A CAUCHY PROBLEM WITH SINGULARITY ALONG THE INITIAL HYPERSURFACE

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy

by
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May, 2011

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ABSTRACT

A Cauchy Problem with Singularity Along the Initial Hypersurface

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We solve a one-sided Cauchy problem with zero right hand side modulo smooth errors for the wave operator associated to a smooth metric which is locally of the form

$$g = tg_{00}dt \otimes dt + 2\sum_{j=1}^{n} tg_{0j}dt \otimes dx^j + \sum_{i,j=1}^{n} g_{ij}dx^i \otimes dx^j.$$

Here t is a defining function of the initial hypersurface, g is a Lorentz metric on $t > 0$, g_{00} is positive and $\sum_{i,j} g_{ij}dx^i \otimes dx^j$ is negative definite.

The degeneracy of the metric at $t = 0$ gives rise to singularities in the wave operator. The initial data must be modified from the classical Cauchy problem to suit the problem at hand. The problem is posed on $t > 0$ and the local solution is constructed using microlocal analysis and the techniques of Fourier Integral Operators.

ACKNOWLEDGEMENTS

I would like to thank Dr. Mendoza for his patience and guidance, without which this would not have been possible. I would also like to thank my friends and family for their perpetual encouragement.

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CHAPTER 1

Introduction

This dissertation concerns the problem of finding solutions modulo smooth functions of the initial value problem

$$\begin{cases} \Box u = 0 \quad \text{in } t > 0 \\ u|_{t=0} = u_0(x) \\ \left(\dfrac{1}{t^{1/2}} \dfrac{\partial u}{\partial t} \right) \Big|_{t=0} = u_1(x) \end{cases} \tag{1.1}$$

in a neighborhood of the initial hypersurface $\{t = 0\}$ where $\Box$ is the D'Alembertian of a smooth symmetric 2-tensor which is a Lorenz metric on $t > 0$, a Riemannian metric on $t < 0$, and degenerates simply along $t = 0$; see Chapter 2 for a complete description of this kind of metric.

Some of the coefficients of $\Box$ blow up at $t = 0$, so (1.1) is a hyperbolic second order Cauchy problem with data prescribed along the singularity set of the operator. The initial conditions are adapted to the situation at hand;

their meaning is elucidated in Chapter 7.

In the absence of singularities, the analogous problem (with the correct initial condition) is strictly hyperbolic; the modern theory underlying such problems, initiated by Lax [7], is by now well developed.

We study the problem adapting the techniques of Fourier Integral Operators (see [2, 3, 5, 7]). From a classical perspective, a major difficulty in our problem is that the light cones of the metric degenerate to 1 dimensional subspaces over $t = 0$.

The following simple example already illustrates the basic features of the problem. In $\mathbb{R}_t \times \mathbb{R}^n_x$ let

$$g = t dt^2 - \sum_{i=1}^{n} dx^i \otimes dx^i. \tag{1.2}$$

This 2-tensor has the properties mentioned above. Its D'Alembertian (or Laplace-Beltrami operator) is

$$\Box = \frac{1}{t}\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial {x^i}^2}.$$

To solve $\Box\, u = 0$, $u|_{t=0} = u_0(x)$, $t^{-1/2}u_t|_{t=0} = u_1(x)$ we make the same ansatz as in the classical case. We seek solutions roughly of the form

$$u = \int e^{i\phi(t,x,y,\theta)} a(t,x,y,\theta) v(y)\, dy\, d\theta$$

where a is a classical symbol in $t > 0$. Since

$$e^{-i\phi}\, \Box(e^{i\phi}a) = (-\frac{1}{t}\phi_t^2 + \sum \phi_{x^i}^2)a + 2i(\phi_t\partial_t - \phi_{x^i}\partial_{x^i})a + \Box(a),$$

the eikonal equation is

$$-\frac{\phi_t^2}{t} + \sum_{i=1}^{n} \phi_{x^i}^2 = 0$$

and the first transport equation is

$$2i(\phi_t \partial_t - \phi_{x^i} \partial_{x^i})a = 0.$$

There are two solutions of the eikonal equation satisfying $\phi|_{t=0} = (x-y)\cdot\theta$, namely

$$\phi^+ = (x-y)\cdot\theta + \frac{2}{3}t^{3/2}|\theta|, \qquad \phi^- = (x-y)\cdot\theta - \frac{2}{3}t^{3/2}|\theta|$$

We then improve the ansatz about the form of the solution to

$$u(t,x) = \int \left[e^{i\phi^+}(a_0^+ u_0(y) + a_1^+ u_1(y)) + e^{i\phi^-}(a_0^- u_0(y) + a_1^- u_1(y))\right] d\theta\, dy$$

with symbols $a_0^\pm$ of order 0 and $a_1^\pm$ of order -1, all classical.

We now ask, following the classical method described in the above mentioned references, that $a_0^\pm$ and $a_1^\pm$ satisfy the first transport equation with the appropriate phase. The solution of these equations are $a_i^\pm =$ constant (that is, independent of t and x). To determine the values of these constants we impose the initial conditions. The value of u at $t = 0$ (really the limit as $t_0 \searrow 0$ of its restriction to the hypersurface $t = t_0$) is $(a_0^+ + a_0^-)u_0 + (a_1^+ + a_1^-)u_1$. For this to equal u_0 with arbitrary u_1, we must have $a_0^+ + a_0^- = 1$ and $a_1^+ + a_1^- = 0$. The top order part in θ of the derivative of u with respect to t is

$$it^{1/2} \int \left[e^{i\phi^+}(a_0^+ u_0(y) + a_1^+ u_1(y))|\theta| - e^{i\phi^-}(a_0^- u_0(y) + a_1^- u_1(y))|\theta|\right] d\theta\, dy.$$

The factor $t^{1/2}$ is a feature of the general problem, not just this example. Its appearance explains the need to divide by $t^{1/2}$ in the second part of the conditions of the Cauchy problem. After dividing by it, we see that the limit as $t \searrow 0$ is

$$\int e^{i(x-y)\cdot\theta}\big[i|\theta|(a_0^+u_0(y)+a_1^+u_1(y)) - i|\theta|(a_0^-u_0(y)+a_1^-u_1(y))\big]\,d\theta\,dy.$$

For this to give u_1 for arbitrary u_0 we must have the equations $a_0^+ - a_0^- = 0$ and $a_1^+ - a_1^- = 1/i|\theta|$. Therefore, we have $a_0^+ = a_0^- = 1/2$, $a_1^+ = 1/2i|\theta|$ and $a_1^- = -1/2i|\theta|$.

The subsequent transport equations (equipped with the correct initial conditions) end up producing only the trivial solution. Thus the solution of the problem (1.1) in the case of the metric (1.2) is

$$u = \int e^{(x-y)\cdot\theta}\Big(\frac{e^{\frac{2i}{3}t^{3/2}|\theta|}+e^{-\frac{2i}{3}t^{3/2}|\theta|}}{2}u_0(y) + \frac{e^{\frac{2i}{3}t^{3/2}|\theta|}-e^{-\frac{2i}{3}t^{3/2}|\theta|}}{2i|\theta|}u_1(y)\Big)\,d\theta\,dy$$

This particular example could have also been solved introducing the change of variables $r = \frac{2}{3}t^{3/2}$ which changes the metric to the standard Minkowsky metric. This change of variables does not work for the general problem for two basic reasons: i) the coefficients of the metric, if they depend on t, become singular, and ii) the presence of cross-terms (such as $dt \otimes dx^i$) in the general case produces extra singularities even if the coefficients do not depend on t.

CHAPTER 2

The Metric

2.1 Definition

Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional paracompact manifold with smooth boundary. The metrics of interest here are the following:

Definition. A *normally simply degenerate Lorentz metric* on $\mathcal{M}$ is a symmetric 2-tensor such that

i) $g|_{\mathring{\mathcal{M}}}$ is a Lorentz metric (see [4, 8]);

ii) $g|_{\partial\mathcal{M}}$ has a 1-dimensional kernel $\mathscr{K} \subset T_{\partial\mathcal{M}}\mathcal{M}$;

iii) if $\imath : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion, then $\imath^* g$ is negative definite;

We show below that *(ii)* and *(iii)* imply that $\mathscr{K}$ is a smooth line subbundle of $T_{\partial\mathcal{M}}\mathcal{M}$. We require, in addition, *(iv)* there is a smooth vector field $\mathcal{T}$ such that $\mathcal{T}|_{\partial\mathcal{M}}$ spans $\mathscr{K}$ and $g(\mathcal{T},\mathcal{T})$ vanishes to precisely first order at $\partial\mathcal{M}$.

