{x}, \boldsymbol{F}) - V(\boldsymbol{x}, \boldsymbol{G})| \leq C(1 + |\boldsymbol{F}|^{p-1} + |\boldsymbol{G}|^{p-1})|\boldsymbol{F} - \boldsymbol{G}|\}, \tag{3.3}$$

where the inequality in (3.3) holds for all $\boldsymbol{x} \in \Omega$, $\{\boldsymbol{F}, \boldsymbol{G}\} \subset \mathbb{M}$ and some constant $C > 0$. The set $\mathcal{L}$ is a linear subspace in X_p. Our regularity assumption on W is

$$W \in \overline{\mathcal{L}}, \tag{3.4}$$

where the closure is taken in X_p. Further discussion can be found in [19].

We remark that if $W \in X_p$ is assumed to be globally quasiconvex (a condition we do not impose in this paper), then W must necessarily belong to $\mathcal{L}$ (see [29] and also [16, p. 120, (7)]).

The following lemma deduces growth conditions for $\overline{W}$ from that of W. We begin by recalling that for ν a continuous Young measure (see (2.2)), $\overline{W}$ is defined as

$$\overline{W}(\boldsymbol{x}, \boldsymbol{G}) = \int_{\mathbb{M}} W(\boldsymbol{x}, \boldsymbol{F} + \boldsymbol{G}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}).$$

The proof of the lemma is an easy consequence of the fact that ν is compactly supported.

Lemma 2 *Given a continuous Young measure ν which is compactly supported, if W belongs to a class satisfying (3.1), (3.2) or (3.4), so does $\overline{W}$.*

PROOF: We first observe that given $R > 0$, we can find constants $c, C > 0$ such that for all $\boldsymbol{G} \in \mathbb{M}$ and $\boldsymbol{F} \in B(\boldsymbol{0}, R)$

$$|\boldsymbol{F} + \boldsymbol{G}|^p \geq c|\boldsymbol{G}|^p - C \tag{3.5}$$

This is because the fraction

$$\frac{|\boldsymbol{F} + \boldsymbol{G}|^p}{|\boldsymbol{G}|^p} \to 1$$

when $|\boldsymbol{G}| \to \infty$, as long as $\boldsymbol{F}$ remains in a bounded set. From (3.5), it is easy to see that if W satisfies (3.1) then so does $\overline{W}$. Moreover if W satisfies (3.2), then

$$|\overline{W}(\boldsymbol{x}, \boldsymbol{G})| \leq C \int_{\mathbb{M}} (1 + |\boldsymbol{F} + \boldsymbol{G}|^p) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) \leq C \int_{\mathbb{M}} (1 + |\boldsymbol{F}|^p + |\boldsymbol{G}|^p) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) \tag{3.6}$$

But since $\nu_{\boldsymbol{x}}$ is compactly supported,

$$\int_{\mathbb{M}} |\boldsymbol{F}|^p d\nu_{\boldsymbol{x}}(\boldsymbol{F}) \leq M, \quad \text{for some } M > 0. \tag{3.7}$$

Using the inequality (3.7) and the fact that $\nu_{\boldsymbol{x}}$ is a probability measure we obtain from (3.6) that $\overline{W}$ also satisfy (3.2).

To prove the last part, first we prove that if $W \in \mathcal{L}$, then so is $\overline{W}$. To this end,

$$|\overline{W}(\boldsymbol{x}, \boldsymbol{F}) - \overline{W}(\boldsymbol{x}, \boldsymbol{G})| \leq \int_{\mathbb{M}} |W(\boldsymbol{x}, \boldsymbol{F} + \zeta) - W(\boldsymbol{x}, \boldsymbol{G} + \zeta)| d\nu_{\boldsymbol{x}}(\zeta)$$

Using the estimate (3.3), we obtain that

$$|\overline{W}(\boldsymbol{x}, \boldsymbol{F}) - \overline{W}(\boldsymbol{x}, \boldsymbol{G})| \leq C\Big(\int_{\mathbb{M}} (1 + |\boldsymbol{F} + \zeta|^{p-1} + |\boldsymbol{G} + \zeta|^{p-1}) d\nu_{\boldsymbol{x}}(\zeta)\Big)|\boldsymbol{F} - \boldsymbol{G}|$$

Using the estimate $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, and the fact that ν_x is compactly supported, we obtain that $\overline{W} \in \mathcal{L}$. Finally, if $W \in \overline{L}$, then there exist a sequence $W_n \in \mathcal{L}$, such that $W_n \to W$ in X_p. Then $\overline{W}_n \in \mathcal{L}$, and $\overline{W_n} \to \overline{W}$ in X_p. This completes the proof of the lemma. ∎

3.1 The main result

Our sufficient conditions for a Young measure local minimum consist of the equilibrium equation (2.13) and a natural strengthening of the necessary conditions (2.7), (2.10) and (2.12).

Theorem 4 *Assume that $W \in \overline{\mathcal{L}}$ satisfies (3.1) and that W is of class C^2 in $\mathcal{O}$. Suppose that ν is a continuous Young measure generated by a bounded sequence of gradients of elements of $\mathcal{A}$ and it satisfies*

(i) Equilibrium equation (2.13) weakly,

(ii) There exists $\beta > 0$ such that

$$\delta^2 E(\boldsymbol{\phi}) = \int_\Omega (\mathsf{L}(\boldsymbol{x})\nabla\boldsymbol{\phi}(\boldsymbol{x}), \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq \beta \int_\Omega |\nabla\boldsymbol{\phi}(\boldsymbol{x})|^2 d\boldsymbol{x} \qquad \text{for all } \boldsymbol{\phi} \in Var(\mathcal{A}).$$

and

(iii) (a) for all $\boldsymbol{x}_0 \in \Omega$

$$\int_{B(\mathbf{0},1)} \overline{W}^\circ(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq \beta \int_{B(\mathbf{0},1)} |\nabla\boldsymbol{\phi}(\boldsymbol{x})|^2 d\boldsymbol{x},$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$

(b) for all $\boldsymbol{x}_0 \in \partial\Omega_2$,

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)} \overline{W}^\circ(\boldsymbol{x}_0, \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq \beta \int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)} |\nabla\boldsymbol{\phi}(\boldsymbol{x})|^2 d\boldsymbol{x},$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(B(\mathbf{0},1);\mathbb{R}^m)$.

Then ν is a Young measure local minimizer of the functional $E(\boldsymbol{y})$.

Remark 9 In case $\nu_{\boldsymbol{x}} = \delta_{\nabla\boldsymbol{y}(\boldsymbol{x})}$, for $\boldsymbol{y} \in C^1(\overline{\Omega};\mathbb{R}^m)$, then we recover the sufficiency results obtained in [19], where examples that illustrate the result can also be found.

This theorem is a rather simple corollary of the following result, whose proof is deferred to Chapter 4.

Theorem 5 *Assume that the Lagrangian W, and the Young measure ν are as in Theorem 4, and satisfy the necessary conditions (2.7), (2.10) and (2.12). Then*

$$\lim_{n\to\infty}\frac{1}{\|\nabla\boldsymbol{\phi}_n\|_2^2}\int_\Omega \overline{W}^\circ(\boldsymbol{x},\nabla\boldsymbol{\phi}_n(\boldsymbol{x}))d\boldsymbol{x}\geq 0$$

for any non-zero strong variation $\{\boldsymbol{\phi}_n\}\subset Var(\mathcal{A})$.

Remark 10 The Young measure ν in Theorem 5 is not required to satisfy the equilibrium equation (2.13).

Proof of Theorem 4: Our goal is to show that for any strong variation $\{\boldsymbol{\phi}_n\}$

$$\Delta E(\boldsymbol{\phi}_n)=\int_\Omega\{\overline{W}(\boldsymbol{x},\nabla\boldsymbol{\phi}_n(\boldsymbol{x}))-\overline{W}(\boldsymbol{x},\mathbf{0})\}d\boldsymbol{x}>0 \tag{3.8}$$

for n large enough. In order to prove this inequality we show that $\delta E(\{\boldsymbol{\phi}_n\})\geq\beta>0$, where

$$\delta E(\{\boldsymbol{\phi}_n\})=\lim_{n\to\infty}\frac{\Delta E(\boldsymbol{\phi}_n)}{\|\nabla\boldsymbol{\phi}_n\|_2^2}.$$

Let $\widehat{W}(\boldsymbol{x},\boldsymbol{F})=\overline{W}(\boldsymbol{x},\boldsymbol{F})-\beta|\boldsymbol{F}|^2$. Then, conditions ($ii$) and ($iii$) of Theorem 4 imply that the Lagrangian $\widehat{W}$ and the Young measure ν satisfy conditions (2.7), (2.10), and (2.12). Therefore, according to Theorem 5

$$\lim_{n\to\infty}\frac{1}{\|\nabla\boldsymbol{\phi}_n\|_2^2}\int_\Omega\widehat{W}^\circ(\boldsymbol{x},\nabla\boldsymbol{\phi}_n(\boldsymbol{x})d\boldsymbol{x}\geq 0. \tag{3.9}$$

Inequality (3.9) reduces to $\delta E(\{\boldsymbol{\phi}_n\})-\beta\geq 0$ since

$$\widehat{W}^\circ(\boldsymbol{x},\boldsymbol{F})=\overline{W}^\circ(\boldsymbol{x},\boldsymbol{F})-\beta|\boldsymbol{F}|^2.$$

Theorem 4 is proved. ∎

Remark 11 i) It is known that conditions (i), and (ii) of Theorem 4 form a set of sufficient conditions for $\nu_{\boldsymbol{x}}=\delta_{\nabla\boldsymbol{y}(\boldsymbol{x})}$ where $\boldsymbol{y}\in C^1$ to be a weak local minimizer. In this case there is no growth assumption on the Lagrangian W, as the gradients of the variations are required to be small.

ii) The same set of sufficient conditions has been formulated for for $\nu_{\boldsymbol{x}} = \delta_{\nabla \boldsymbol{y}(\boldsymbol{x})}$ where $\boldsymbol{y} \in C^1$ to be a local minimizer in the sense that for each strong variation $\boldsymbol{\phi}_n$ with $\|\nabla \boldsymbol{\phi}_n\|_{L^\infty} \leq C$, $E(\boldsymbol{y}) \leq E(\boldsymbol{y} + \boldsymbol{\phi}_n)$, for large n. Again in this case there is no restriction on the Lagrangian W, as the competing functions are required to have a gradient that belongs to a ball of fixed radius in L^∞. See [18].

iii) When $\min(m,d) = 1$, then Theorem 4 reduces to the sufficiency result in [[41], Theorem 3.2], for Lagrangians that satisfy the growth conditions imposed. Indeed, it suffices to show that condition (1) of Theorem 3.2 of [41] is a consequence of condition (iii) of our theorem. Here we take $\nu_{\boldsymbol{x}} = \delta_{\nabla \boldsymbol{y}(\boldsymbol{x})}$ for $\boldsymbol{y} \in \overline{\Omega}$. To this end, condition (iii) says that $\widehat{W}(\boldsymbol{x}, \boldsymbol{F}) = W(\boldsymbol{x}, \boldsymbol{F}) - \beta |\boldsymbol{F}|^2$ is quasiconvex at $\boldsymbol{F}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \overline{\Omega}$. Applying Proposition 4, we see that

$$\widehat{W}(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{f}) - \widehat{W}(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})) - (\widehat{W}_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})), \boldsymbol{f}) \geq 0 \tag{3.10}$$

for all $\boldsymbol{x} \in \overline{\Omega}$, and all $\boldsymbol{f} \in \mathbb{R}^{m \times d}$. We then obtain from (3.10) that

$$W(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{f}) - W(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})) - (W_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})), \boldsymbol{f}) \geq \beta |\boldsymbol{f}|^2,$$

for all $\boldsymbol{x} \in \overline{\Omega}$, and all $\boldsymbol{f} \in \mathbb{R}^{m \times d}$, which is exactly condition (1) of Theorem 3.2 of [41].

3.2 Example

Consider the variational problem of minimizing

$$E(y) = \int_\Omega y_{x_1}^2 + (y_{x_2}^2 - 1)^2 dx_1 dx_2 \tag{3.11}$$

over the set of all $y \in W^{1,\infty}(\Omega; \mathbb{R})$, where $\Omega = [0,1] \times [0,1]$. For $\epsilon > 0$, define the piecewise differentiable function h_ϵ on $\mathbb{R}$ in such a way that $h'_\epsilon = -1$ on $[i-1, i-\epsilon]$, and $h'_\epsilon = 0$ on $[i-\epsilon, i]$ for any integer i. Then the sequence

$$y_n(x_1, x_2) = \frac{1}{n} h_\epsilon(n x_2)$$

is bounded in $W^{1,\infty}(\Omega;\mathbb{R})$ and the sequence of gradients generate the Young measure $\nu = (1-\epsilon)\delta_A + \epsilon\delta_B$, where $A = (0,-1)$, and $B = \mathbf{0}$.

We claim that for sufficiently small ϵ the Young measure ν and the Lagrangian satisfy our sufficiency result and hence ν is a Young measure local minimizer. The Lagrangian corresponding to the variational problem is

$$W(\boldsymbol{F}) = f_1^2 + (f_2^2 - 1)^2, \quad \text{where } \boldsymbol{F} = (f_1, f_2),$$

and

$$\overline{W}(\boldsymbol{F}) = \int_{\mathbb{R}^2} W(\boldsymbol{F}+\boldsymbol{G})d\nu(\boldsymbol{G}) = (1-\epsilon)W(\boldsymbol{A}+\boldsymbol{F}) + \epsilon W(\boldsymbol{F}),$$

Moreover,

$$\overline{W}_{\boldsymbol{F}}(\boldsymbol{F}) = (1-\epsilon)W_{\boldsymbol{F}}(\boldsymbol{A}+\boldsymbol{F}) + \epsilon W_{\boldsymbol{F}}(\boldsymbol{F})$$

$$\overline{W}_{\boldsymbol{FF}}(\boldsymbol{F}) = (1-\epsilon)W_{\boldsymbol{FF}}(\boldsymbol{A}+\boldsymbol{F}) + \epsilon W_{\boldsymbol{FF}}(\boldsymbol{F})$$

Plugging $\boldsymbol{F} = \mathbf{0}$, we get $\overline{W}(\mathbf{0}) = \epsilon, \overline{W}_{\boldsymbol{F}}(\mathbf{0}) = \mathbf{0}$, and

$$\overline{W}_{\boldsymbol{FF}}(\mathbf{0}) = (1-\epsilon)W_{\boldsymbol{FF}}(\boldsymbol{A}) + \epsilon W_{\boldsymbol{FF}}(\mathbf{0}).$$

Then trivially the equilibrium equations are satisfied. Moreover the Hessian is given by

$$\overline{W}_{\boldsymbol{FF}}(\mathbf{0}) = \begin{pmatrix} 2 & 0 \\ 0 & 8-12\epsilon \end{pmatrix}$$

and is positive definite for small ϵ.

To prove the uniform quasiconvexity condition we show the uniform positivity of the Weierstrass function. To this end, let us calculate the Weierstrass excess function around $\boldsymbol{F} = \mathbf{0}$:

$$\overline{W}^{\circ}(\boldsymbol{F}) = \overline{W}(\boldsymbol{F}) - \overline{W}(\mathbf{0}) - (\overline{W}_{\boldsymbol{F}}(\mathbf{0}), \boldsymbol{F}).$$

Then we have

$$\overline{W}^{\circ}(\boldsymbol{F}) = (1-\epsilon)W(\boldsymbol{A}+\boldsymbol{F}) + \epsilon W(\boldsymbol{F}) - \epsilon.$$

This is because $\overline{W}_{\boldsymbol{F}}(\mathbf{0}) = \mathbf{0}$, and $\overline{W}(\mathbf{0}) = \epsilon$. We then calculate $\overline{W}^{\circ}(\boldsymbol{F})$ explicitly:

$$\overline{W}^{\circ}(\boldsymbol{F}) = f_1^2 + f_2^4 - 4(1-\epsilon)f_2^3 + 2f_2^2(2-3\epsilon).$$

But then

$$f_2^4 - 4(1-\epsilon)f_2^3 \geq -4(1-\epsilon)^2 f_2^2.$$

Therefore,

$$\overline{W}^{\circ}(\boldsymbol{F}) \geq f_1^2 + (2(2-3\epsilon) - 4(1-\epsilon)^2 f_2^2$$

Simplifying the right hand side, we get

$$\overline{W}^{\circ}(\boldsymbol{F}) \geq f_1^2 + 2\epsilon(1-2\epsilon)f_2^2.$$

For a sufficiently small ϵ, we get a positive number $\beta = \beta(\epsilon)$ such that

$$\overline{W}^{\circ}(\boldsymbol{F}) \geq \beta|\boldsymbol{F}|^2,$$

as desired. Observe that we can construct a sequence of functions $z_n \in W^{1,\infty}$ such that $E(z_n) \to 0$. Therefore, ν is a local but not a global minimizer of E as $\overline{W}(0) = \epsilon > 0$.

Another example in this spirit is minimizing the functional in (3.11) over the set of functions $y \in W_0^{1,\infty}(\Omega; \mathbb{R})$. Here one can easily show that given ϵ the Young measure

$$\nu = \frac{1-\epsilon}{2}\delta_{\boldsymbol{A}} + \epsilon\delta_{\boldsymbol{B}} + \frac{1-\epsilon}{2}\delta_{\boldsymbol{C}},$$

where $\boldsymbol{A} = (0,-1)$, $\boldsymbol{B} = \mathbf{0}$, and $\boldsymbol{C} = (0,1)$ can be generated by a sequence of gradients of functions in $W_0^{1,\infty}(\Omega; \mathbb{R})$. Moreover a similar calculation as above yields ν is a Young measure local but not global minimizer of E .

CHAPTER 4

PROOF OF THE SUFFICIENCY RESULT

4.1 The method

The method of proof of Theorem 5 is direct. For strong variations we evaluate

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \frac{1}{\|\nabla\boldsymbol{\phi}_n\|_2^2}\int_\Omega \overline{W}^\circ(\boldsymbol{x}, \nabla\boldsymbol{\phi}_n(\boldsymbol{x}))d\boldsymbol{x}, \tag{4.1}$$

and show that the non-negativity of second variation and quasiconvexity inequalities imply that $\delta' E(\{\boldsymbol{\phi}_n\}) \geq 0$. This method was developed in [18] for a simpler case of uniformly bounded sequence $\nabla\boldsymbol{\phi}_n$. It was also used in [19] to obtain sufficiency results for strong local minimizers by modifying it to handle all strong variations, provided the Lagrangian W is as in Theorem 4. In this thesis we apply the method to variational problems that may have minimizing sequences described by Young measure but not necessarily classical minimizers.

First, observe that the coercivity condition (3.1) on W implies that a strong variation whose gradients are unbounded in L^p, has the property that

$$\lim_{n\to\infty} \Delta E(\boldsymbol{\phi}_n) = +\infty,$$

where as defined in (3.8)

$$\Delta E(\boldsymbol{\phi}_n) = \int_\Omega \{\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}_n(\boldsymbol{x})) - \overline{W}(\boldsymbol{x}, \boldsymbol{0}\} d\boldsymbol{x}. \tag{4.2}$$

Hence, $\delta' E(\{\boldsymbol{\phi}_n\}) \geq 0$. Thus, we may restrict our attention only to variations $\{\boldsymbol{\phi}_n\}$ for which the sequences $\beta_n = \|\nabla\boldsymbol{\phi}_n\|_p$ are bounded. In particular, by means of extracting a subsequence we may assume, without loss of generality, that $\boldsymbol{\phi}_n$ converges to zero in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m)$, and $\boldsymbol{\phi}_0 \to 0$ strongly in $L^p(\Omega;\mathbb{R}^m)$.

Furthermore, from the definition of $\overline{W}^\circ$ we notice that it has a zero of order two at $\boldsymbol{F} = \boldsymbol{0}$. Combining this fact with the coercivity inequality (3.1), and the fact that the Young measure ν is compactly supported we deduce that there are positive constants c_1 and c_2 such that for all $\boldsymbol{F} \in \mathbb{M}$ and all $\boldsymbol{x} \in \overline{\Omega}$ we have

$$\overline{W}^\circ(\boldsymbol{x}, \boldsymbol{F}) \geq c_1|\boldsymbol{F}|^2(|\boldsymbol{F}|^{p-2} - c_2). \tag{4.3}$$

Let $\alpha_n = \|\nabla\boldsymbol{\phi}_n\|_2$. Then

$$\delta' E(\{\boldsymbol{\phi}_n\}) \geq \varliminf_{n\to\infty} \frac{c_1}{\alpha_n^2}\int_\Omega |\nabla\boldsymbol{\phi}_n(\boldsymbol{x})|^2(|\nabla\boldsymbol{\phi}_n(\boldsymbol{x})|^{p-2} - c_2) d\boldsymbol{x} = c_1\left(\varliminf_{n\to\infty}\frac{\beta_n^p}{\alpha_n^2} - c_2|\Omega|\right).$$

By Hölder inequality we have $\alpha_n \leq \beta_n|\Omega|^{(p-2)/2p}$. Thus, we need to consider only those strong variations $\{\boldsymbol{\phi}_n\}$ for which either $\alpha_n \to \alpha_0 > 0$ and $\beta_n \to \beta_0 < +\infty$, or both α_n and β_n go to zero, while $\beta_n^p/\alpha_n^2 \to \gamma < \infty$.

Let us first consider the case $\alpha_0 > 0$. We have that

$$\varliminf_{n\to\infty}\int_\Omega \overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x} \geq \varliminf_{n\to\infty}\int_\Omega Q\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x},$$

where $Q\overline{W}(\boldsymbol{x}, \boldsymbol{F})$ is the quasiconvexification of $\overline{W}(\boldsymbol{x}, \boldsymbol{F})$, [8]. It is easy to verify that $Q\overline{W}$ is bounded from below, since the Lagrangian $\overline{W}$ is bounded from below by Lemma 2. The theorem of Acerbi and Fusco [1] then says that the functional

$$\boldsymbol{\phi} \mapsto \int_\Omega Q\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}) d\boldsymbol{x}$$

is $W^{1,p}$ sequentially lower semicontinuous, and thus,

$$\varliminf_{n\to\infty}\int_\Omega Q\overline{W}(\boldsymbol{x}, \nabla\boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x} \geq \int_\Omega Q\overline{W}(\boldsymbol{x}, \boldsymbol{0}) d\boldsymbol{x}.$$

Finally, the quasiconvexity condition (2.10) can be rewritten as $Q\overline{W}(\boldsymbol{x},\mathbf{0}) = \overline{W}(\boldsymbol{x},\mathbf{0})$. It then follows that

$$\int_\Omega Q\overline{W}(\boldsymbol{x},\mathbf{0})d\boldsymbol{x} = \int_\Omega \overline{W}(\boldsymbol{x},\mathbf{0})d\boldsymbol{x}.$$

We conclude that

$$\delta E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \frac{1}{\alpha_0^2}\int_\Omega \left(\overline{W}(\boldsymbol{x},\nabla\boldsymbol{\phi}_n) - \overline{W}(\boldsymbol{x},\mathbf{0})\right) d\boldsymbol{x} \geq 0.$$

The case

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0, \qquad \lim_{n\to\infty}\frac{\beta_n^p}{\alpha_n^2} = \gamma < \infty \tag{4.4}$$

is the heart of the matter. The rest of the thesis is devoted to the application of our method to the proof of Theorem 5 in this case. The proof proceeds in steps. In Section 4.2 we rewrite the normalized increment in a form that is convenient to apply our analysis. In Sections 4.3 and 4.4 we study behavior of limits of sequences and separate the “oscillation” and “concentration” effects. In Section 4.5 we prove a representation formula for $\delta' E(\{\boldsymbol{\phi}_n\})$ that shows the effects of the “oscillation” part and the “concentration” part of a variation on the two terms. In Section 4.6 we prove the localization principle that enables us to connect the effect of “concentrations” on the variational functional to the quasiconvexity conditions. Lastly, we combine our calculus of variations and the necessary conditions (2.7),(2.10) and (2.12) to show that $\delta' E(\{\boldsymbol{\phi}_n\}) \geq 0$.

4.2 Reformulation of the integral increment

According to our method, developed in [18], it will be convenient for us to represent $\overline{W}^\circ$ in the form that shows the quadratic term in its Tayer expansion around $\boldsymbol{F} = \mathbf{0}$ explicitly, because it appears in the formula for the second variation:

$$\overline{W}^\circ(\boldsymbol{x},\boldsymbol{F}) = \frac{1}{2}(\mathsf{L}(\boldsymbol{x})\boldsymbol{F},\boldsymbol{F}) + U(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2, \tag{4.5}$$

where we recall $\mathsf{L}(\boldsymbol{x}) = \overline{W}^\circ_{\boldsymbol{FF}}(\boldsymbol{x},\mathbf{0}) = \int_{\mathbb{M}} W_{\boldsymbol{FF}}(\boldsymbol{x},\boldsymbol{F})d\nu_{\boldsymbol{x}}(\boldsymbol{x})$, and

$$U(\boldsymbol{x},\boldsymbol{F}) = \frac{1}{|\boldsymbol{F}|^2}(\overline{W}^\circ(\boldsymbol{x},\boldsymbol{F}) - \frac{1}{2}(\mathsf{L}(\boldsymbol{x})\boldsymbol{F},\boldsymbol{F})) \tag{4.6}$$

is continuous on $\overline{\Omega}\times\mathbb{M}$ and vanishes on $\overline{\Omega}\times\{\mathbf{0}\}$. We can then rewrite $\delta' E(\{\boldsymbol{\phi}_n\})$ in the following form

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \int_\Omega \left(\frac{1}{2}(\mathsf{L}(\boldsymbol{x})\nabla\boldsymbol{\psi}_n, \nabla\boldsymbol{\psi}_n) + U(\boldsymbol{x}, \alpha_n\nabla\boldsymbol{\psi}_n)|\nabla\boldsymbol{\psi}_n|^2 \right) d\boldsymbol{x}, \quad (4.7)$$

where $\boldsymbol{\psi}_n(\boldsymbol{x}) = \boldsymbol{\phi}_n(\boldsymbol{x})/\alpha_n$.