By the kernel of g we mean

$$\{v \in T_{p_0}\mathcal{M} : g(v,w) = 0\,\forall w \in T_{p_0}\mathcal{M}\} \tag{2.1}$$

where $p_0 \in \partial\mathcal{M}$.

Remark. The conditions on the metric are expressed without reference to coordinates. Therefore, they are invariant by definition.

Conditions *(ii)* and *(iii)* imply that any nonzero vector in $\mathscr{K}$ is nontangential. Indeed, fix $p_0 \in \partial\mathcal{M}$. If $v \in T_{p_0}\mathcal{M}$ is tangential to the boundary and $v \neq 0$, then $g(v,v) = \imath^* g(v,v) \neq 0$, by property *(iii)*. Therefore, if v is a nonzero vector with $v \in \mathscr{K}$, then v is nontangential.

Moreover, in any local coordinates (t,x) with t a defining function of the boundary, the metric can be written as

$$g = \tilde{g}_{00} dt \otimes dt + 2\sum_{j=1}^{n} g_{0j} dt \otimes dx^j + \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j$$

and the matrix $[g_{ij}]_{i,j=1,\ldots,n}$ is negative definite at the boundary. Therefore, it is negative definite in a neighborhood of p_0.

To see that conditions *(ii)* and *(iii)* imply that $\mathscr{K}$ is a smooth line bundle, let $\mathcal{T}$ be some nowhere zero vector field, not necessarily smooth, that spans $\mathscr{K}$. Pick coordinates (t,x) arbitrarily with t vanishing on $\partial\mathcal{M}$. In these coordinates we may write the metric as

$$g = g_{00} dt \otimes dt + 2\sum_{j=1}^{n} g_{0j} dt \otimes dx^j + \sum_{i,j=1}^{n} g^{ij} dx^i \otimes dx^j$$

where the coefficients are smooth. Also, we may write

$$\mathcal{T} = \alpha \partial_t + \sum_{j=1}^{n} \beta^j \partial_{x^j}$$

without the assumption that α or β^j are smooth.

Note that if $\alpha = 0$, then $\mathcal{T}|_{\partial \mathcal{M}}$ is tangential and can not span the kernel. Therefore, $\alpha \neq 0$ and it suffices to consider the vector field with $\alpha = 1$. That is, we may assume $\mathcal{T}$ is of the form

$$\mathcal{T} = \partial_t + \sum_{j=1}^{n} \beta^j \partial_{x^j},$$

again with no assumption on the smoothness of the β^j. We now have

$$g(\mathcal{T}, \cdot) = \big(g_{00} + \sum_{j=1}^{n} g_{0j}\beta^j\big) dt + \sum_{j=1}^{n} \big(g_{0j} + \sum_{i=1}^{n} g_{ij}\beta^i\big) dx^j = 0.$$

This gives, in particular, the conditions $g_{0j} = -\sum_{i=1}^{n} g_{ij}\beta^i$ for $j = 1, \ldots, n$. Write g_0 for the column vector with entries g_{0j}, write G for the negative definite matrix with entries g_{ij} for $i, j = 1, \ldots, n$, and let β be the column vector with entries β^j. Then these conditions can be written as $g_0 = G\beta$. Since G is negative definite with smooth entries, it is invertible and the entries of G^{-1} are also smooth. Therefore, $\beta = G^{-1} g_0$, so the β^j are smooth and $\mathcal{T}$ is a smooth vector field. Furthermore, $\mathcal{T}$ spans the one-dimensional kernel $\mathscr{K}$. Therefore, $\mathscr{K}$ is a smooth line bundle.

It should also be noted that conditions *(i)* and *(iii)* imply that the signature of $g|_{\mathring{\mathcal{M}}}$ is $(+ - \ldots -)$. Since $\imath^* g$ is negative definite, there are at least n negative eigenvalues at $\partial\mathcal{M}$. The metric is smooth, so there remain at least n negative eigenvalues in the interior near the boundary. The metric is Lorentz, so there must be precisely n eigenvalues with one sign, and one with the other. Therefore, the signature must be $(+ - \ldots -)$.

Another feature is that if $p_0 \in \partial\mathcal{M}$, $U \subset \mathcal{M}$ is a neighborhood of p_0 and $\mathcal{T}'$ is a smooth nonvanishing vector field such that $\mathcal{T}'|_{\partial\mathcal{M}}$ is a section of $\mathscr{K}$, then again $g(\mathcal{T}', \mathcal{T}')$ vanishes to precisely first order at $\partial\mathcal{M}$. To see this, let t be a defining function of $\partial\mathcal{M}$ (we will always assume that such defining functions are positive in the interior). Since $\mathscr{K}$ is one-dimensional, $\mathcal{T}'|_{\partial\mathcal{M}} = c\mathcal{T}|_{\partial\mathcal{M}}$ for some smooth function c. Furthermore, since $\mathcal{T}'$ is nonzero, $c \neq 0$. Thus, we may write $V = c\mathcal{T} + tX$ for some vector field X, giving

$$g(\mathcal{T}', \mathcal{T}') = c^2 g(\mathcal{T}, \mathcal{T}) + 2tcg(\mathcal{T}, X) + t^2 g(X, X).$$

Since $\mathcal{T}$ is in the kernel of g along $\partial\mathcal{M}$, $g(\mathcal{T}, X)$ vanishes at $t = 0$ so $2tcg(\mathcal{T}, X)$ vanishes to second order at $\partial\mathcal{M}$. Clearly, $t^2 g(X, X)$ vanishes to second order at $\partial\mathcal{M}$, and by hypothesis $g(\mathcal{T}, \mathcal{T})$ vanishes to first order. Hence, $g(\mathcal{T}', \mathcal{T}')$ vanishes to first order at $\partial\mathcal{M}$.

2.2 Coordinates

Lemma 2.1. *For any normally simply degenerate Lorentz metric there are coordinates* (t, x) *in a neighborhood of each point of the boundary, with* t *a defining function of the boundary, in which the representation of the metric is of the form*

$$g = tg_{00}dt \otimes dt + tg_{0j}dt \otimes_s dx^j + g_{ij}dx^i \otimes dx^j, \tag{2.2}$$

where $\otimes_s$ *is the symmetric product. Moreover, the matrix* $[g_{ij}]_{i,j=1,\ldots,n}$ *is negative definite,* $g_{00} > 0$*, and each term* g_{00}*,* g_{0j}*, and* g_{ij} *is smooth.*

Proof. Fix some smooth nowhere vanishing inward pointing vector field $\mathcal{T}$ on $\mathcal{M}$ near $\partial\mathcal{M}$ such that $\mathcal{T}|_{\partial\mathcal{M}}$ spans $\mathscr{K}$. Then $g(\mathcal{T}, \mathcal{T})$ vanishes to precisely first order on $\partial\mathcal{M}$. Let t be such that $\mathcal{T}t = 1$ in a neighborhood of p_0 and $t|_{\partial\mathcal{M}} = 0$. Let $x_1, \ldots, x_n$ be coordinates on $\partial\mathcal{M}$ near p_0 and extend these functions to a neighborhood of p_0 so that $\mathcal{T}|_{\partial\mathcal{M}}x^j = 0$ for $j = 1, \ldots, n$. Then $t, x^1, \ldots, x^n$ form a coordinate chart in a neighborhood $U \subset \mathcal{M}$ of p_0. In these coordinates, $\mathcal{T} = \partial_t$ and t is a defining function of $\partial\mathcal{M}$.

We may write $g = \tilde{g}_{00}dt \otimes dt + \sum_{j=1}^{n} \tilde{g}_{0j}dt \otimes_s dx^j + \sum_{i,j=1}^{n} g_{ij}dx^i \otimes dx^j$, where all coefficients are smooth, owing to the fact that g is smooth. The coefficient matrix of the pull-back of g to the boundary is $[g_{ij}]_{i,j=1,\ldots,n}$. By property *(iii)* in the definition of g, this is negative definite. Therefore, there is a neighborhood of p_0 in which $[g_{ij}]_{i,j=1,\ldots,n}$ is negative definite.

By construction, $\mathcal{T} = \partial_t$ and $\partial_t|_{\partial\mathcal{M}}$ spans $\mathscr{K}$. For any vector field W, $g(\mathcal{T}, W) = 0$ on $\partial\mathcal{M}$. In particular, $g(\mathcal{T}, \partial_{x^j}) = \tilde{g}_{0j}$ for $j = 1, \ldots, n$. It follows that for each $j = 1, \ldots, n$, $\tilde{g}_{0j} = tg_{0j}$.

Furthermore, $g(\mathcal{T}, \mathcal{T}) = \tilde{g}_{00}$ vanishes to precisely first order at $t = 0$, giving $\tilde{g}_{00} = tg_{00}$ with $g_{00} \neq 0$ at $t = 0$.

Finally, the fact that the metric is Lorentz in the interior with signature $(+-\ldots-)$ gives that the determinant of the coefficient matrix must have sign $(-1)^n$. Using Laplace's expansion of the determinant we see that

$$\det(g) = tg_{00}\det([g_{ij}]_{i,j=1,\ldots,n}) + O(t^2). \tag{2.3}$$

Since t is a defining function of the boundary (and positive on the interior), for t sufficiently small, the sign is determined by $g_{00}\det([g_{ij}]_{i,j=1,\ldots,n})$. The matrix $[g_{ij}]_{i,j=1,\ldots,n}$ is negative definite, and thus has sign $(-1)^n$. Therefore, g_{00} must be positive for t small. Possibly shrinking U, we have $g_{00} > 0$ on U.

Hence, there are coordinates (t, x) in a neighborhood $U \subset \mathcal{M}$ of $p_0 \in \partial\mathcal{M}$ in which the metric is represented as in (2.2) with coefficients as indicated in the statement of the lemma. $\square$

One of the questions is how much freedom there is in choosing coordinates while retaining the representation of the metric. The key feature is property (2.4) below:

Lemma 2.2. *Suppose* $(\tilde{t}, \tilde{x})$ *are coordinates in a neighborhood* $U \subset \mathcal{M}$ *of*

$p_0 \in \partial\mathcal{M}$ with $\tilde{t}$ a defining function of the boundary (and positive on the interior) such that

$$\partial_{\tilde{t}}|_{\partial\mathcal{M}} \text{ spans } \mathscr{K}. \tag{2.4}$$

Then, the representation of the metric in these coordinates is of the form (2.2)*, and the coefficients have the indicated properties.*

Proof. Put $\tilde{\mathcal{T}} = \partial_{\tilde{t}}$ and $X^j = \partial_{\tilde{x}^j}$. The representation of g has the appropriate form if $g(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}) = \tilde{t}\tilde{g}_{00}$ for some $g_{00} > 0$, $g(\tilde{\mathcal{T}}, X^j) = \tilde{t}\tilde{g}_{0j}$, and $[g(X^i, X^j)]$ is negative definite. The first condition is immediate since $\tilde{\mathcal{T}}|_{\partial\mathcal{M}}$ is inward pointing and spans $\mathscr{K}$. The second is immediate again due to the fact that $\tilde{\mathcal{T}}|_{\partial\mathcal{M}}$ spans $\mathscr{K}$. The last condition is evident from the fact that the $x^j|_{\partial\mathcal{M}}$ form local coordinates in $\partial\mathcal{M}$ so that at the boundary, $[g(X^i, X^j)] = [\imath^* g(X^i, X^j)]$ is negative definite by condition *(iii)*. Therefore, $[g(X^i, X^j)]$ is negative definite in a neighborhood of p_0. □

For the remainder of the paper, we will use the coordinates as in Lemma 2.1 unless explicitly stated otherwise, i.e. (t, x) are coordinates in a neighborhood of a point in the boundary, and t is a defining function of $\partial\mathcal{M}$ with $\partial_t|_{\partial\mathcal{M}}$ inward pointing and spanning $\mathscr{K}$.

Notation. We will adopt the convention that the indices μ, ν have range $0, \ldots, n$ with $t = x^0$, and indices i, j have range $1, \ldots, n$, unless explicitly stated otherwise.

Lemma 2.3. *With the coordinates as in Lemma 2.1, the inverse of the matrix associated to the metric is of the form*

$$[g^{\mu\nu}] = \begin{bmatrix} \frac{1}{t}\hat{g}^{00} & g^{0j} \\ g^{j0} & g^{ij} \end{bmatrix}$$

where $\hat{g}^{00}$, g^{0j}, g^{j0} *and* g^{ij} *are smooth, the submatrix* $[g^{ij}]$ *is negative definite, and* $\hat{g}^{00} > 0$.

Proof. Starting with the coordinates from Lemma 2.1, write the matrix associated to the metric as

$$[g_{\mu\nu}] = \begin{bmatrix} tg_{00} & tg_{0j} \\ tg_{j0} & g_{ij} \end{bmatrix} = \begin{bmatrix} tg_{00} & t\beta \\ t\beta^{\dagger} & h \end{bmatrix},$$

i.e., β is the row vector $(g_{01}, \ldots, g_{0n})$ and h is the negative definite matrix $[g_{ij}]$ with $1 \leq i, j \leq n$. With this notation, it is easily verified that the inverse of the matrix is

$$[g^{\mu\nu}] = \frac{1}{g_{00} - t\beta h^{-1}\beta^{\dagger}} \begin{bmatrix} \frac{1}{t} & -\beta h^{-1} \\ -(\beta h^{-1})^{\dagger} & (g_{00} - t\beta h^{-1}\beta^{\dagger})h^{-1} + t(\beta h^{-1})^{\dagger}(\beta h^{-1}) \end{bmatrix}.$$

The entries of the matrix h are smooth, and h is negative definite. Therefore, the entries of h^{-1} are smooth. The components of β are also smooth, being coefficients of the metric. Thus, $g_{00} - t\beta h^{-1}\beta^{\dagger}$ is smooth. Moreover, for t sufficiently small, $g_{00} - t\beta h^{-1}\beta^{\dagger} > 0$, owing to g_{00} being positive. It follows that $1/(g_{00} - t\beta h^{-1}\beta^{\dagger})$ is smooth for t sufficiently small.

With $\hat{g}^{00} = 1/(g_{00} - t\beta h^{-1}\beta^\dagger)$, we have the matrix

$$[g^{\mu\nu}] = \begin{bmatrix} \frac{1}{t}\hat{g}^{00} & g^{0j} \\ g^{j0} & g^{ij} \end{bmatrix}$$

where

$$g^{j0} = g^{0j} = -\frac{g_{0i}h^{ij}}{g_{00} - t\beta h^{-1}\beta^\dagger}$$

and

$$g^{ij} = h^{ij} + t\frac{1}{g_{00} - t\beta h^{-1}\beta^\dagger}\sum_{k,\ell=1}^{n} h^{ik}g_{k0}g_{0\ell}h^{\ell j}$$

where h^{ij} is the i,j entry of the matrix h^{-1}. Clearly g^{0j} and g^{ij} are smooth, as they are sums and products of smooth functions. □

2.3 Null Geodesics

The null geodesics of our metric have a curious behavior as they reach the boundary which is in part the source of some difficulties. We describe their behavior near $\partial\mathcal{M}$ in order to have a better understanding of the issue.