In [18] we did not have the regularity conditions on the function U that are inherited from W, except continuity. Hence the analysis was carried out first for a Lipschitz continuous (or even smooth) function V and then used the fact that any continuous function U can be approximated, uniformly on compact sets, by smooth functions V. This was sufficient for the purposes of [18] because the sequence $\{\nabla\boldsymbol{\phi}_n\}$ was assumed to be uniformly bounded. Here we use exactly the same approach. However, since the sequence $\{\nabla\boldsymbol{\phi}_n\}$ is no longer uniformly bounded, we need to approximate U not just with smooth functions, but with functions V that satisfy

$$|V(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2 - V(\boldsymbol{x},\boldsymbol{G})|\boldsymbol{G}|^2| \le C(|\boldsymbol{F}|+|\boldsymbol{G}|+|\boldsymbol{F}|^{p-1}+|\boldsymbol{G}|^{p-1})|\boldsymbol{F}-\boldsymbol{G}|. \quad (4.8)$$

On any compact set this inequality reduces to the inequality [18, formula (7.8)]. The set of functions W that produce functions U satisfying (4.8) is exactly the subspace $\mathcal{L}$ defined by (3.3). Hence, our requirement that W belong to $\overline{\mathcal{L}}$. We observe that if $W \in \overline{\mathcal{L}}$, so also $\overline{W}$. The lemma below provides a rigorous statement corresponding to our intuition.

Lemma 3 *Assume that $W \in \overline{\mathcal{L}}$ and that $W_{\boldsymbol{F}}$ and $W_{\boldsymbol{FF}}$ are continuous functions on $\overline{\Omega}\times\mathcal{O}$. Then we can find a sequence of continuous functions $V_n(\boldsymbol{x},\boldsymbol{F})$ satisfying the inequality (4.8) and such that $V_n(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2 \to U(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2$ in X_p.*

Proof: Applying Lemma 2, $\overline{W} \in \mathcal{L}$. By assumption there exists a sequence $W_n \in \mathcal{L}$ such that $W_n \to \overline{W}$ in X_p. Let $\mathcal{R}_\delta = B(\mathbf{0},\delta)$, where δ is chosen in such away that $\overline{W}$ is twice differentiable in $B(\mathbf{0},\delta)$. Let $\rho(\boldsymbol{F})$ be a smooth function which is equal to 0 on $\mathcal{R}_{\delta/2}$ and 1 on the complement of $\mathcal{R}_\delta$.

Then the functions

$$\widetilde{W}^{(n)} = \rho(\boldsymbol{F})W_n(\boldsymbol{x},\boldsymbol{F}) + (1-\rho(\boldsymbol{F}))\overline{W}(\boldsymbol{x},\boldsymbol{F})$$

also belong to $\mathcal{L}$, for each $n \geq 1$, and converge to $\overline{W}$ in X_p. Let

$$U_n(\boldsymbol{x},\boldsymbol{F}) = \frac{\widetilde{W}^{(n)}(\boldsymbol{x},\boldsymbol{F}) - \widetilde{W}^{(n)}(\boldsymbol{x},\boldsymbol{0}) - (\overline{W}_{\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{0}),\boldsymbol{F}) - \frac{1}{2}(\mathsf{L}(\boldsymbol{x})\boldsymbol{F},\boldsymbol{F})}{|\boldsymbol{F}|^2}.$$

When $|\boldsymbol{F}| < \delta/2$, we have $U_n(\boldsymbol{x},\boldsymbol{F}) = U(\boldsymbol{x},\boldsymbol{F})$ for all $\boldsymbol{x} \in \Omega$. When $|\boldsymbol{F}| \geq \delta/2$, we have

$$\|U_n - U\|_{X_{p-2}} \leq \frac{4}{\delta^2}(\|\widetilde{W}^{(n)} - \overline{W}\|_{X_p}),$$

Finally, by construction, functions $U_n(\boldsymbol{x},\boldsymbol{F})$ satisfy (4.8). ∎

To simplify notation we will use the shorthand

$$\mathcal{F}(\boldsymbol{x},\alpha,\boldsymbol{G}) = \frac{1}{\alpha^2}\overline{W}^{\circ}(\boldsymbol{x},\alpha\boldsymbol{G}).$$

Then in terms of $\mathcal{F}$

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \int_\Omega \mathcal{F}(\boldsymbol{x},\alpha_n,\nabla\boldsymbol{\psi}_n(\boldsymbol{x}))d\boldsymbol{x}. \tag{4.9}$$

4.3 The decomposition lemma

A key tool in proving Theorem 5 is a decomposition result in [15], and [26] for sequences of gradients that are bounded in L^p. It states that a sequence of gradient of functions that is bounded in L^p can be written as a sum of sequence of gradients one being p-equiintegrable and the other vanishing except on a set of small measure and converging weakly in L^p.

Lemma 4 *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set and let $\{\boldsymbol{w}_n\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. There exists a subsequence, $\{\boldsymbol{w}_j\}$, and a sequence $\{\boldsymbol{z}_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$|\{\boldsymbol{z}_j \neq \boldsymbol{w}_j \text{ or } \nabla\boldsymbol{z}_j \neq \nabla\boldsymbol{w}_j\}| \to 0$$

as $j \to \infty$, and $\{|\nabla\boldsymbol{z}_j|^p\}$ is equi-integrable. If Ω is Lipschitz then each $\boldsymbol{z}_j$ may be chosen to be a Lipschitz function.

In [18] we used this result for $p = 2$ and the sequence $\{\boldsymbol{w}_j\}$ replaced by $\{\boldsymbol{\psi}_n\}$. The reason for that was we considered strong variations $\{\boldsymbol{\phi}_n\}$ which have a uniformly bounded gradient and with the right scaling , $\alpha_n = \|\nabla\boldsymbol{\phi}_n\|_2$, it was enough to study the Lagrangian $W(\boldsymbol{x}, \boldsymbol{F})$ on a bounded set. Here the situation is different as we allow strong variations $\{\boldsymbol{\phi}_n\}$ with unbounded gradients. Taking into consideration the fact that the Lagrangian W is super-quadratic, we have different scaling of $\boldsymbol{\phi}_n$ to be applied corresponding to the property of W near $\mathbf{0}$ and infinity. Let $\boldsymbol{\zeta}_n = \frac{\boldsymbol{\phi}_n}{\beta_n}$, a scaling by $\beta_n = \|\nabla\boldsymbol{\phi}_n\|_p$. Observe that $\|\nabla\boldsymbol{\zeta}_n\|_p = 1$ and $\boldsymbol{\zeta}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{\psi}_n$. Accordingly Lemma 4 has to be modified so as to include a decomposition for $\boldsymbol{\zeta}_n$. Namely in addition to the decomposition for $\boldsymbol{\psi}_n$, the modified version should provide us a similar decomposition for $\boldsymbol{\zeta}_n$ that preserves this scaling. Let $r_n = \frac{\alpha_n}{\beta_n}$.

Theorem 6 (Decomposition Theorem) *Suppose $\boldsymbol{\psi}_n \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^m)$ and the sequence $r_n > 0$ is such that $\boldsymbol{\zeta}_n = r_n\boldsymbol{\psi}_n$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $n(j)$, sequences $\boldsymbol{z}_j$ and $\boldsymbol{v}_j$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$, and subsets R_j of Ω such that*

(a) $\boldsymbol{\psi}_{n(j)} = \boldsymbol{z}_j + \boldsymbol{v}_j$.

(b) For all $\boldsymbol{x} \in \Omega \setminus R_j$ *we have* $\boldsymbol{z}_j(\boldsymbol{x}) = \boldsymbol{\psi}_{n(j)}(\boldsymbol{x})$ *and* $\nabla\boldsymbol{z}_j(\boldsymbol{x}) = \nabla\boldsymbol{\psi}_{n(j)}(\boldsymbol{x})$.

(c) The sequence $\{|\nabla\boldsymbol{z}_j|^2\}$ *is equiintegrable.*

(d) $\boldsymbol{v}_j \rightharpoonup 0$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^m)$.

(e) $|R_j| \to 0$ *as* $j \to \infty$.

In addition, the sequences $\boldsymbol{t}_j = r_{n(j)}\boldsymbol{v}_j$ and $\boldsymbol{s}_j = r_{n(j)}\boldsymbol{z}_j$ are bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and satisfy

(a') $\boldsymbol{\zeta}_{n(j)} = \boldsymbol{s}_j + \boldsymbol{t}_j$.

(b') For all $\boldsymbol{x} \in \Omega \setminus R_j$ *we have* $\boldsymbol{s}_j(\boldsymbol{x}) = \boldsymbol{\zeta}_{n(j)}(\boldsymbol{x})$ *and* $\nabla\boldsymbol{s}_j(\boldsymbol{x}) = \nabla\boldsymbol{\zeta}_{n(j)}(\boldsymbol{x})$.

(c') The sequence $\{|\nabla\boldsymbol{s}_j|^p\}$ *is equiintegrable.*

(d') $\boldsymbol{t}_j \rightharpoonup 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$.

We will refer to $\boldsymbol{z}_j$ as the weak part of the variation and to $\boldsymbol{v}_j$ as the strong part.

Proof: The proof is a slight modification of the proof of Lemma 4 which is found in [15]. This modification is needed to prove the added new properties of the sequence that are mentioned in the modified Lemma.

We prove the lemma in several steps. We recap the proof of Lemma 4 to select an appropriate subsequence $\boldsymbol{\psi}_{n(k)}$ that will give the abovementioned relations between the sequences.

Step 1. The proof of the lemma uses properties of Maximal functions. We have to make sense of the maximal function of $\boldsymbol{\psi}_n$ and $\boldsymbol{\zeta}_n$ as the functions are defined on a bounded set $\Omega \subset \mathbb{R}^d$. Since the domain Ω is a smooth domain we can extend Sobolev functions defined on Ω to functions defined on the whole space $\mathbb{R}^d$. In fact, there exists a linear extension operator

$$\boldsymbol{X} : W^{1,p}(\Omega;\mathbb{R}^m) \to W^{1,p}(\mathbb{R}^d;\mathbb{R}^m), \quad 1 \le p \le \infty$$

and a constant $C > 0$ independent of p such that

$$\boldsymbol{X}(\boldsymbol{\psi})(\boldsymbol{x}) = \boldsymbol{\psi}(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega \quad \text{and } \|\boldsymbol{X}(\boldsymbol{\psi})\|_{W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)} \le C\|\boldsymbol{\psi}\|_{W^{1,p}(\Omega;\mathbb{R}^m)}$$

for all $\boldsymbol{\psi} \in W^{1,p}(\Omega;\mathbb{R}^m)$. Identifying $\boldsymbol{\zeta}_n$ with its extension $\boldsymbol{X}(\boldsymbol{\zeta}_n)$, the sequence of maximal function $\{M(\nabla\boldsymbol{\zeta}_n)\}$ is bounded in L^p, since $\nabla\boldsymbol{\zeta}_n$ is bounded in L^p. Let $\eta = \{\eta_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Omega}$ be the Young measure generated by a subsequence $\{M(\nabla\boldsymbol{\zeta}_{n(k)})\}$. Consider the truncation map $T_j : \mathbb{R} \to \mathbb{R}$ given by

$$T_j(s) = \begin{cases} s & |s| \le j \\ j\frac{s}{|s|} & |s| > j. \end{cases}$$

Then for each j, the truncation T_j is bounded and therefore, the sequence $\{|T_j(M(\nabla\boldsymbol{\zeta}_{n(k)}))|^p\}$ is equiintegrable. It follows that for each j

$$|T_j(M(\nabla\boldsymbol{\zeta}_{n(k)}))|^p \rightharpoonup \int_{\mathbb{R}} |T_j(s)|^p d\eta_{\boldsymbol{x}}(s), \quad as \;\; k \to \infty$$

weakly in $L^1(\Omega)$. Let

$$\overline{f}(\boldsymbol{x}) = \int_{\mathbb{R}} |s|^p d\eta_{\boldsymbol{x}}(s).$$

Then $\overline{f} \in L^1(\Omega)$. Applying dominated convergence theorem, because $T_j(s)| \leq |s|$, we have

$$\int_{\mathbb{R}} |T_j(s)|^p d\eta_{\boldsymbol{x}}(s) d\boldsymbol{x} \rightharpoonup \overline{f}(\boldsymbol{x}), \quad as \quad j \to \infty$$

weakly in $L^1(\Omega)$. It turns out that it is possible to choose a subsequence, not relabeled, such that

$$|T_k(M(\nabla \boldsymbol{\zeta}_{n(k)}))|^p \rightharpoonup \overline{f} \quad as \quad k \to \infty$$

weakly in $L^1(\Omega)$ (See [15]).