In general, the geodesic of a metric are the projections of integral curves of the Hamiltonian vector field of the metric on the cotangent bundle, in our case, the Hamiltonian of

$$p_2(t,x,\tau,\xi) = \frac{1}{t}\hat{g}^{00}\tau^2 + 2\sum_{j=1}^{n} g^{0j}\tau\xi_j + \sum_{i,j=1}^{n} g^{ij}\xi_i\xi_j.$$

We are interested in the null geodesics, so we may multiply by t and divide by $\hat{g}^{00}$ which has the effect of reparametrizing the geodesics. Write

$$p = \tau^2 + 2t \sum_{j=1}^{n} \tilde{g}^{0j}\tau\xi_j + t \sum_{i,j=1}^{n} \tilde{g}^{ij}\xi_i\xi_j$$

where $\tilde{g}^{0j} = g^{0j}/\hat{g}^{00}$ and $\tilde{g}^{ij} = g^{ij}/\hat{g}^{00}$. Since $\hat{g}^{00} > 0$, $\tilde{g}^{0j}$ is smooth, and $[\tilde{g}^{ij}]$ is still negative definite. We will henceforth omit the tilde and discuss the integral curves of

$$\begin{aligned} H_p =& 2(\tau + t\sum_{j=1}^{n} g^{0j}\tau\xi_j)\partial_t \\ &+ 2t\sum_{j=1}^{n}(g^{0j}\tau + \sum_{i=1}^{n} g^{ij}\xi_i)\partial_{x^j} \\ &- \big(2\sum_{j=1}^{n} g^{0j}\tau\xi_j + \sum_{i,j=1}^{n} g^{ij}\xi_i\xi_j + 2t\sum_{j=1}^{n}\frac{\partial g^{0j}}{\partial t}\tau\xi_j + t\sum_{i,j=1}^{n}\frac{\partial g^{ij}}{\partial t}\xi_i\xi_j\big)\partial_\tau \\ &- t\big(2\sum_{j=1}^{n}\frac{\partial g^{0j}}{\partial x^k}\tau\xi_j + \sum_{i,j=1}^{n}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j\big)\partial_{\xi_k}. \end{aligned}$$

For convenience, take $t(0) = 0$. Since $p = 0$, we must have $\tau(0) = 0$. Therefore, we seek curves issuing from points of the form $(0, x_0, 0, \theta)$. Since we are not interested in the zero section of the cotangent bundle, we take $\theta \neq 0$. The integral curves of H_p are the solutions of the system

$$\begin{cases} \dot{t} = 2(\tau + t\sum_{j=1}^{n} g^{0j}\tau\xi_j) \\ \dot{x}^j = 2t\sum_{j=1}^{n}(g^{0j}\tau + \sum_{i=1}^{n} g^{ij}\xi_i) \\ \dot{\tau} = -2\sum_{j=1}^{n} g^{0j}\tau\xi_j + \sum_{i,j=1}^{n} g^{ij}\xi_i\xi_j + 2t\sum_{j=1}^{n}\frac{\partial g^{0j}}{\partial t}\tau\xi_j + t\sum_{i,j=1}^{n}\frac{\partial g^{ij}}{\partial t}\xi_i\xi_j \\ \dot{\xi}_k = -t\big(2\sum_{j=1}^{n}\frac{\partial g^{0j}}{\partial x^k}\tau\xi_j + \sum_{i,j=1}^{n}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j\big) \end{cases}$$

subject to the initial conditions $t(0) = \tau(0) = 0$, $x(0) = x_0$, $\xi(0) = \theta$. Here, we discuss only the nature of the solutions as they approach the boundary. For a full treatment of this problem, see Chapter 4.

Since $\tau(0) = t(0) = 0$, $\dot{t}|_{s=0} = 0$. Furthermore, $\ddot{t}|_{s=0} = -2\dot{\tau}|_{s=0} = -2\sum g^{ij}|_{s=0}\theta_i\theta_j$. Finally, $\dot{x}^j|_{s=0} = \ddot{x}^j|_{s=0} = 0$, but the third derivative of x^j with respect to s is $2\ddot{t}|_{s=0}\sum g^{ij}|_{s=0}\theta_i$. Writing the second approximation of t gives that $t \sim -\sum g^{ij}|_{s=0}\theta_i\theta_j s^2$. Since $[g^{ij}]$ is negative definite, we may write $|\theta|_g^2 = -\sum g^{ij}|_{s=0}\theta_i\theta_j$ so $t \sim 2|\theta|_g^2 s^2$. Writing the approximation for x^j gives $x^j \sim x_0^j + (1/3)s^3|\theta|_g^2 \sum g^{ij}|_{s=0}\theta_i$. Therefore, we have the approximation

$$x^j \sim x_0^j \pm C|\theta|_g \sum_{i=1}^{n} g^{ij}|_{s=0}\theta_i t^{3/2}.$$

The $\pm$ comes from the fact that $t \sim Cs^2$ so that $s \sim \pm\tilde{C}t^{1/2}$. This plays a significant role in Chapter 4. Since the power of t is $3/2 > 1$, the null geodesics come in to the boundary with a cusp.

2.4 Examples

There is a wide class of nearly trivial examples. Let X be a compact, connected manifold of dimension n. Being compact, X admits a Riemannian metric. Let $\mathcal{M} = X \times [0, \infty)$. With $x^1, \ldots, x^n$ coordinates on X, extend them as constant to $\mathcal{M}$. Let t be the standard coordinate on $[0, \infty)$. If

$\sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j$ is a Riemannian metric on X, then

$$g = t\, dt \otimes dt + \sum_{j=1}^{n} t g_{0j}\, dt \otimes_s dx^j - \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j$$

with any smooth functions g_{0j} is a metric that has the desired properties.

We are not making any assumptions that the boundary be connected. There are topological restrictions to the existence of a Lorentz metric on a closed manifold. Notably, the tangent bundle must split in order to have the desired signature.

In the case at hand, since $\partial\mathcal{M} \neq \emptyset$, there is a global nonvanishing vector field $\mathcal{T}$ on $\mathcal{M}$. If there is such a vector field which is also transverse to the boundary, then we can construct a metric which degenerates simply near the boundary by gluing together a Lorentz metric in the interior for which $\mathcal{T}$ is timelike with a metric of the kind given in the previous paragraph.

As a nonexample, the closed unit ball in $\mathbb{R}^2$ does not admit such a metric. Suppose there is a normally simply degenerate Lorentz metric on the closed unit ball, g. Then there is a symmetric linear map ρ for which $g(u,v) = g_e(\rho u, v)$ for all v, where g_e is the Euclidean metric on $\mathbb{R}^2$. The eigenvalues of ρ are real, and depend smoothly on u and on the base point. By the properties of g, there must be a negative eigenvalue at every point; call this λ. The kernel of $\rho - \lambda$ is a smooth globally defined line subbundle of $T\mathcal{M}$ that is tangential to $\partial\mathcal{M}$. Since the space is contractible, the subbundle is trivial. Therefore, there is a globally defined nonvanishing continuous section of the tangent bundle that is tangential to $\partial\mathcal{M}$.

By gluing together two closed balls along the boundary and pulling the interiors apart, we construct S^2. This gives rise to a continuous (in fact smooth) nonvanishing section of the tangent bundle of S^2. This is a contradiction since S^2 admits no such section. Hence the closed unit ball in $\mathbb{R}^2$ does not admit such a metric.

This can be modified and extended to show that the closed unit ball in $\mathbb{R}^{2n}$ does not admit such a metric.

CHAPTER 3

Statement of the Problem

Definition. For a Lorentz metric given in the local coordinates $(x^0, \dots, x^n)$ by $g = \sum_{\mu,\nu=0}^{n} g_{\mu\nu}\, dx^\mu \otimes dx^\nu$, the D'Alembertian is the Laplace-Beltrami operator,

$$\Box = -\frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} |g|^{1/2} \frac{\partial}{\partial x^\nu} \tag{3.1}$$

where $|g|$ is the absolute value of the determinant of the coefficient matrix of g, and $g^{\mu\nu}$ is the μ, ν entry of the inverse of the coefficient matrix (see [4]). The D'Alembertian is independent of coordinates.

The inverse of the coefficient matrix is of the form

$$[g^{\mu\nu}] = \begin{bmatrix} \frac{1}{t}\hat{g}^{00} & g^{0j} \\ g^{j0} & g^{ij} \end{bmatrix}$$

where $\hat{g}^{00}$, $g^{0j} = g^{j0}$ and g^{ij} are smooth.