Step 2. Now consider the subsequence $\boldsymbol{\psi}_{n(k)}$. A similar argument as in **Step 1** produces a subsequence $k(j)$, chosen to be greater than j, such that $|T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))}|^2$ is weakly convergent in $L^1(\Omega)$. Set

$$R'_j = \{\boldsymbol{x} \in \Omega : M(\nabla \boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x}) \geq j\}.$$

Since Ω is bounded and $M(\nabla \boldsymbol{\psi}_{n(k(j))})$ is bounded in L^2, we have $|R'_j| \to 0$ as $j \to \infty$. It is proved in [12, p. 255, Claim #2] that there exist Lipschitz functions $\boldsymbol{z}'_k$ such that

$$\boldsymbol{z}'_j = \boldsymbol{\psi}_{n(k(j))} \quad \text{a.e on} \quad \Omega \setminus R'_j, \qquad |\nabla \boldsymbol{z}'_j(\boldsymbol{x})| \leq Cj \quad \text{a.e on } \boldsymbol{x} \in \mathbb{R}^d$$

Let $R_j = R'_j \cup \{\boldsymbol{x} \in \Omega : \nabla \boldsymbol{z}'_j \neq \nabla \boldsymbol{\psi}_{n(k(j))}\}$. Then $|R_j| \to 0$ as $j \to \infty$. We observe that for $x \in \Omega \setminus R_j$ we have the inequality

$$|\nabla \boldsymbol{z}'_j(\boldsymbol{x})| = |\nabla \boldsymbol{\psi}_{n(k(j))}(\boldsymbol{x})| \leq |M(\nabla \boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x})| = |T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x}))|$$

while if $\boldsymbol{x} \in R'_j$, then

$$|\nabla \boldsymbol{z}'_j(\boldsymbol{x})| \leq Cj = C|T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x}))|.$$

We conclude that

$$|\nabla \boldsymbol{z}'_j(\boldsymbol{x})|^2 \leq C|T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x}))|^2 \quad \text{a.e} \quad \boldsymbol{x} \in \Omega$$

which together with the weak convergence of $|T_j(M(\nabla\boldsymbol{\psi}_{n(k(j))})(\boldsymbol{x}))|^2$, yields the equi-integrability of $\{|\nabla \boldsymbol{z}_j'|^2\}$.

Let $\boldsymbol{v}_j' = \boldsymbol{\psi}_{(n(k(j)))} - \boldsymbol{z}_j'$. Then $\nabla \boldsymbol{v}_j'$ is bounded in L^2 because so are $\nabla\boldsymbol{\psi}_{n(k(j))}$ and $\nabla \boldsymbol{z}_j'$. Now let $\langle \boldsymbol{v}_j' \rangle$ be the average of the field $\boldsymbol{v}_j'$ over Ω and let

$$\boldsymbol{z}_j = \boldsymbol{z}_j' + \langle v_j' \rangle, \quad \boldsymbol{v}_j - \langle \boldsymbol{v}_j' \rangle.$$

Then, by Poincare' inequality, $\boldsymbol{v}_j$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^m)$. Thus, $\boldsymbol{z}_j$ is also bounded in $W^{1,2}$. Moreover, by the Cauchy-Schwartz inequality

$$\begin{aligned} |\int_\Omega (\boldsymbol{\varphi}, \boldsymbol{v}_j(\boldsymbol{x}))d\boldsymbol{x} \ &+ \ \int_\Omega (\nabla\boldsymbol{\varphi}(\boldsymbol{x}), \nabla \boldsymbol{v}_j(\boldsymbol{x}))d\boldsymbol{x}| \\ &\le \left(\int_{R_j} |\boldsymbol{\varphi}(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{1/2} \|\boldsymbol{v}_j\|_{L^2} + \left(\int_{R_j} |\nabla\boldsymbol{\varphi}(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{1/2} \|\nabla \boldsymbol{v}_j\|_{L^2} \end{aligned}$$

which goes to 0 as $j \to \infty$ for any $\boldsymbol{\varphi} \in W^{1,2}(\Omega;\mathbb{R}^m)$ since the sequence $\boldsymbol{v}_j$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^m)$, and $R_j \to 0$. That is, $\boldsymbol{v}_j \rightharpoonup 0$ in $W^{1,2}(\Omega;\mathbb{R}^m)$. By compact embedding , we can select a subsequence, not relabeled such that $\boldsymbol{v}_j \to 0$ in $L^2(\Omega;\mathbb{R}^m)$.

Step 3. In this step we prove (i) and (ii) of the lemma. We recall that $\alpha_n \le \beta_n |\Omega|^{(p-2)/2p}$ for all n, and by choice $j \le k(j)$. Thus, $j \le c_0 \frac{\beta_{n(k(j))}}{\alpha_{n(k(j))}} k(j)$ for all j, where $c_0 = |\Omega|^{(p-2)/2p} > 0$. On the one hand,

$$\{\boldsymbol{x} : M(c_0 \nabla\boldsymbol{\psi}_{n(k(j))}) < j\} \subset \{\boldsymbol{x} : M(\nabla\boldsymbol{\psi}_{n(k(j))}) < \frac{\beta_{n(k(j))}}{\alpha_{n(k(j))}} k(j)\}$$

On the other hand,

$$\{\boldsymbol{x} : M(\nabla\boldsymbol{\psi}_{n(k(j))}) < \frac{\beta_{n(k(j))}}{\alpha_{n(k(j))}} k(j)\} = \{\boldsymbol{x} : M(\nabla\boldsymbol{\zeta}_{n(k(j))}) < k(j)\}.$$

Working with the sequence $c_0\boldsymbol{\psi}_{n(k)}$ in stead of $\boldsymbol{\psi}_{n(k)}$ in **Step 2**, WLOG we may assume that $c_0 = 1$ (Otherwise, we will change Lebesgue measure $d\boldsymbol{x}$ to the normalized Lebesgue measure $d\boldsymbol{x}/|\Omega|$.). Define now

$$\boldsymbol{s}_j(\boldsymbol{x}) = \frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} \boldsymbol{z}_j(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \Omega.$$

Then for some positive constant C,

$$\boldsymbol{s}_j(\boldsymbol{x}) = \boldsymbol{\zeta}_{n(k(j))}(\boldsymbol{x}) \quad \text{a.e. on } \Omega \setminus R_j, \qquad \text{and} \qquad |\nabla \boldsymbol{s}_{k(j)}| \leq Ck(j) \text{ for a.e. } \boldsymbol{x} \in \Omega.$$

Now for a.e. $\boldsymbol{x} \in \Omega \setminus R_j$ we have

$$|\nabla \boldsymbol{s}_j(\boldsymbol{x})| = |\nabla \boldsymbol{\zeta}_{n(k(j))}(\boldsymbol{x})| \leq |M(\nabla \boldsymbol{\zeta}_{n(k(j))}(\boldsymbol{x}))| = T_{k(j)}(M(\nabla \boldsymbol{\zeta}_{n(k(j))}(\boldsymbol{x}))),$$

while if $\boldsymbol{x} \in R_j'$, then

$$|\nabla \boldsymbol{s}_j(\boldsymbol{x})| = \frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} |\nabla \boldsymbol{z}_j(\boldsymbol{x})| \leq C \frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} j = C \frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))}(\boldsymbol{x}))|$$

But using the identity $rT_a(s) = T_{ra}(rs)$, valid for $r \geq 0$, we have

$$\frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} T_j(M(\nabla \boldsymbol{\psi}_{n(k(j))}(\boldsymbol{x}))| = T_{\frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} j}\left(\frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} M(\nabla \boldsymbol{\psi}_{n(k(j))})\right)$$

Therefore,

$$|\nabla \boldsymbol{s}_j(\boldsymbol{x})| \leq CT_{\frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} j}\left(\frac{\alpha_{n(k(j))}}{\beta_{n(k(j))}} M(\nabla \leq \boldsymbol{\psi}_{n(k(j))})\right) \leq CT_{k(j)}(M(\nabla \boldsymbol{\zeta}_{n(k(j))})).$$

We conclude that

$$|\nabla \boldsymbol{s}_j(\boldsymbol{x})|^p \leq C|T_{k(j)}(M(\nabla \boldsymbol{\zeta}_{n(k(j))}))|^p \quad \text{a.e } \boldsymbol{x} \in \Omega,$$

which yields the equiintegrability of $\{|\nabla \boldsymbol{s}_{k(j)}|^p\}$. We can also show that $\boldsymbol{t}_j \rightharpoonup \mathbf{0}$ as $j \to \infty$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, in a similar way we showed $\boldsymbol{v}_j \rightharpoonup 0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^m)$. ▮

4.4 The orthogonality principle

We recall that the strong variation $\{\boldsymbol{\phi}_n\}$ yields the normalized sequence $\{\boldsymbol{\psi}_n\}$ with the right scaling to fit our formulation. From the Decomposition Lemma it follows that $\boldsymbol{\psi}_n = \boldsymbol{v}_n + \boldsymbol{z}_n$, where the sequence have the properties mentioned in the Lemma. In this section we will show that the two terms $\boldsymbol{v}_n$ and $\boldsymbol{z}_n$ act on $\mathcal{F}$ in a non-interacting way.

Lemma 5 (Orthogonality)

$$\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{z}) \to 0$$

strongly in $L^1(\Omega)$ *as* $n \to \infty$.

PROOF: Let

$$I_n(\boldsymbol{x}, \mathcal{F}) = \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{z}_n)$$

From the Decomposition Lemma in the previous section we see that $I_n(\boldsymbol{x}, \mathcal{F}) = 0$ for $\boldsymbol{x} \in \Omega \setminus R_n$ where R_n is the sequence of set on which $\boldsymbol{\psi}_n = \boldsymbol{z}_n$ and its measure $|R_n| \to 0$ as $n \to \infty$. Thus to prove the lemma it suffices to show that

$$\int_{R_n} |I_n(\boldsymbol{x}, \mathcal{F})| d\boldsymbol{x} \to 0$$

as $n \to \infty$. First, let us assume that U satisfies (4.8). Then there exists a positive constant C such that

$$|\mathcal{F}(\boldsymbol{x}, \alpha, \boldsymbol{G}_1) - \mathcal{F}(\boldsymbol{x}, \alpha, \boldsymbol{G}_2)| \leq C(|\boldsymbol{G}_1| + |\boldsymbol{G}_2| + \alpha^{p-2}(|\boldsymbol{G}_1|^{p-1} + |\boldsymbol{G}_2|^{p-1})|\boldsymbol{G}_1 - \boldsymbol{G}_2| \tag{4.10}$$

for all $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ in $\mathbb{R}^{m \times d}$. Moreover we observe there is a positive constant C such that

$$|\mathcal{F}(\boldsymbol{x}, \alpha, \boldsymbol{G})| \leq C|\boldsymbol{G}|^2(1 + |\alpha \boldsymbol{G}|^{p-2}) \tag{4.11}$$

for all $\boldsymbol{G}$. Then

$$\int_{R_n} |I_n(\boldsymbol{x}, \mathcal{F})| d\boldsymbol{x} \leq \int_{R_n} |\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n)| d\boldsymbol{x} + \int_{R_n} |\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{z}_n)| d\boldsymbol{x} \tag{4.12}$$

We show both terms in the right hand side of the above inequality go to zero separately. Applying (4.11) to the integrand of the second term we get

$$\int_{R_n} |\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{z}_n)| d\boldsymbol{x} \leq C \int_{R_n} |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 (1 + |\alpha_n \nabla \boldsymbol{z}_n(\boldsymbol{x})|^{p-2}) d\boldsymbol{x}$$

Using the relation between $\boldsymbol{z}_n$ and $\boldsymbol{s}_n$ from Lemma 6, namely $\boldsymbol{s}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{z}_n$, the last inequality can be written as

$$\int_{R_n} |\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{z}_n)| d\boldsymbol{x} \leq C \left(\int_{R_n} |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 d\boldsymbol{x} + \frac{\beta_n^p}{\alpha_n^2} \int_{R_n} |\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p d\boldsymbol{x} \right).$$

The two terms in the right hand side converge to 0 because $|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2$ and $|\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p$ are equiintegrable and the sequence of numbers $\frac{\beta_n^p}{\alpha_n^2}$ is bounded. Next let

$$d_n(\boldsymbol{x}) = |\mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n)|$$

be the integrand of the first term of (4.12). Then applying (4.10), we have

$$d_n(\boldsymbol{x}) \leq C(|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})| + |\nabla \boldsymbol{v}_n(\boldsymbol{x})| + \alpha_n^{p-2}(|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})|^{p-1} + |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^{p-1})|\nabla \boldsymbol{z}_n(\boldsymbol{x})|$$

Applying Cauchy-Schwarz inequality and using the relations $\boldsymbol{\zeta}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{\psi}_n$ and $\boldsymbol{t}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{v}_n$ from Lemma 6, we get

$$\int_{R_n} d_n(\boldsymbol{x}) d\boldsymbol{x} \leq C(\|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})\|_2 + \|\nabla \boldsymbol{v}_n(\boldsymbol{x})\|_2) \left(\int_{R_n} |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 d\boldsymbol{x} \right)^{1/2}$$
$$+ \frac{\beta_n^p}{\alpha_n^2}(1 + \|\nabla \boldsymbol{t}_n(\boldsymbol{x})\|_p^{p-1}) \left(\int_{R_n} |\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p \right)^{1/p}.$$

Again equiintegrability of $|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2$ and $|\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p$, and our boundedness assumption on $\frac{\beta_n^p}{\alpha_n^2}$ imply the last two terms go to 0, as $n \to \infty$.