For notational ease, let $|g| = t\gamma(t,x)$. From (2.3), the determinant of the metric vanishes to precisely first order at $t = 0$, so that $\gamma(0,x) \neq 0$. The

D'Alembertian in these coordinates (on $t > 0$) is

$$\begin{aligned}\Box = -\Bigg[&\frac{1}{t}\hat{g}^{00}\frac{\partial^2}{\partial t^2} + 2\sum_{j=1}^{n} g^{0j}\frac{\partial^2}{\partial t \partial x^j} + \sum_{i,j=1}^{n} g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \\ &+ \Bigg(-\frac{1}{2t^2}\hat{g}^{00} + \frac{1}{t}\frac{\partial \hat{g}^{00}}{\partial t} + \frac{1}{2t\gamma}\frac{\partial \gamma}{\partial t} + \sum_{j=0}^{n}\Big(\frac{\partial g^{0j}}{\partial x^j} + \frac{g^{0j}}{2\gamma}\frac{\partial \gamma}{\partial x^j}\Big)\Bigg)\frac{\partial}{\partial t} \\ &+ \sum_{j=1}^{n}\Bigg(\frac{\partial g^{0j}}{\partial t} + \frac{g^{0j}}{2t} + \frac{g^{0j}}{2\gamma}\frac{\partial \gamma}{\partial t} + \sum_{i=1}^{n}\Big(\frac{\partial g^{ij}}{\partial x^i} + \frac{g^{ij}}{2\gamma}\frac{\partial \gamma}{\partial x^i}\Big)\Bigg)\frac{\partial}{\partial x^j}\Bigg]\end{aligned}$$

The operator we will study is the negative of the D'Alembertian. It is a singular second order linear operator with singularity along $\partial\mathcal{M}$. In the coordinates of Lemma 2.3, the operator we study is

$$\begin{aligned}\Box = &\frac{1}{t}\hat{g}^{00}\frac{\partial^2}{\partial t^2} + 2\sum_{j=1}^{n} g^{0j}\frac{\partial^2}{\partial t \partial x^j} + \sum_{i,j=1}^{n} g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \\ &+ \Bigg(-\frac{1}{2t^2}\hat{g}^{00} + \frac{1}{t}\frac{\partial \hat{g}^{00}}{\partial t} + \frac{1}{2t\gamma}\frac{\partial \gamma}{\partial t} + \sum_{j=0}^{n}\Big(\frac{\partial g^{0j}}{\partial x^j} + \frac{g^{0j}}{2\gamma}\frac{\partial \gamma}{\partial x^j}\Big)\Bigg)\frac{\partial}{\partial t} \\ &+ \sum_{j=1}^{n}\Bigg(\frac{\partial g^{0j}}{\partial t} + \frac{g^{0j}}{2t} + \frac{g^{0j}}{2\gamma}\frac{\partial \gamma}{\partial t} + \sum_{i=1}^{n}\Big(\frac{\partial g^{ij}}{\partial x^i} + \frac{g^{ij}}{2\gamma}\frac{\partial \gamma}{\partial x^i}\Big)\Bigg)\frac{\partial}{\partial x^j}\end{aligned} \tag{3.2}$$

The precise problem we wish to study is as follows. Let $\mathcal{T}$ be a smooth vector field such that $\mathcal{T}$ spans $\mathscr{K}$, let t be a defining function of $\partial\mathcal{M}$ with $\partial_t|_{\partial\mathcal{M}} = \mathcal{T}$, and let (t, x) be coordinates as in the construction in Lemma 2.1. With these coordinates, we find a distribution u that, modulo smooth functions, locally solves the initial value problem

$$\begin{cases} \Box u = 0 \quad \text{in } t > 0 \\ \lim_{t\searrow 0} u = u_0 \\ \lim_{t\searrow 0}(t^{-1/2}\partial_t u) = u_1 \end{cases} \tag{3.3}$$

where $u_0, u_1 \in C_c^{-\infty}(\partial\mathcal{M})$.

Remark. It appears that the description of the problem depends on the choice of coordinates, due to the fact that they are written explicitly using the t coordinate. However, the condition that $\partial_t|_{\partial\mathcal{M}} = \mathcal{T}$ gives that it depends only on $\mathcal{T}$. Suppose $\tilde{t}, \tilde{x}$ are other coordinates, with $\tilde{t}$ a defining function of $\partial\mathcal{M}$ such that $\partial_{\tilde{t}}|_{\partial\mathcal{M}} = \mathcal{T}$. Then, since $\partial_{\tilde{t}}|_{\partial\mathcal{M}} = c(\tilde{x})\partial_t|_{\partial\mathcal{M}}$, we must have $c(\tilde{x}) = 1$. Both being defining functions of $\partial\mathcal{M}$, we have $\tilde{t} = tT(t,x)$. But then, $\partial_{\tilde{t}}|_{\partial\mathcal{M}} = T(0,x)\partial_t|_{\partial\mathcal{M}}$. Therefore, $T(0,x) = 1$, so $\tilde{t} = t + t^2\tilde{T}(t,x)$. Note that $\tilde{t}^{-1/2} = (t+t^2\tilde{T}(t,x))^{-1/2} = t^{-1/2}(1+t\tilde{T}(t,x))^{-1/2}$. Now, $(1+t\tilde{T}(t,x))^{-1/2}$ is a smooth function of t near $t = 0$. Therefore, $(1+t\tilde{T}(t,x))^{-1/2} = 1+tR(t,x)$ for some smooth function R. Therefore, $\tilde{t}^{-1/2} = t^{-1/2} + t^{1/2}R(t,x)$ for some smooth function R. Since $t^{1/2}R(t,x) = 0$ on $\partial\mathcal{M}$, we have

$$\lim_{\tilde{t}\to 0} \tilde{t}^{-1/2}\partial_{\tilde{t}}u = \lim_{t\searrow 0} t^{-1/2}\partial_t u.$$

Clearly, since t and $\tilde{t}$ are defining functions of $\partial\mathcal{M}$,

$$\lim_{\tilde{t}\searrow 0} u = \lim_{t\searrow 0} u.$$

Hence, the initial conditions only depend on the choice of vector field $\mathcal{T}$, and not on the coordinates.

CHAPTER 4

The Phase Function

4.1 Introduction

The theory of Fourier Integral Operators uses geometric techniques to study partial differential equations. The theory does not directly apply to the problem at hand due to the singularities. The key ingredients are a phase, an amplitude, the critical set of ϕ and the lagrangian submanifold determined by ϕ. In this chapter, we will discuss the phase function and the lagrangian. We treat the amplitude in Chapter 5 and 6.

Definition. Let Γ be an open conic subset of $X \times \mathbb{R}^m \backslash 0$. A smooth function $\phi : \Gamma \to \mathbb{R}$ is a *phase function* if $\phi(x, \theta)$ is homogeneous of degree 1 in θ and $d\phi \neq 0$ on Γ.

The *critical set* of ϕ, denoted C_ϕ, is the set

$$C_\phi = \{(x,\theta) \in \Gamma : \phi_\theta = 0\}$$

where Γ is an open conic subset of $X \times \mathbb{R}^m \backslash 0$.

A phase function ϕ is *nondegenerate* if the differentials of the functions ϕ_{θ_j} are independent on C_ϕ.

Finally, the *lagrangian* associated to a nondegenerate phase function ϕ is the set

$$\Lambda_\phi = \{(x, \phi_x(x,\theta)) : \phi_{\theta_j}(x,\theta) = 0,\ j = 1, \dots m\}.$$

If ϕ is a nondegenerate phase function, then C_ϕ is a submanifold of $X \times \mathbb{R}^m$ of codimension m and Λ_ϕ is an immersed (thus locally an embedded) submanifold of T^*X.

4.2 The Eikonal Equation

Definition. The *principal symbol* of $\Box$ is

$$p_2(t,x,\tau,\xi) = -\Big(\frac{\tau^2}{t}\hat{g}^{00} + \sum_{j=1}^{n} 2g^{0j}\tau\xi_j + \sum_{i,j=1}^{n} g^{ij}\xi_i\xi_j\Big). \tag{4.1}$$

We follow Lax (see [7]) and take the phase $\phi(t,x,y,\theta)$ to be a solution of the eikonal equation with initial data $\phi|_{t=0} = (x-y)\cdot\theta$. The eikonal equation is $p_2(t,x,\phi_t(t,x,y,\theta),\phi_x(t,x,y,\theta)) = 0$, subject to the initial condition.

Explicitly, we must solve

$$\frac{1}{t}\hat{g}^{00}\phi_t^2 + 2\sum_{j=1}^{n} g^{0j}\phi_t\phi_{x^j} + \sum_{i,j=1}^{n} g^{ij}\phi_{x^i}\phi_{x^j} = 0 \tag{4.2}$$

after multiplying the equation by -1.

We multiply by $t/\hat{g}^{00}$ and solve

$$\phi_t^2 + 2t\sum_{j=1}^{n}\tilde{g}^{0j}\phi_t\phi_{x^j} + t\sum_{i,j=1}^{n}\tilde{g}^{ij}\phi_{x^i}\phi_{x^j} = 0$$

with the given initial data. Since $\hat{g}^{00}$ is positive, the matrix $[\tilde{g}^{ij}]$ is negative definite, and the functions $\tilde{g}^{0j}$ are smooth. We will henceforth omit the tilde.