Finally since $W \in \overline{\mathcal{L}}$ by Lemma 3 there exist a sequence of functions $U_k(\boldsymbol{x}, \boldsymbol{F})|\boldsymbol{F}|^2$ satisfying (4.8) such that $U_k(\boldsymbol{x}, \boldsymbol{F})|\boldsymbol{F}|^2 \to U(\boldsymbol{x}, \boldsymbol{F})|\boldsymbol{F}|^2$ in X_p. Let $\mathcal{F}_k(\boldsymbol{x}, \alpha, \boldsymbol{F}) = \dfrac{1}{2}(\mathsf{L}(\boldsymbol{x})\boldsymbol{F}, \boldsymbol{F}) + U_k(\boldsymbol{x}, \alpha\boldsymbol{F})|\boldsymbol{F}|^2$. Given $\epsilon > 0$ there exists k_0 such that

$$|\mathcal{F}_{k_0}(\boldsymbol{x}, \alpha, \boldsymbol{F}) - \mathcal{F}(\boldsymbol{x}, \alpha, \boldsymbol{F})| \leq \epsilon |\boldsymbol{F}|^2(1 + |\alpha \boldsymbol{F}|^{p-2})$$

for all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{F} \in \mathbb{M}$. Then we have

$$|I_n(\boldsymbol{x}, \mathcal{F})| \leq |I_n(\boldsymbol{x}, \mathcal{F}_{k_0})| + |I_n(\boldsymbol{x}, \mathcal{F}) - I_n(\boldsymbol{x}, \mathcal{F}_{k_0})|$$

where $|I_n(\boldsymbol{x}, \mathcal{F}) - I_n(\boldsymbol{x}, \mathcal{F}_{k_0})|$ is dominated by $\epsilon r_n(\boldsymbol{x})$ and

$$r_n(\boldsymbol{x}) = C(|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})|^2 + |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^2 + |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 + \alpha_n^{p-2}(|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})|^p + |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^p + |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^p))$$

is bounded in L^1. Thus by way of what we proved earlier we have

$$\lim_{n\to\infty}\int_\Omega |I_n(\boldsymbol{x},\mathcal{F})|d\boldsymbol{x} \le \epsilon C.$$

That finishes the proof of the lemma. ∎

4.5 Representation formula

Our aim is proving the nonnegativity of $\delta' E(\{\boldsymbol{\phi}_n\})$ under the assumptions of Theorem 5, where

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty}\int_\Omega \mathcal{F}(\boldsymbol{x},\alpha_n,\nabla\boldsymbol{\psi}_n)d\boldsymbol{x}.$$

The Orthogonality Principle that we proved in the previous section allow us to decompose $\delta' E(\{\boldsymbol{\phi}_n\})$ as

$$\delta' E(\{\boldsymbol{\phi}_n\}) \ge \varliminf_{n\to\infty}\int_\Omega \mathcal{F}(\boldsymbol{x},\alpha_n,\nabla\boldsymbol{z}_n)d\boldsymbol{x} + \varliminf_{n\to\infty}\int_\Omega \mathcal{F}(\boldsymbol{x},\alpha_n,\nabla\boldsymbol{v}_n)d\boldsymbol{x} \tag{4.13}$$

where from the Decomposition Lemma, $\boldsymbol{\psi}_n = \boldsymbol{z}_n+\boldsymbol{v}_n$, $\nabla\boldsymbol{z}_n$ is square-equiintegrable and $\boldsymbol{v}_n$ is non zero on a set of small measure. We note the two terms in (4.13) reflect two different effects of the $\{\boldsymbol{\psi}_n\}$ on $\mathcal{F}$. In this section we will drive representation formulas for each of the terms on the right hand side of (4.13).

Let us start with the first term

$$T_1 = \varliminf_{n\to\infty}\int_\Omega \mathcal{F}(\boldsymbol{x},\alpha_n,\nabla\boldsymbol{z}_n)d\boldsymbol{x}.$$

Because of the equiintegrability of $|\nabla\boldsymbol{z}_n|^2$ this term can be represented using Young measures. In fact applying Corollary 1.3, the two sequences $\{\nabla\boldsymbol{z}_n\}$ and $\{\nabla\boldsymbol{\psi}_n\}$ generate the same gradient young measures. Let $\nu^0 = \{\nu^0_{\boldsymbol{x}}\}$ be the gradient Young measure generated by $\{\nabla\boldsymbol{z}_n\}$. Equiintegrability of $|\nabla\boldsymbol{z}|^2$ and the vanishing of $U(\boldsymbol{x},\boldsymbol{F})$ on $\overline{\Omega}\times\{\boldsymbol{0}\}$ imply that when $\alpha_n\to 0$, the the action of $\boldsymbol{z}_n$ on $\mathcal{F}$ can be described only by the gradient Young measure $\nu^0_{\boldsymbol{x}}$.

Lemma 6 *There exists a subsequence, not relabeled, such that*

$$T_1 = \frac{1}{2}\int_{\overline{\Omega}}\int_{\mathbb{R}^{m\times d}} (\mathsf{L}(\boldsymbol{x})\boldsymbol{F},\boldsymbol{F})d\nu^0_{\boldsymbol{x}}(\boldsymbol{F})d\boldsymbol{x} \tag{4.14}$$

We call $\boldsymbol{z}_n$ the weak part of the variation because its action on the functional is described by the second variation of E.

PROOF OF LEMMA 6: It only suffices to show that, there exists a subsequence such that

$$\lim_{n\to\infty}\int_\Omega U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x}))|\nabla\boldsymbol{z}_n(\boldsymbol{x})|^2=0$$

because applying Theorem 1 and from the equiintegrability of $|\nabla\boldsymbol{z}_n|^2$ we have

$$(\mathsf{L}(\boldsymbol{x})\nabla\boldsymbol{z}_n(\boldsymbol{x}),\nabla\boldsymbol{z}_n(\boldsymbol{x}))\ \rightharpoonup\ \int_{\mathbb{R}^{m\times d}}(\mathsf{L}(\boldsymbol{x})\boldsymbol{F},\boldsymbol{F})d\nu^0_{\boldsymbol{x}}(\boldsymbol{F}),$$

weakly in $L^1(\Omega)$. To show this we use the equiintegrability of $|\nabla\boldsymbol{z}_n|^2$ and its scaled relation $|\nabla\boldsymbol{s}_n|^p$. Let $\epsilon>0$ and $\delta>0$ be such that

$$\sup_n\int_E|\nabla\boldsymbol{s}_n|^p d\boldsymbol{x}<\epsilon \text{ and } \sup_n\int_E|\nabla\boldsymbol{z}_n|^2 d\boldsymbol{x}<\epsilon, \tag{4.15}$$

whenever E is measurable and $|E|<\delta$. Since $\nabla\boldsymbol{z}_n$ is bounded in L^2 and $\alpha_n\to 0$, we have $\alpha_n\nabla\boldsymbol{z}_n\to\boldsymbol{0}$ in L^2, and we can find a subsequence, not relabeled, such that $\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x})\to\boldsymbol{0}$ a.e. $\boldsymbol{x}\in\Omega$. Applying Egorov's theorem, we can find $E\subset\Omega$ such that $|E|<\delta$ and $\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x})\to\boldsymbol{0}$ uniformly on $\Omega\setminus E$. Now on $\Omega\setminus E$, $\|\alpha_n\nabla\boldsymbol{z}_n\|_\infty$ is uniformly bounded, and we can find N such that $|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x}))|\le\epsilon$ for all $n\ge N$ and for all $\boldsymbol{x}\in\Omega\setminus E$ since $U(\boldsymbol{x},\boldsymbol{0})=0$. Also for all $\boldsymbol{x}\in E$, $|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x})|\le C(1+|\alpha_n\nabla\boldsymbol{z}_n(\boldsymbol{x})|^{p-2})$ for some constant C. Then for all $n\ge N$ dividing the integral over the two sets, we have

$$\int_\Omega|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n)||\nabla\boldsymbol{z}_n)|^2d\boldsymbol{x}\le\epsilon\int_{\Omega\setminus E}|\nabla\boldsymbol{z}_n|^2d\boldsymbol{x}+\int_E|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n)||\nabla\boldsymbol{z}_n)|^2d\boldsymbol{x}$$

We can estimate the second integral

$$\int_E|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n)||\nabla\boldsymbol{z}_n)|^2d\boldsymbol{x}\le C\int_E|\nabla\boldsymbol{z}_n|^2d\boldsymbol{x}+C\frac{\beta_n^p}{\alpha_n^2}\int_E|\nabla\boldsymbol{s}_n|^pd\boldsymbol{x}$$

By (4.15) we conclude that

$$\int_\Omega|U(\boldsymbol{x},\alpha_n\nabla\boldsymbol{z}_n)||\nabla\boldsymbol{z}_n)|^2d\boldsymbol{x}\le\epsilon C$$

for all $n\ge N$, which finishes the proof of the lemma. ∎

Next we compute the second term of the right hand side of (4.13)

$$T_2 = \underline{\lim}_{n\to\infty} \int_\Omega \mathcal{F}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x}.$$

Passing to a subsequence, not relabeled, and expanding the expression we may write

$$T_2 = T_{21} + T_{22}$$

where

$$T_{21} = \lim_{n\to\infty} \frac{1}{2} \int_\Omega (\mathsf{L}(\boldsymbol{x})\nabla \boldsymbol{v}_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x} \text{ and } T_{22} = \lim_{n\to\infty} \int_\Omega U(\boldsymbol{x}, \alpha \nabla \boldsymbol{v}_n(\boldsymbol{x})) |\nabla \boldsymbol{v}_n| dx$$

The first term T_{21} cannot be written in terms of Young measures as $|\nabla \boldsymbol{v}|^2$ is not equiintegrable. Instead, we apply Fonseca's Varifold Theorem, Theorem 2 and the remark following it to compute the limit. We note that the function $f(\boldsymbol{x}, \boldsymbol{F}) = (\mathsf{L}(\boldsymbol{x})\boldsymbol{F}, \boldsymbol{F})$ is a continuous function which is homogeneous of degree 2 in the second variable. Thus, we can find a family of probability measures $\lambda = \{\lambda_{\boldsymbol{x}}\}$ on the unit sphere $\mathbb{S}$ of $\mathbb{M}$ and a nonnegative measure π on $\overline{\Omega}$ such that

$$\lim_{n\to\infty} \frac{1}{2} \int_\Omega (\mathsf{L}(\boldsymbol{x})\nabla \boldsymbol{v}_n(\boldsymbol{x})), \nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x} = \int_{\overline{\Omega}} Q(\boldsymbol{x}) d\pi(\boldsymbol{x}) \tag{4.16}$$

where $Q(\boldsymbol{x}) = \dfrac{1}{2} \displaystyle\int_{\mathbb{S}} (\mathsf{L}(\boldsymbol{x})\boldsymbol{F}, \boldsymbol{F}) d\lambda_x(\boldsymbol{F})$. The nonnegative measure π on $\overline{\Omega}$ is the weak-* limit of the sequence of measures $|\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x}$.

To compute the second term T_{22} we follow the approach of DiPerna and Majda [10]. Let us rewrite the integrand in terms of the bounded continuous function

$$B(\boldsymbol{x}, \boldsymbol{F}) = \frac{U(\boldsymbol{x}, \boldsymbol{F})}{1 + |\boldsymbol{F}|^{p-2}}.$$

Then it follows that

$$U(\boldsymbol{x}, \boldsymbol{F})|\boldsymbol{F}|^2 = B(\boldsymbol{x}, \boldsymbol{F})|\boldsymbol{F}|^2(1 + |\boldsymbol{F}|^{p-2})$$

and

$$T_{22} = \lim_{n\to\infty} \int_\Omega B(\boldsymbol{x}, \alpha_n \nabla \boldsymbol{v}_n(\boldsymbol{x})) |\nabla \boldsymbol{v}_n(\boldsymbol{x}))|^2 (1 + |\alpha_n \nabla \boldsymbol{v}_n(\boldsymbol{x}))|^{p-2}) d\boldsymbol{x}$$

We prove the following representation formula.

Lemma 7 *There exist a subsequence, not relabeled, a nonnegative measure σ on $\overline{\Omega}$ and a family of probability measure $\mu = \{\mu_{\boldsymbol{x}}\}$ on the Stone-Čech compactification of $\mathbb{M}$ such that*

$$B(\boldsymbol{x}, \alpha_n \nabla \boldsymbol{v}_n)(1 + |\alpha_n \nabla \boldsymbol{v}_n|^{p-2})|\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x} \stackrel{*}{\rightharpoonup} [\int_{\beta\mathbb{M}} B(\boldsymbol{x}, \boldsymbol{F}) d\mu_{\boldsymbol{x}}(\boldsymbol{F})] d\sigma,$$

in the sense of measures. In particular,

$$T_{22} = \int_{\overline{\Omega}} \int_{\beta\mathbb{M}} B(\boldsymbol{x}, \boldsymbol{F}) d\mu_{\boldsymbol{x}}(\boldsymbol{F}) d\sigma(\boldsymbol{x}). \tag{4.17}$$

Proof: The lemma is a consequence of the fact that the space $C_B(\mathbb{M})$ of bounded continuous functions on $\mathbb{M}$, is isometrically isomorphic to the space of continuous functions on the Stone-Čech compactification $\beta\mathbb{M}$ of $\mathbb{M}$. Here we ignore canonical isomorphisms between spaces of continuous functions and do not make a notational distinction between $g \in C(\beta\mathbb{M})$ and its isomorphic image $g \in C_B(\mathbb{M})$. Thus, we write

$$C_B(\overline{\Omega} \times \mathbb{M})^* = \mathcal{M}(\overline{\Omega} \times \beta\mathbb{M}), \tag{4.18}$$

where $\mathcal{M}(\overline{\Omega} \times \beta\mathbb{M})$ is the space of Radon measures on the compact Hausdorff space $\overline{\Omega} \times \beta\mathbb{M}$. For each fixed n, the functional

$$\Lambda_n(B) = \int_{\Omega} B(\boldsymbol{x}, \alpha_n \nabla \boldsymbol{v}_n)(1 + |\alpha_n \nabla \boldsymbol{v}_n|^{p-2})|\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x}$$

is a linear and continuous functional on $C_B(\overline{\Omega} \times \mathbb{M})$. Moreover, the sequence Λ_n corresponds to the bounded sequence of positive Radon measures on $\overline{\Omega} \times \beta\mathbb{M}$ via the isomorphism (4.18). By the Banach-Alaoglu theorem there exists a subsequence, not relabeled, and a non-negative Radon measure $\Lambda \in \mathcal{M}(\overline{\Omega} \times \beta\mathbb{M})$ such that $\Lambda_n \stackrel{*}{\rightharpoonup} \Lambda$ in the sense of measures. Setting $B(\boldsymbol{x}, \boldsymbol{F}) = B(\boldsymbol{x})$, we get that the projection σ of Λ onto $\overline{\Omega}$ is a weak-* limit (in the sense of measures) of $(1 + |\alpha_n \nabla \boldsymbol{v}_n|^{p-2})|\nabla \boldsymbol{v}_n|^2$.