Note that for $t > 0$, ϕ is a solution of this equation if and only if it is a solution of the eikonal equation. Direct verification then shows that it is a solution for $t = 0$.

We solve this nonlinear first order partial differential equation using the Hamilton-Jacobi method (see [1], [6]). With $\tau = \phi_t$ and $\xi_j = \phi_{x^j}$, we are solving $(t/\hat{g}^{00})p_2(t, x, \tau, \xi) = 0$. The Hamiltonian vector field for $(t/\hat{g}^{00})p_2$ is

$$\begin{aligned}
H =& \Big(2\tau + 2t\sum_{j=1}^{n} g^{0j}\xi_j\Big)\partial_t \\
&+ \sum_{j=1}^{n}\left(2tg^{0j}\tau + \sum_{i=1}^{n}\left(2tg^{ij}\xi_i\right)\right)\partial_{x^j} \\
&- \left(\sum_{j=1}^{n}\left(2g^{0j} + 2t\frac{\partial g^{0j}}{\partial t}\right)\tau\xi_j + \sum_{i,j=1}^{n}\left(g^{ij} + t\frac{\partial g^{ij}}{\partial t}\right)\xi_i\xi_j\right)\partial_\tau \\
&- \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(2t\frac{\partial g^{0j}}{\partial x^k}\tau\xi_j\right) + \sum_{i,j=1}^{n}\left(t\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j\right)\right)\partial_{\xi_k}
\end{aligned}$$

We seek the integral curves of this vector field with some initial conditions. We ask that at $t = 0$, $\phi = (x - y)\cdot\theta$. Let r be a parameter for the parametrization of the integral curves. Take $t(0) = 0$. Then, in order that $p_2|_{t=0} = 0$, we must have $\tau(0) = 0$. The initial x is arbitrary, so write $x(0) = x_0$. For the initial condition on ξ note that $\phi(0) = (x - y) \cdot \theta$, so that $\phi_x|_{r=0} = \xi(0) = \theta$. We seek a solution of the following system with the given initial conditions:

$$\begin{aligned}
\dot{t} &= 2\tau + 2t \sum_{j=1}^{n} g^{0j} \xi_j \\
\dot{x}^j &= 2tg^{0j}\tau + \sum_{i=1}^{n} \left(2tg^{ij}\xi_i \right) \\
\dot{\tau} &= -\left(\sum_{j=1}^{n} \left(2g^{0j} + 2t \frac{\partial g^{0j}}{\partial t} \right) \tau \xi_j + \sum_{i,j=1}^{n} \left(g^{ij} + t \frac{\partial g^{ij}}{\partial t} \right) \xi_i \xi_j \right) \\
\dot{\xi}_k &= \sum_{j=1}^{n} \left(2t \frac{\partial g^{0j}}{\partial x^k} \tau \xi_j \right) + \sum_{i,j=1}^{n} \left(t \frac{\partial g^{ij}}{\partial x^k} \xi_i \xi_j \right) \\
\dot{\phi} &= 2tp_2 = 0 \\
t(0) &= 0, \qquad \tau(0) = 0 \\
x(0) &= x_0, \qquad \xi(0) = \theta \\
\phi(0) &= (x_0 - y) \cdot \theta
\end{aligned} \tag{4.3}$$

where the dot represents differentiation with respect to a parameter r. Immediately one has that ϕ and y are constant along the integral curves.

This is a system of $2n + 3$ first order ordinary differential equations with $2n + 3$ initial conditions. The right hand side of each equation is smooth in all variables. Therefore, there exists an $\epsilon > 0$ and a unique C^∞ curve

$$r \mapsto (t(r), x(r), y(r), \tau(r), \xi(r))$$

that solves (4.3) for $r \in (-\epsilon, \epsilon)$ with

$$(t(0), x(0), y(0), \tau(0), \xi(0)) = (0, x_0, y, 0, \theta).$$

Furthermore, the solution depends smoothly on (x_0, θ). Therefore, t, x, τ and ξ all depend smoothly on r, x_0, θ.

We first investigate the relationship between the coordinate t and the parameter r. The function $(r, x_0, \theta) \mapsto t(r, x_0, \theta)$ is smooth. Furthermore, $\dot{t}(0, x_0, \theta) = 0$ but $\ddot{t}(0, x_0, \theta) = 2\dot{\tau}(0, x_0, \theta) = -\sum_{i,j=1}^{n} g^{ij}(0, x_0)\theta_i\theta_j > 0$, owing to the fact that g^{ij} is negative definite. By Taylor's theorem,

$$t(r, x_0, \theta) = -\frac{1}{2}r^2 g^{ij}(0, x_0)\theta_i\theta_j + \frac{1}{2}\int_0^r (r-s)^2 \frac{d^3t}{dr^3}(s, x_0, \theta)\, ds.$$

Therefore, we may write $t(r, x_0, \theta) = r^2 T(r, x_0, \theta)$ where $T(r, x_0, \theta)$ is smooth. Furthermore, since $\ddot{t}(0, x_0, \theta) > 0$, there is a neighborhood $U \subset (-\epsilon, \epsilon) \times \mathbb{R}^n \times \mathbb{R}^n$ of $(0, x_0, \theta)$ on which $t(r, x_0, \theta) > 0$. Therefore, $T(r, x_0, \theta) > 0$ and $\sqrt{T(r, x_0, \theta)}$ is smooth on U. Let $s = r\sqrt{T(r, x_0, \theta)}$.

Now, $\dot{s}(0, x_0, \theta) = \sqrt{T(0, x_0, \theta)} = \sqrt{-\sum_{i,j=1}^{n} g^{ij}(0, x_0)\theta_i\theta_j} > 0$. By the implicit function theorem, there is a neighborhood $\tilde{U}$ of $(0, x_0, \theta) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with U of the form $(-\varepsilon, \varepsilon) \times \mathbb{R}^n \times \mathbb{R}^n$, a neighborhood $V \subset \mathbb{R}$ of 0 and a unique smooth function $R : \tilde{U} \to V$ for which $R(s, x_0, \theta) = r$ with $R(0, x_0, \theta) = 0$.

Now, our task is to determine x_0 in terms of s, x, θ. Note that by Taylor's theorem we may write $x = x_0 + rX(r, x_0, \theta)$ where X is smooth. Since r is a smooth in s, x_0, θ, we have that $x = x_0 + R(s, x_0, \theta)X(R(s, x_0, \theta), x_0, \theta)$.

The determinant of the Jacobian of the map $(s, x_0, \theta) \mapsto (s, x, \theta)$ at $(0, x_0, \theta)$ is 1. Therefore, for (s, x, θ) in a neighborhood of $(0, x_0, \theta)$ we can write $x_0 = X_0(s, x, \theta)$. Moreover, this neighborhood can be taken to be of the form $(-\epsilon, \epsilon) \times V \times \mathbb{R}^n$ where V is an open subset of $\mathbb{R}^n$. Possibly shrinking ϵ, we may assume that for $(s, x, \theta) \in U$, $x_0 = X_0(s, x, \theta)$, where U is of the form $(-\epsilon, \epsilon) \times V \times \mathbb{R}^n$. Again, this function X_0 is unique.

On $(-\epsilon, \epsilon) \times V \times \mathbb{R}^n \times \mathbb{R}^n$, we have the unique solution of (4.3)

$$\phi(s, x, y, \theta) = (X_0(s, x, \theta) - y) \cdot \theta \tag{4.4}$$

where X_0 is smooth. Hence, ϕ is a smooth function of (s, x, y, θ).

Lemma 4.1. *There are two solutions of the eikonal equation* (4.2) *subject to the initial condition* $\phi|_{t=0} = (x - y) \cdot \theta$*. They are of the form* $\phi^{\pm}(t, x, y, \theta) = (x - y) \cdot \theta + t^{3/2}\psi^{\pm}(t, x, \theta)$ *where* $\psi^{\pm}(t, x, \theta)$ *are smooth functions of* $t^{1/2}$, x *and* θ.

Proof. From what we have done, for any (x_0, θ), there is a neighborhood $U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of $(0, x_0, \theta)$ and a smooth function $X_0(s, x, \theta)$ on U such that $\phi(s, x, y, \theta) = (X_0(s, x, \theta) - y) \cdot \theta$, where $s = r\sqrt{T(r, x_0, \theta)}$. Moreover, on U, this solution is unique. Observe that $s^2 = t$. For $s \in [0, \epsilon)$, put $X_0^+ = X_0(\sqrt{t}, x, \theta)$ and for $s \in (-\epsilon, 0)$ put $X_0^- = X_0(-\sqrt{t}, x, \theta)$. Now, define $\phi^+(t, x, y, \theta) = (X_0^+(\sqrt{t}, x, \theta) - y) \cdot \theta$ and $\phi^-(t, x, y, \theta) = (X_0^-(\sqrt{t}, x, \theta) - y) \cdot \theta$. Then, ϕ^+ and ϕ^- are the only solutions of the eikonal equation satisfying the initial conditions. Thus, there are two solutions of (4.3), denoted $\phi^{\pm}$, both of which are smooth functions of $\sqrt{t}, x, y, \theta$.

Both ϕ^{+} and ϕ^{-} are smooth functions of $t^{1/2}$, and are so that $\phi_t^{\pm}|_{t=0} = 0$, and $\phi^{\pm}|_{t=0} = (x-y)\cdot\theta$. By Taylor's theorem, we have that

$$\phi^{\pm} = (x-y)\cdot\theta + t^{1/2}\psi_1^{\pm}(x,\theta) + t\psi_2^{\pm}(x,\theta) + t^{3/2}\psi^{\pm}(x,\theta).$$

However, $\phi_t^{\pm}|_{t=0} = 0$. Thus, $\psi_1^{\pm}(x,\theta) = \psi_2^{\pm}(x,\theta) = 0$, and ϕ is as claimed. □

It should be noted that ϕ is a solution of the eikonal equation in a neighborhood of C_ϕ near the boundary.

Lemma 4.2. *The function ϕ is homogeneous of degree 1 in θ.*

Proof. Since $\dot{\phi} = 0$ along the integral curves, and ϕ is homogeneous of degree 1 in θ at $t = 0$, it remains homogeneous of degree 1 along each integral curve. □

Lemma 4.3. *The differentials of ϕ_{θ_j} are independent in a neighborhood of the boundary.*

Proof. We have that for each $j = 1, \ldots, n$, $\phi^{\pm}_{\theta_j} = x^j - y^j + t^{3/2}\psi^{\pm}_{\theta_j}$. The differentials of these at $t = 0$ are $dx^j - dy^j$, and these are clearly independent. Therefore, they are independent in a neighborhood of the boundary. □

Corollary 4.4. *The functions $\phi^{\pm}$ are nondegenerate phase functions in the interior of $\mathcal{M}$ near the boundary.*

Proof. In the interior, the phase functions are smooth and homogeneous of degree 1 in θ. Moreover, the differentials of ϕ_{θ_j} are independent. □

The technical difficulty is in what happens at the boundary. The problem is the functions fail to be smooth in the t coordinate.

Lemma 4.5. *Temporarily changing the C^∞ structure so that $s = t^{1/2}$ is a smooth function, the functions $\phi^\pm$ are nondegenerate phase functions in coordinates (s, x, y, θ) in a neighborhood $U \subset \mathcal{M}$ of any point $p_0 \in \partial\mathcal{M}$.*

Proof. Omitting the $\pm$, write $\phi = \phi(s, x, y, \theta) = (x - y)\cdot\theta + s^3\psi(s, x, \theta)$, where s is as in Lemma 4.1. In these coordinates ϕ is smooth up to and including the boundary, and homogeneous of degree 1 in θ. The differentials of ϕ_{θ_j} are

$$d\phi_{\theta_j} = (1 + s^3\psi_{\theta_j,x^j})dx^j - dy^j + s^3\psi_{\theta_j,x^k}dx^k + 3s^2\psi_{\theta_j} + s^3\psi_{\theta_j,s}ds.$$

At the boundary, we have $d\phi_{\theta_j} = dx^j - dy^j$, which are independent. Therefore, the differentials are independent in a neighborhood of $s = 0$. Finally, $d\phi = 0$ only when $\theta = 0$. Hence, ϕ is a nondegenerate phase function in a neighborhood of any point on $\partial\mathcal{M}$ in the coordinates (s, x, y, θ). □

This gives us the insight that in some sense the s coordinate is the correct one to use for the geometric analysis, since with this coordinate we have a nondegenerate phase function down to the boundary. On the interior, it is sufficient to work in the t coordinate.

We now seek a better understanding of the phase functions. We would like to write a few terms of the their expansions in powers of $t^{1/2}$. Write $\phi^{\pm} = (x-y)\cdot\theta + t^{3/2}\psi_3^{\pm} + t^2\psi_4^{\pm} + t^{5/2}\tilde{\psi}^{\pm}$. Using the equation

$$\hat{g}^{00}(\phi_t^{\pm})^2 + 2t\sum_{j=1}^{n} g^{0j}\phi_t^{\pm}\phi_{x^j}^{\pm} + t\sum_{i,j=1}^{n} g^{ij}\phi_{x^i}^{\pm}\phi_{x^j}^{\pm} = 0,$$

we expand the coefficients in powers of t. Collecting terms of like order in t, we can determine the coefficients appearing in the expansion of $\phi^{\pm}$. This gives

$$\begin{aligned} &tp_2(t, x, \phi_t, \phi_x) \\ &\quad = t\left(\frac{9}{4}\hat{g}^{00}\left(\psi_3^{\pm}\right)^2 + \sum_{i,j=1}^{n} g^{ij}\theta_i\theta_j\right) \\ &\qquad + t^{3/2}\left(6\hat{g}^{00}\psi_3^{\pm}\psi_4^{\pm} + 3\psi_3^{\pm}\sum_{j=1}^{n} g^{0j}\theta_j\right) \\ &\qquad + O(t^2) \end{aligned}$$

where each term from the metric is evaluated at $t = 0$. From these, we can determine $\psi_3^{\pm}$ and $\psi_4^{\pm}$. The $\psi_3^{\pm}$ appears quadratically, so we must choose the correct sign, as demanded by the definition of $\phi^{\pm}$. Explicitly, we have

$$\psi_3^{\pm} = \pm\frac{2}{3\sqrt{\hat{g}^{00}}}\sqrt{-\sum_{i,j=1}^{n} g^{ij}\theta_i\theta_j}$$

$$\psi_4^{\pm} = -\frac{1}{2\hat{g}^{00}}\sum_{j=1}^{n} g^{0j}\theta_j$$

where, again, each term from the metric is evaluated at $t = 0$.

Remark. Momentarily changing the C^∞ structure of $\mathcal{M}$ so that $s = t^{1/2}$ is a smooth function we have that the lagrangians parametrized by $\phi^{\pm}$ are smooth. This is just because $\phi^{\pm}$ is a smooth function of $t^{1/2}$.

The lagrangians are closed conic lagrangian submanifolds of $T^*\mathcal{M}$, due to the fact that our phase functions are nondegenerate (see [5]).

CHAPTER 5

The Transport Equations

5.1 Set-up

We have a suitable phase function, and its associated lagrangian. Now, we seek as solution a distribution of the form

$$u = \int [e^{i\phi^+}(a^{0,+}u_0 + a^{1,+}u_1) + e^{i\phi^-}(a^{0,-}u_0 + a^{1,-}u_1)]\, d\theta\, dy$$

where $u_0 = u_0(y)$ and $u_1 = u_1(y)$ are the initial data, and $a^{i,\pm} = a^{i,\pm}(t, x, y, \theta)$ are symbols, in some sense.

We will work in a coordinate patch U in $\mathcal{M}$ that contains $x^0 \in \partial\mathcal{M}$ and use the coordinates from Lemma 2.1.