The slicing decomposition theorem for the measure Λ (see [14]) allows us to represent the measure Λ in terms of its projection σ onto $\overline{\Omega}$ and a family of probability measures $\{\mu_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \overline{\Omega}}$ on $\beta\mathbb{M}$, such that $\Lambda = \mu_{\boldsymbol{x}} \otimes \sigma$, i.e

$$T_{22} = \int_{\overline{\Omega}} \int_{\beta\mathbb{M}} B(\boldsymbol{x}, \boldsymbol{F}) d\mu_{\boldsymbol{x}}(\boldsymbol{F}) d\sigma(\boldsymbol{x}).$$

The proof of the lemma is complete. ▮

Remark 12 For $B(\boldsymbol{x}, \boldsymbol{F}) \in C_B(\overline{\Omega} \times \mathbb{M})$ define

$$T(B)(\boldsymbol{x}) = \int_{\beta\mathbb{M}} B(\boldsymbol{x}, \boldsymbol{F}) d\mu_{\boldsymbol{x}}(\boldsymbol{F})$$

In particular, for $\boldsymbol{x}_0 \in \overline{\Omega}$, $B(\boldsymbol{x}_0, \cdot) \in C_B(\overline{\Omega} \times \mathbb{M})$, and

$$T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}) = \int_{\beta\mathbb{M}} B(\boldsymbol{x}_0, \boldsymbol{F}) d\mu_{\boldsymbol{x}}(\boldsymbol{F}).$$

Evaluating at $\boldsymbol{x}_0$, we obtain that $T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}_0) = T(B)(\boldsymbol{x}_0)$

Remark 13 The measures π, σ and $\mu_{\boldsymbol{x}}$ are related. In fact, applying Lemma 7, we have

$$|\nabla \boldsymbol{v}_n|^2 = \frac{1}{1 + |\alpha_n \nabla \boldsymbol{v}_n|^{p-2}} (1 + |\alpha_n \nabla \boldsymbol{v}_n|^{p-2}) |\nabla \boldsymbol{v}_n|^2 \overset{*}{\rightharpoonup} \tau(\boldsymbol{x}) d\sigma \tag{4.19}$$

where

$$\tau(\boldsymbol{x}) = \int_{\beta\mathbb{M}} \frac{1}{1 + |\boldsymbol{F}|^{p-2}} d\mu_{\boldsymbol{x}}(\boldsymbol{F}).$$

Thus, π is absolutely continuous with respect to σ, and

$$\pi = \tau(\boldsymbol{x}) \sigma. \tag{4.20}$$

Combining (4.16), (4.17) and (4.20) we have the following representation formula for T_2.

$$T_2 = \int_{\overline{\Omega}} \left(Q(\boldsymbol{x}) \tau(\boldsymbol{x}) + T(B)(\boldsymbol{x}) \right) d\sigma(\boldsymbol{x}) \tag{4.21}$$

In summary we have shown that $\delta' E(\{\boldsymbol{\phi}_n\}) \geq T_1 + T_2$, where T_1 is given by (4.14) and T_2 is given by (4.21). Our goal now is to show that $T_1 \geq 0$, and $T_2 \geq 0$.

The validity of the first inequality follows from the non-negativity of second variation. To see that, $\|\nabla \boldsymbol{\psi}_n\|_2 = 1$ and $\boldsymbol{\psi}_n|_{\partial\Omega_1} = 0$, imply that there exists $\boldsymbol{\psi}_0 \in W^{1,2}(\Omega; \mathbb{R}^m)$ satisfying $\boldsymbol{\psi}_0|_{\partial\Omega_1} = 0$ and a subsequence $\{\boldsymbol{\psi}_n\}$, not relabeled, such that $\boldsymbol{\psi}_n \rightharpoonup \boldsymbol{\psi}_0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^m)$. Since $\boldsymbol{v}_n \rightharpoonup 0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^m)$, we have $\boldsymbol{z}_n \rightharpoonup \boldsymbol{\psi}_0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^m)$. By Lemma 1 we can find a sequence $\widetilde{\boldsymbol{z}}_n$ such that $\widetilde{\boldsymbol{z}}_n - \psi_0 \in W_0^{1,2}(\Omega;\mathbb{R}^m)$ and $\nabla \boldsymbol{z}_n$ and $\nabla\widetilde{\boldsymbol{z}}_n$ generate the same Young measure $\nu^0 = \{\nu_{\boldsymbol{x}}^0\}_{\boldsymbol{x}\in\Omega}$. We observe that $\widetilde{\boldsymbol{z}}_n \in W^{1,2}(\Omega;\mathbb{R}^m)$ satisfying $\widetilde{\boldsymbol{z}}_n|_{\partial\Omega_1} = 0$. From the non-negativity of the second variation, condition (ii) of the theorem

$$0 \le \int_\Omega (\mathsf{L}(\boldsymbol{x})\nabla\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}), \nabla\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}))d\boldsymbol{x}$$

Taking limit as $n\to\infty$ in the above inequality we have

$$0 \le \frac{1}{2}\lim_{n\to\infty}\int_\Omega (\mathsf{L}(\boldsymbol{x})\nabla\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}), \nabla\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}))d\boldsymbol{x} = \frac{1}{2}\int_\Omega\int_{\mathbb{R}^{m\times d}} (\mathsf{L}(\boldsymbol{x})\boldsymbol{F}\boldsymbol{F})d\nu_{\boldsymbol{x}}^0(\boldsymbol{F})d\boldsymbol{x} = T_1.$$

The validity of the second inequality, $T_2 \ge 0$, is not obvious, and follows from the quasiconvexity conditions. However, T_2, by its definition, has a geometrically global character, while the quasiconvexity conditions have local character. In order to exhibit the local character of T_2 and link it to the quasiconvexity conditions, we need the localization principle.

4.6 Localization

Theorem 7 (Localization principle in the interior)) *Let $\boldsymbol{x}_0 \in \Omega\cup\overline{\partial\Omega_1}$. Let the cut-off functions $\theta_k^r(\boldsymbol{x}) \in C_0^\infty(B_\Omega(\boldsymbol{x}_0,r))$ be such that $\theta_k^r(\boldsymbol{x}) \to \chi_{B_\Omega(\boldsymbol{x}_0,r)}(\boldsymbol{x})$ for all $\boldsymbol{x}\in\overline{\Omega}$, as $k\to\infty$ and $0\le\theta_k^r\le 1$. Then for σ a.e. $\boldsymbol{x}_0 \in \Omega\cup\overline{\partial\Omega_1}$, we have*

$$\lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\sigma(B_\Omega(\boldsymbol{x}_0,r))}\int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})))d\boldsymbol{x}$$
$$= Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B(\boldsymbol{x}_0,\cdot))(\boldsymbol{x}_0)$$

In order to formulate the localization principle for the free boundary we have to take care of the geometry of the domain. The reason is the quasiconvexity at the free boundary inequality requires to have a domain with a "flat" part of the boundary with the outer unit normal $\boldsymbol{n}(\boldsymbol{x}_0)$. Let us define the set

$$B_r^- = \frac{B_\Omega(\boldsymbol{x}_0,r)-\boldsymbol{x}_0}{r}.$$

Then B_r^- is almost the half ball $B_{\boldsymbol{n}(\boldsymbol{x}_0)}(\mathbf{0},1)$. As $r \to 0$ the set B_r^- "converges" to $B_{\boldsymbol{n}(\boldsymbol{x}_0)}^-(\mathbf{0},1)$. To be precise we have the following lemma whose proof is postponed for later.

Lemma 8 *There exist functions* $\boldsymbol{f}_r \in C^1(\overline{B_r^-};\mathbb{R}^d)$ *such that* $\boldsymbol{f}_r$ *are diffeomorphism between* B_r^- *and* $B_{\boldsymbol{n}(\boldsymbol{x}_0)}^-(\mathbf{0},1)$ *and* $\boldsymbol{f}_r(\boldsymbol{x}) \to \boldsymbol{x}$ *and* $\nabla \boldsymbol{f}_r(\boldsymbol{x}) \to \boldsymbol{I}$ *uniformly as* $r \to 0$.

Let

$$\boldsymbol{v}_n^r(\boldsymbol{x}) = \frac{\boldsymbol{v}_n(\boldsymbol{x}_0 + r\boldsymbol{f}_r(\boldsymbol{x})) - C_n^r(\boldsymbol{x}_0)}{r} \tag{4.22}$$

be the blown-up version of $\boldsymbol{v}_n$ defined on $B_{\boldsymbol{n}(\boldsymbol{x}_0)}^+(\mathbf{0},1)$, where C_n^r is chosen in such a way that the average of $\boldsymbol{v}_n^r$ is zero. Denote $\boldsymbol{t}_n^r(\boldsymbol{x}) = \frac{\alpha_n}{\beta_n}\boldsymbol{v}_n^r(\boldsymbol{x})$.

Theorem 8 (Localization at a boundary point) *Let* $\boldsymbol{x}_0 \in \partial\Omega_2 \cap supp(\sigma)$ *Let the cut-off functions* $\theta_k(\boldsymbol{x}) \in C_0^\infty(B(\mathbf{0},1))$ *be such that* $\theta_k(\boldsymbol{x}) \to \chi_{B(\mathbf{0},1)}(\boldsymbol{x})$, *as* $k \to \infty$ *and* $0 \le \theta_k(\boldsymbol{x}) \le 1$. *Let* $\xi_{n,k}^r(\boldsymbol{x}) = \theta_k(\boldsymbol{x})\boldsymbol{v}_n^r(\boldsymbol{x})$. *Then*

$$\begin{aligned}\lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty}\frac{r^d}{\sigma(B_\Omega(\boldsymbol{x}_0,r))}\int_{B_{\boldsymbol{n}(\boldsymbol{x}_0)}^-(\mathbf{0},1)} &\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla\boldsymbol{\xi}_{n,k}^r(\boldsymbol{x}))d\boldsymbol{x} \\ &= Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B(\boldsymbol{x}_0,\cdot))(\boldsymbol{x}_0)\end{aligned} \tag{4.23}$$

for σ *a.e.* $\boldsymbol{x}_0 \in \partial\Omega_2$.

The remaining part of this section focus on proving the above stated results.

Proof of Localization at the Interior: We prove the theorem for $U(\boldsymbol{x},\boldsymbol{F})$ satisfying (4.8). For a general $U(\boldsymbol{x},\boldsymbol{F})$ corresponding to W, we can approximate it by functions satisfying (4.8) and the proof of the theorem can be carried out in a similar manner as in the proof of Lemma 5. This approximation is possible because of Lemma 3. We carry out the proof in several steps

Step 1. First we show that the gradient of the cut off functions do not appear in the limit.

Lemma 9 *For each k, and r*

$$\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x} = \lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}$$

PROOF: Let

$$S_{n,k,r}(\boldsymbol{x}) = |\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x}))) - \mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))|$$

Then we show that $S_{n,k,r} \to 0$ strongly in L^1 as $n \to \infty$. From the estimate (4.10), we get a positive constant $C = C(k,r)$ such that

$$\begin{aligned} S_{n,k,r}(\boldsymbol{x}) \le\; & C(|\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x}))| + |\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x})|)|\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})| \\ & + C(\alpha_n^{p-2}(|\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x}))|^{p-1} + |\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x})|^{p-1})|\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})|) \end{aligned}$$

Then using the relation $\boldsymbol{t}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{v}_n$, and the assumption that $\frac{\beta_n^p}{\alpha_n^2}$ is bounded we have

$$\begin{aligned} \int_{B_\Omega(\boldsymbol{x}_0,r)} S_{n,k,r}(\boldsymbol{x})d\boldsymbol{x} \le & C(\|\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x}))\|_2 + \|\nabla\boldsymbol{v}_n(\boldsymbol{x})\|_2)\|\boldsymbol{v}_n\|_2) \\ & + C(\|\nabla(\theta_k^r(\boldsymbol{x})\boldsymbol{t}_n(\boldsymbol{x}))\|_p^{p-1} + \|\nabla\boldsymbol{t}_n(\boldsymbol{x})\|_p^{p-1})\|\boldsymbol{t}_n\|_p. \end{aligned}$$

The lemma is proved since$\boldsymbol{v}_n \rightharpoonup 0$ in $W^{1,2}$ and $\boldsymbol{t}_n = \frac{\alpha_n}{\beta_n}\boldsymbol{v}_n \rightharpoonup 0$ weakly in $W^{1,p}$. ∎

Step 2. The limit in the right hand side of the above lemma can be computed using our representation formula (4.21). To this end, we begin with the following lemma.

Lemma 10 *For each $r > 0$,*

$$\begin{aligned} & \lim_{k\to\infty}\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x} \\ & \qquad = \lim_{k\to\infty}\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} \theta_k^r(\boldsymbol{x})^2\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x} \end{aligned}$$

PROOF: Let

$$T_{n,k,r}(\boldsymbol{x}) = |\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x})) - \theta_k^r(\boldsymbol{x})^2\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla\boldsymbol{v}_n(\boldsymbol{x}))|.$$

To prove the lemma it suffices to show that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} T_{n,k,r}(\boldsymbol{x})d\boldsymbol{x}=0$$

We notice that

$$T_{n,k,r}(\boldsymbol{x})=|U(\boldsymbol{x}_0,\alpha_n\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))|\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2-\theta_k^r(\boldsymbol{x})^2U(\boldsymbol{x}_0,\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x}))||\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2.$$

But then $U(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2=B(\boldsymbol{x},\boldsymbol{F})|\boldsymbol{F}|^2(1+|\boldsymbol{F}|^{p-2})$ where B is a bounded continuous function on $\Omega\times\mathbb{M}$. Then we have

$$T_{n,k,r}(\boldsymbol{x})=\theta_k^r(\boldsymbol{x})^2|B(\boldsymbol{x},\alpha_n\theta_k^r(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))-B(\boldsymbol{x},\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x}))||\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2(1+|\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x})|^{p-2})$$

Because U satisfies (4.8), we can find a positive constant C such that for any $\boldsymbol{x}$, $\boldsymbol{F}$ and θ

$$|B(\boldsymbol{x},\theta\boldsymbol{F})-B(\boldsymbol{x},\boldsymbol{F})|\le C|\theta-1|.$$

Then it follows that

$$T_{n,k,r}(\boldsymbol{x})\le C\theta_k^r(\boldsymbol{x})^2|\theta_k^r(\boldsymbol{x})-1||\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2(1+|\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x})|^{p-2}).$$

Applying the representation theorem Theorem 7 to the right hand side of the above inequality, it follows that

$$\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} T_{n,k,r}(\boldsymbol{x})d\boldsymbol{x}\le C\int_{B_\Omega(\boldsymbol{x}_0,r)}\theta_k^r(\boldsymbol{x})^2|\theta_k^r(\boldsymbol{x})-1|d\sigma(\boldsymbol{x}).$$

By bounded convergence theorem and using the assumption that $\theta_k^r(\boldsymbol{x})\to\chi_{B_\Omega(\boldsymbol{x}_0,r)}(\boldsymbol{x})$, we have

$$\lim_{k\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)}\theta_k^r(\boldsymbol{x})^2|\theta_k^r(\boldsymbol{x})-1|d\sigma(\boldsymbol{x})=0.$$

That finishes the proof of the lemma. ∎

Step 3. Let us compute the limit in the right hand side of the above lemma. Taking first the limit as $n\to\infty$, and then the limit as $k\to\infty$, we obtain by bounded convergence theorem

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty}&\int_{B_\Omega(\boldsymbol{x}_0,r)}\theta_k^r(\boldsymbol{x})^2\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}\\ &=\int_{B_\Omega(\boldsymbol{x}_0,r)}(Q(\boldsymbol{x})\tau(\boldsymbol{x})+T(B(\boldsymbol{x}_0,\cdot))(\boldsymbol{x}))d\sigma(\boldsymbol{x})\end{aligned}\tag{4.24}$$

Step 4. In order to finish the proof of Theorem 7 we need to divide both sides of (4.24) by $\sigma(B_\Omega(\boldsymbol{x}_0, r))$, and take the limit as $r \to 0$. The result is a corollary of Lebesgue differentiation theorem. Indeed, for σ a.e. $\boldsymbol{x}_0 \in \overline{\Omega}$,

$$\lim_{r\to 0} \frac{1}{\sigma(B_\Omega(\boldsymbol{x}_0, r))} \int_{B_\Omega(\boldsymbol{x}_0,r)} (Q(\boldsymbol{x})\tau(\boldsymbol{x}) + T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}))d\sigma(\boldsymbol{x}) \\ = Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}_0)$$

This completes the proof of the theorem. ∎

Proof of Localization at the Boundary: The proof follows the same sequence of steps as in the proof of the last theorem, except that here we need to handle the deformations $\boldsymbol{f}_r$.

Step 1. Let us show that the gradient of the cut of functions do not enter in the limit (4.23).

Lemma 11

$$\lim_{n\to\infty} \int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{0},1)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \nabla \boldsymbol{\xi}^r_{n,k}(\boldsymbol{x}))d\boldsymbol{x} = \lim_{n\to\infty} \int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{0},1)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{x})\nabla \boldsymbol{v}^r_{n,k}(\boldsymbol{x}))d\boldsymbol{x} \tag{4.25}$$

The proof is very similar to the proof of Lemma 9 of Theorem 7.

Step 2. Let us compute the limit as $n \to \infty$ and then take the limit as $k \to \infty$. We make the change of variables

$$\boldsymbol{x}' = \boldsymbol{x}_0 + r\boldsymbol{f}_r(\boldsymbol{x})$$

in the right hand side of (4.25). Solving for $\boldsymbol{x}$ in the above equation we get

$$\boldsymbol{x} = \boldsymbol{p}_r(\boldsymbol{x}') = \boldsymbol{f}_r^{-1}((\boldsymbol{x}' - \boldsymbol{x}_0)/r).$$

Then

$$\lim_{n\to\infty} \int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{0},1)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{x})\nabla \boldsymbol{v}^r_{n,k}(\boldsymbol{x}))d\boldsymbol{x} \\ = \lim_{n\to\infty} \int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{p}_r(\boldsymbol{x}'))\nabla \boldsymbol{v}_n(\boldsymbol{x}')\boldsymbol{J}_r(\boldsymbol{x}')) \frac{J_r^{-1}(\boldsymbol{x}')}{r^d} d\boldsymbol{x}' \tag{4.26}$$

where

$$\boldsymbol{J}_r(\boldsymbol{x}') = (\nabla \boldsymbol{f}_r)(\boldsymbol{f}_r^{-1}(\boldsymbol{x}' - \boldsymbol{x}_0)/r)$$

and $J_r(\boldsymbol{x}') = \det \boldsymbol{J}_r(\boldsymbol{x}')$. We represent the expression under the integral as the function $\mathcal{F}$ constructed with $\widehat{\mathsf{L}}^{(k,r)}_{\boldsymbol{x}_0}$ and $\widehat{U}^{k,r}_{\boldsymbol{x}_0}$ replacing L and U, where

$$(\widehat{\mathsf{L}}^{(k,r)}_{\boldsymbol{x}_0}(\boldsymbol{x})\boldsymbol{F}, \boldsymbol{F}) = \frac{\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))^2}{r^d J_r(\boldsymbol{x})}(\mathsf{L}(\boldsymbol{x}_0)\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x}), \mathsf{L}(\boldsymbol{x}_0)\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x}))$$

and

$$\widehat{U}^{k,r}_{\boldsymbol{x}_0}(\boldsymbol{x}, \boldsymbol{F}) = \frac{\theta(\boldsymbol{p}_r(\boldsymbol{x}))^2}{r^d J_r(\boldsymbol{x})} U(\boldsymbol{x}_0, \theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})) \frac{|\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})|^2}{|\boldsymbol{F}|^2}.$$

For notational convenience we have dropped the prime in the variable $\boldsymbol{x}$. Let

$$\widehat{\mathcal{F}}(\boldsymbol{x}, \alpha_n, \boldsymbol{F}) = \frac{1}{2}(\widehat{\mathsf{L}}^{(k,r)}_{\boldsymbol{x}_0}(\boldsymbol{x})\boldsymbol{F}, \boldsymbol{F}) + \widehat{U}^{k,r}_{\boldsymbol{x}_0}(\boldsymbol{x}, \alpha_n \boldsymbol{F})|\boldsymbol{F}|^2.$$

Again we would like to use our representation formula (4.21) to compute the limit in the right hand side of (4.26). The following lemma gives an arrangement of the limit so that we can apply the representation result.

Lemma 12 *For $\sigma - a.e$ $\boldsymbol{x}_0 \in \overline{\Omega} \cap supp(\sigma)$*

$$\lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty} \frac{r^d}{\sigma(B_\Omega(\boldsymbol{x}_0, r))} \int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{x}_0,r)} \widehat{\mathcal{F}}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x}$$
$$= \lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\sigma(B_\Omega(\boldsymbol{x}_0, r))} \int_{B_\Omega(\boldsymbol{x}_0,r)} \theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))\mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x}$$

PROOF: The lemma is a consequence of the estimate

$$d^r_{n,k} \le C(|\theta_k(\boldsymbol{x}) - 1| + |\boldsymbol{J}_r - \boldsymbol{I}| + |J_r^{-1} - 1|)\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))|\boldsymbol{F}|^2(1 + |\alpha_n \boldsymbol{F}|^{p-2}) \quad (4.27)$$

for some constant C where that $d^r_{n,k}$ is the difference

$$d^r_{n,k} = |r^d \widehat{\mathcal{F}}(\boldsymbol{x}, \alpha_n, \boldsymbol{F}) - \theta^2(\boldsymbol{p}_r(\boldsymbol{x}))\mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})|.$$

Before verifying (4.27), let us complete the proof of the lemma as a consequence of (4.27). Let

$$I_{n,k,r}(\boldsymbol{x}) = |r^d \widehat{\mathcal{F}}(\boldsymbol{x}, \alpha_n, \nabla \boldsymbol{v}_n(\boldsymbol{x})) - \theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))\mathcal{F}(\boldsymbol{x}_0, \alpha_n, \theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x}|.$$

Then we only need to show that for $\sigma - a.e\ \boldsymbol{x}_0 \in \overline{\Omega} \cap supp(\sigma)$

$$\lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\sigma(B_\Omega(\boldsymbol{x}_0,r))}\int_{B_\Omega(\boldsymbol{x}_0,r)} I_{n,k,r}(\boldsymbol{x})d\boldsymbol{x} = 0. \tag{4.28}$$

Using the inequality (4.27) we obtain that

$$\begin{aligned}I_{n,k,r}(\boldsymbol{x}) \le C(&|\theta_k(\boldsymbol{x})-1|)\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))|\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2(1+|\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x})|^{p-2}\\ &+(|\boldsymbol{J}_r(\boldsymbol{x})-\boldsymbol{I}|+|j_r^{-1}(\boldsymbol{x})-1|)\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))|\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2(1+|\alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x})|^{p-2})\end{aligned}$$

Integrating both sides of the inequality over $B_\Omega(\boldsymbol{x}_0,r)$ and taking the limit as $n\to\infty$ first and $k\to\infty$ next we get

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{x}_0,r)} I_{n,k,r}(\boldsymbol{x})d\boldsymbol{x} \le \int_{B_\Omega(\boldsymbol{x}_0,r)} C(|\boldsymbol{J}_r(\boldsymbol{x})-\boldsymbol{I}|+|j_r^{-1}(\boldsymbol{x})-1|)d\sigma(\boldsymbol{x})$$

Here we use the representation result proved in Section 4.5 and the fact that $\theta_k\to 0$ in L^∞. Last, we divide by $\sigma(B_\Omega(\boldsymbol{x}_0,r))$ and take the limit as $r\to 0$. Applying Lebesgue differentiation theorem for $\sigma - a.e\ \boldsymbol{x}\in\overline{\Omega}\cap supp(\sigma)$

$$\begin{aligned}\lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty}&\frac{1}{\sigma(B_\Omega(\boldsymbol{x}_0,r))}\int_{B_\Omega(\boldsymbol{x}_0,r)} I_{n,k,r}(\boldsymbol{x})d\boldsymbol{x}\\ &\le C\lim_{r\to 0}\sup_{\boldsymbol{y}\in B_\Omega(\boldsymbol{x}_0,r)}|\boldsymbol{J}_r(\boldsymbol{x})-\boldsymbol{I}|+|j_r^{-1}(\boldsymbol{x})-1|\end{aligned}$$

The right hand side is zero from the properties of the deformations $\boldsymbol{f}_r$. Next we verify (4.27). Observe that

$$r^d\widehat{\mathcal{F}}(\boldsymbol{x},\alpha_n,\boldsymbol{F}) = J_r^{-1}(\boldsymbol{x})\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}\boldsymbol{J}_r)$$

Then application of triangular inequality yields

$$\begin{aligned}d_{n,k}^r \le |J_r^{-1}(\boldsymbol{x})(&\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}\boldsymbol{J}_r)-\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}))|\\ &+|J_r^{-1}(\boldsymbol{x})-1|\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})+|\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))-1|\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})\end{aligned} \tag{4.29}$$

The last two terms of (4.29) can be estimated using (4.11) as we can find a constant C such that

$$|\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})| \le C\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))|\boldsymbol{F}|^2(1+|\alpha_n\boldsymbol{F}|^{p-2}). \tag{4.30}$$

The first term of (4.29) can be estimated using (4.10). It follows that we can find a constant C such that

$$|J_r^{-1}(\boldsymbol{x})\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})) - J_r^{-1}(\boldsymbol{x})\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})| \\ \le C\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))(|\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})| + |\boldsymbol{F}| + \alpha_n^{p-2}|\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})|^{p-1} + \alpha_n^{p-2}|\boldsymbol{F}|^{p-1})|\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x}) - \boldsymbol{F}|.$$

Simplification of the right hand side yields

$$|J_r^{-1}(\boldsymbol{x})\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F}\boldsymbol{J}_r(\boldsymbol{x})) - J_r^{-1}(\boldsymbol{x})\mathcal{F}(\boldsymbol{x}_0,\alpha_n,\theta_k(\boldsymbol{p}_r(\boldsymbol{x}))\boldsymbol{F})| \\ \le C\theta_k^2(\boldsymbol{p}_r(\boldsymbol{x}))|\boldsymbol{J}_r(\boldsymbol{x}) - \boldsymbol{I}||\boldsymbol{F}|^2(1 + |\alpha_n\boldsymbol{F}|^{p-2}). \tag{4.31}$$

Combination of (4.30) and (4.31) yields the desired inequality (4.27). ▮

Step 3. The remaining part of the proof is very similar to Step 3, and Step 4 of the proof of the Theorem 7. ▮

PROOF OF LEMMA 8 : We may assume that without loss of generality that $\boldsymbol{x}_0 = \boldsymbol{0}$ and that the tangent plane to $\partial\Omega$ at $\boldsymbol{x}_0$ has the equation $x_d = 0$ with outer unit normal $\boldsymbol{n} = -\boldsymbol{e}_d$. Let $\boldsymbol{x}' = (x_1, \ldots, x_{d-1})$. There exists $\delta > 0$ so that the C^1 surface $\partial\Omega \cap B(\boldsymbol{0}, \delta)$ has the equation $x_d = \phi(\boldsymbol{x}')$, where ϕ is a function of class C^1, satisfying $\phi(\boldsymbol{0}) = 0$ and $\nabla\phi(\boldsymbol{0}) = \boldsymbol{0}$.

For $r < \delta$ the domain B_r is described as

$$B_r = \{(\boldsymbol{z}', z_d) \in B(\boldsymbol{0},1) : \psi_r(\boldsymbol{z}') \le z_d \le \sqrt{1-|\boldsymbol{z}'|^2}, |\boldsymbol{z}'|^2 + \psi_r(\boldsymbol{z}')^2 \le 1\}, \tag{4.32}$$

where $\psi_r(\boldsymbol{z}') = r^{-1}\phi(r\boldsymbol{z}')$. We observe that

$$\psi_r \to 0, \quad \nabla\psi_r \to \boldsymbol{0} \tag{4.33}$$

uniformly in $\boldsymbol{z}'$. Let $\boldsymbol{y} = \boldsymbol{f}_r(\boldsymbol{z})$ be defined by

$$\boldsymbol{y}' = \frac{\boldsymbol{z}'}{\sqrt{1-\psi_r(\boldsymbol{z}')^2}}, \quad y_d = \frac{(z_d - \psi_r(\boldsymbol{z}'))\sqrt{1-|\boldsymbol{z}'|^2-\psi_r(\boldsymbol{z}')^2}}{(\sqrt{1-|\boldsymbol{z}'|^2}-\psi_r(\boldsymbol{z}'))\sqrt{1-\psi_r(\boldsymbol{z}')^2}} \tag{4.34}$$

Equation (4.34) can be rewritten in the form

$$\boldsymbol{y}' = \frac{\boldsymbol{z}'}{\sqrt{1-\psi_r(\boldsymbol{z}')^2}}, \quad \frac{y_d}{\sqrt{1-|\boldsymbol{y}'|^2}} = \frac{z_d - \psi_r(\boldsymbol{z}')}{\sqrt{1-|\boldsymbol{z}'|^2}-\psi_r(\boldsymbol{z}')}$$

It follows that the first inequality in (4.32) is equivalent to

$$0 \leq \frac{y_d}{\sqrt{1-|\boldsymbol{y}'|^2}} \leq 1,$$

while the second inequality in (4.32) is equivalent to $|\boldsymbol{y}'| \leq 1$. Hence $\boldsymbol{f}_r$ maps B_r onto $B^-_{-\boldsymbol{e}_d}$. It is easy to see that in view of $((4.33))_1$ $\boldsymbol{f}_r(\boldsymbol{z}') \to \boldsymbol{z}'$ uniformly. One can also show in a straight forward calculation that $\nabla\phi_r \to \boldsymbol{I}$ uniformly. ∎

4.7 Proof of the main result

In this section we use the analytic tools developed sofar to prove Theorem 5. In order to prove the theorem it suffices to show that $T_1 \geq 0$, and $T_2 \geq 0$, where T_1, and T_2 are given by (4.14) and (4.21), respectively. We have shown already that $T_1 \geq 0$.

Next we prove that $T_2 \geq 0$. To show that it suffices to show that

$$Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B)(\boldsymbol{x}_0) \geq 0 \quad \text{for } \sigma - a.e.\boldsymbol{x}_0 \in \overline{\Omega}. \tag{4.35}$$

For $\boldsymbol{x}_0 \in \Omega \cup \partial\Omega_1$ we have that the functions $\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})$ vanish on $\partial B_\Omega(\boldsymbol{x}_0, r)$ and therefore by the quasiconvexity inequality, we have

$$\int_{B_\Omega(\boldsymbol{x}_0,r)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \nabla(\theta_k(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})))d\boldsymbol{x} \geq 0,$$

for all n, k and r. Applying Theorem 7 we obtain that that $Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}_0) \geq 0$ for $\sigma - a.e$ $\boldsymbol{x}_0 \in \Omega \cup \overline{\partial\Omega_1}$. For points $\boldsymbol{x}_0$ on $\partial\Omega_2$, we use the sequence of functions $\{\boldsymbol{\xi}^r_{n,k}(\boldsymbol{x})\}$ as defined in Theorem 8. These functions are defined on the half-ball $B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{0}, 1)$ and vanish on the "round" part of the boundary part of the half-ball. Then applying the quasiconvexity inequality at the boundary, we have

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}(\boldsymbol{0},1)} \mathcal{F}(\boldsymbol{x}_0, \alpha_n, \nabla\boldsymbol{\xi}^r_{n,k}(\boldsymbol{x}))d\boldsymbol{x} \geq 0$$

for all n, k, r. Applying Theorem 8 we also obtain that $Q(\boldsymbol{x}_0)\tau(\boldsymbol{x}_0) + T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}_0) \geq 0$ for $\sigma - a.e.$ $\boldsymbol{x}_0 \in \partial\Omega_2$. Recalling the remark following the proof of Theorem (7), we have $T(B(\boldsymbol{x}_0, \cdot))(\boldsymbol{x}_0) = T(B)(\boldsymbol{x}_0)$. Thus we proved (4.35).

REFERENCES CITED

[1] E. Acerbi and N. Fusco. Semicontinuity problems in the calculus of variations. Arch. Ration. Mech. Anal., 86(2):125-145, 1984.

[2] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal., 63(4):337-403, 1976/77.

[3] J. M. Ball. The calculus of variations and material science. Quart. Appl. Math., 56(4):719-740, 1998. Current and future challenges in the applications of mathematics (Providence, RI, 1997).

[4] J.M. Ball and R. James. Fine phase mixtures as minimizers of energy. Arch. Ration. Mech. Anal.,100, 13-52

[5] J. M. Ball and J. E. Marsden. Quasiconvexity at the boundary, positivity of the second variation and elastic stability. Arch. Ration. Mech. Anal., 86(3):251-277, 1984.

[6] M. Chipot, The use of Young measures for constructing minimizing sequences in the Calculus of Variations, Journal of Applied Analysis 1, 1, 71-81, 1995.

[7] M. Chipot and D. Kinderlehrer. Equilibrium configurations of Cystals. Arch. Ration. Mech. Anal.,100(3) 237-277, 1988.

[8] B. Dacorogna. Quasiconvexity and relaxation of nonconvex problems in the calculus of variations. J. Funct. Anal., 46(1):102-118, 1982.

[9] B. Dacorogna and P. Marcellini. Existence of minimizers for non quasi-convex integrals. Arch. Ration. Mech. Anal., 131(4):359-3999, 1995.

[10] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. Comm. Math. Phys., 108(4):667-689, 1987.

[11] L.C. Evans. Weak convergence methods for Nonlinear partial differential equations, CBMS Regional Conference Series in Mathematics, Number 74.

[12] L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[13] H. Federer. Geometric measure theory. Die Grundlehren der mathemactischen Wisenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[14] I. Fonseca. Lower semicontinuity of surface energies. Proc. Roy. Soc. Edinburgh Sect. A, 120(1-2):99-115, 1992.

[15] I. Fonseca, S. Müller, and P. Pedregal. Analysis of concentrations and oscillation effects generated by gradients. SIAM J. Math. Anal., 29(3):736 - 756 (electronic), 1998.

[16] M. Giaquinta, G. Modica, and J. Soucek. Cartesian currents in the calculus of variations. II, volume 38 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998. Variational integrals.

[17] D. Gilbarg and N.S. Trudinger. Elliptic partial differential equations of

second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[18] Y. Grabovsky and T. Mengesha. Direct approach to the problem of strong local minima in calculus of variations. Calc. Var. and PDE, 29: 59-83, 2007.

[19] Y. Grabovsky and T. Mengesha. Sufficient conditions for strong local minima: the case of C^1 extremals, submitted.

[20] Y. Grabovsky and L. Truskinovsky. Metastability in nonlinear elasticity. In preparation.

[21] M.E.Gurtin. Two phase deformations of elastic solids. Arch. Ration. Mech. Anal., 84(1):1-29, 1983.

[22] M. R. Hestenes. Sufficient conditions for multiple integral problems in the calculus of variations. Amer. J. Math., 70: 239-276, 1948.

[23] M. R. Hestenes. Calculus of variations and optimal control theory. Jonh Wiley & Sons Inc. New York, 1966.

[24] R. V. Kohn and P. Sternberg. Local minimizers and singular perturbations. Proc. Roy. Soc. Edinburgh Sect. A, 111(1-2):69-84, 1989.

[25] R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. Comm. Pure Appl. Math., 39:113-137, 139-182 and 353-377, 1986.

[26] J. Kristensen. Finite functional and Young measures generated by gradients of sobolev functions. Technical Report Mat-Report No. 1994-34, Mathematical Institute, Technical University of Denmark, 1994.

[27] J. Kristensen and A. Taheri. Partial regularity of strong local minimizers in multi-dimensional calculus of variations. Arch. Ratio. Mech. Anal., 170(1):63-89, 2003.

[28] E.Levi. Sui criterii sufficienti per il massimo e per il minimo nel Calcolo delle Variazioni. Annali di Matematica Pura ed Applicata (3), vol.21 (1913), 173-218

[29] P. Marcellini. Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. Manuscripta Math., 51(1-3):1-28, 1985.

[30] N. G. Meyers Quasi-convexity and lower semicontinuity of multiple integrals of any order, Trans. Amer. Math. Soc. 119, 225-249, 1965.

[31] C. B. Morrey, Jr. Quasiconvexity and the lower semicontinuity of multiple integrals. Pacific J. Math., 2: 25-53, 1952.

[32] C.B. Morrey, Jr. Multiple integrals in the calculus of variations,

[33] S. Muller. Variational models for microstructure and phase transitions. Lectures at the C.I.M.E. summer school 'Calculus of Variations and geometric evolution problems', Cetraro, 1996.

[34] S. Muller and V. Sverak. Convex integration for Lipschitz mappings and counterexamples to regularity. Ann. of Math. (2), 157(3):715-742, 2003.

[35] P. Pedregal. Parameterized measures and variational principles. Progress in Nonlinear Differential Equations and their applications, 30. Birkhäuser Verlag, Basel, 1997.

[36] K.D.E Post and J. Sivaloganathan. On homotopy conditions and the existence of multiple equilibria in finite elasticity. Proc. Roy. Soc. Edinburgh Sect. A, 127(3):595-614, 1997.

[37] W. Reid. Sufficient conditions by expansion methods for the problem of Bolza in the Calculus of Variations. Annals of Mathematics, vol. 38, No. 3 July , 1937.

[38] W. Rudin. Real and complex analysis. McGraw-Hill Book Co., New York, third edition, 1987.

[39] L. Szekelyhidi Jr. The regularity of critical points of polyconvex functionals. Arch. Ration. Mech. Anal., 172(1):133-152, 2004

[40] V. Sverak Rank one convexity does not imply quasiconvexity, 1992

[41] Ali Taheri. Sufficiency theorems for local minimizers of the multiple integrals of the calculus of variations. Proc. Roy. Soc. Edinburgh Sect. A, 131(1):155-184, 2001.

[42] Ali Taheri. Local minimizers and quasiconvexity- the impact of topology. Arch. Ration. Mech. Anal., 176(3):363-414, 2005.

[43] L. C. Young Lectures on the calculus of variations and optimal control theory. W.B. Saunders Co., Philadelphia, 1969. Forward by Wendell H. Fleming.

[44] L. Tartar. Compensated Compactness and applications to partial differential equations, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium Vol. IV, (R. Knops, ed.), Pitman, 1979, 136-212.

[45] K. Zhang. Remark on quasiconevxity and stability of equilibria for variational integrals. Proc. Amer. Math. Soc., 114(4): 927-930, 1992.