Lemma 5.1. *The operator acts on $e^{i\phi}a$ as*

$$\Box(e^{i\phi}a) = e^{i\phi}\big[p_2(t, x, \phi_t, \phi_x)a + i[\Box, M_\phi](a) + \Box(a)\big]$$

where $[\Box, M_\phi](a) = \Box(\phi a) - \phi\,\Box(a)$*, that is, the commutator of* $\Box$ *and multiplication by* ϕ*, applied to* a*.*

Proof. For convenience, write the operator as

$$\Box = \frac{1}{t}\hat{g}^{00}\frac{\partial^2}{\partial t^2} + 2\sum_{j=1}^{n} g^{0j}\frac{\partial^2}{\partial t\partial x^j} + \sum_{i,j=1}^{n} g^{ij}\frac{\partial^2}{\partial x^i\partial x^j} + A_0\partial_t + \sum_{j=1}^{n} A_j\partial_{x^j}.$$

Now, direct computation, and using that g^{ij} is symmetric, gives

$$\begin{aligned}\Box(e^{i\phi}a) = \Bigg[&\frac{1}{t}\hat{g}^{00}\Big(- \phi_t^2 a + 2i\phi_t a_t + i\phi_{tt}a + a_{tt}\Big)\\ &+ \sum_{j=1}^{n} g^{0j}\Big(- \phi_t\phi_{x^j}a + i\phi_t a_{x^j} + i\phi_{x^j}a_t + i\phi_{t\,x^j}a + a_{t\,x^j}\Big)\\ &+ \sum_{i,j=1}^{n} g^{ij}\Big(- \phi_{x^i}\phi_{x^j} + 2i\phi_{x^i}a_{x^j} + i\phi_{x^ix^j}a + a_{x^ix^j}\Big)\\ &+ A_0\Big(i\phi_t a + a_t\Big) + \sum_{j=1}^{n} A_j\Big(i\phi_{x^j}a + a_{x^j}\Big)\Bigg]e^{i\phi}\end{aligned}$$

and rearranging the terms and grouping according to the order of homogeneity in θ we have

$$\begin{aligned}\Box(e^{i\phi}a) = \Bigg[&-\Big(\frac{1}{t}\hat{g}^{00}\phi_t^2 + \sum_{j=1}^{n} 2g^{0j}\phi_t\phi_{x^j} + \sum_{i,j=1}^{n} g^{ij}\phi_{x^i}\phi_{x^j}\Big)a\\ &+ i\Big(2\frac{1}{t}\hat{g}^{00}\phi_t\partial_t + 2\sum_{j=1}^{n} g^{0j}(\phi_t\partial_{x^j} + \phi_{x^j}\partial_t) + 2\sum_{i,j=1}^{n} g^{ij}\phi_{x^i}\partial_{x^j}\\ &+ \frac{\hat{g}^{00}}{t}\phi_{tt} + 2\sum_{j=1}^{n} g^{0j}\phi_{t\,x^j} + \sum_{i,j=1}^{n} g^{ij}\phi_{x^ix^j} + A_0\phi_t + \sum_{j=1}^{n} A_j\phi_{x^j}\Big)a\\ &+ \frac{\hat{g}^{00}}{t}a_{tt} + \sum_{j=1}^{n} g^{0j}a_{t\,x^j} + \sum_{i,j=1}^{n} g^{ij}a_{x^ix^j} + A_0 a_t + \sum_{j=1}^{n} A_j a_{x^j}\Bigg]e^{i\phi}.\end{aligned}$$

This simplifies to

$$\Box(e^{i\phi}a) = \Bigg[p_2(t,x,\phi_t,\phi_x)a + i\Bigg(2\frac{\hat{g}^{00}}{t}\phi_t\partial_t + 2\sum_{j=1}^{n} g^{0j}(\phi_t\partial_{x^j} + \phi_{x^j}\partial_t) + 2\sum_{i,j=1}^{n} g^{ij}\phi_{x^i}\partial_{x^j} + \Box(\phi)\Bigg)a + \Box(a)\Bigg]e^{i\phi}.$$

Now, observe that

$$\begin{aligned}\big[\Box, M_\phi\big](a) &= \Box(\phi)a + \phi\,\Box(a) \\ &\quad + 2\frac{\hat{g}^{00}}{t}\phi_t a_t + 2\sum_{j=1}^{n} g^{0j}(\phi_t a_{x^j} + \phi_{x^j}a_t) + 2\sum_{i,j=1}^{n} g^{ij}\phi_{x^i}a_{x^j} - \phi\,\Box(a) \\ &= \Box(\phi)a + 2\frac{\hat{g}^{00}}{t}\phi_t a_t + 2\sum_{j=1}^{n} g^{0j}(\phi_t a_{x^j} + \phi_{x^j}a_t) + 2\sum_{i,j=1}^{n} g^{ij}\phi_{x^i}a_{x^j}.\end{aligned}$$

Therefore, we have

$$\Box(e^{i\phi}a) = \Big[p_2(t,x,\phi_t,\phi_x)a + i[\Box, M_\phi](a) + \Box(a)\Big]e^{i\phi}$$

as desired. $\Box$

The equation above is often written in terms of the principal symbol and the subprincipal symbol. However, when dealing with the operator acting on functions, the subprincipal symbol is not invariant under changes of coordinates. One must study the operator acting on half-densities in order for the subprincipal symbol to be invariant.

The commutator of $\Box$ and multiplication by ϕ is a first order differential operator. The vector field is tangent to C_ϕ, and when transferred to the lagrangian is precisely the hamiltonian vector field of p_2.

We formally write $a^{i,\pm}$, $i = 0, 1$ as expansions of symbols of decreasing order. That is, $a^{i,\pm} \sim \sum_{j=0}^{\infty} a_{-j}^{i,\pm}$ where $a_{-j}^{i,\pm}$ is homogeneous in θ of degree $-j-i$. Then, we collect terms of the same order in θ and ask that they satisfy the transport equation. That is, we write

$$\begin{aligned}\Box(e^{i\phi^{\pm}}a^{i,\pm}) =& p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_0^{i,\pm} \\ &+ p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-1}^{i,\pm} + i[\Box, M_{\phi}^{\pm}](a_0^{i,\pm}) \\ &+ p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-2}^{i,\pm} + i[\Box, M_{\phi}^{\pm}](a_{-1}^{i,\pm}) + \Box(a_0^{i,\pm}) \\ &+ \sum_{j=3}^{\infty}\Big(p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-j}^{i,\pm} + i[\Box, M_{\phi}^{\pm}](a_{-(j-1)}^{i,\pm}) + \Box(a_{-(j-2)}^{i,\pm})\Big).\end{aligned}$$

We have $\phi^{\pm}$ that satisfy $p_2(t,x,\phi_t^{\pm},\phi_x^{\pm}) = 0$ in a neighborhood of C_{ϕ}. Therefore, $p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_0^{i,\pm} = 0$ in a neighborhood of C_{ϕ}. Next, we solve $p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-1}^{i,\pm} + i[\Box, M_{\phi}^{\pm}](a_0^{i,\pm}) = 0$ in a neighborhood of C_{ϕ}, with some initial conditions. This is equivalent to solving $i[\Box, M_{\phi}^{\pm}](a_0^{i,\pm}) = 0$, and is the homogeneous transport equation.

After solving the homogeneous transport equation, we have that

$$p_2(t,x,\phi_t,\phi_x)a_0 + p_2(t,x,\phi_t,\phi_x)a_1 + i[\Box, M_{\phi}](a_0) = 0$$

in a neighborhood of C_{ϕ}. We continue in this fashion for $j \geq 1$ and solve

$$p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-j-2}^{i,\pm} + i[\Box, M_{\phi}^{\pm}](a_{-j-1}^{i,\pm}) + \Box(a_{-j}^{i,\pm}) = 0$$

in a neighborhood of C_{ϕ}. We already have that $p_2(t,x,\phi_t^{\pm},\phi_x^{\pm})a_{-j-2}^{i,\pm} = 0$. Therefore, for $j \geq 1$ we seek a solution of

$$[\Box, M_{\phi}^{\pm}](a_{-j}^{i,\pm}) = \frac{1}{i}\Box(a_{-(j-1)}^{i,\pm}). \tag{5.1}$$

in a neighborhood of C_ϕ, with some sort of "initial conditions." These are the inhomogeneous transport equations.

5.2 The Homogeneous Transport Equation

The first transport equation is $i[\Box, M_\phi^\pm](a_0^{i,\pm}) = 0$. We seek a solution on C_ϕ. In fact, we will solve for (t, x) in a neighborhood U of $p_0 \in \partial\mathcal{M}$ (and thus in a neighborhood of C_ϕ near the boundary). This is a homogeneous first order partial differential equation.

Lemma 5.2. *There is a unique solution of*

$$i[\Box, M_{\phi^\pm}](a) = 0$$

$$a|_{t=0} = a_0(x, y, \theta).$$

for $(t, x) \in U$, and arbitrary y, θ. The solution is of the form $a = a_0(x, y, \theta) + t\tilde{a}(t, x, y, \theta)$ where $\tilde{a}$ is a smooth function of $t^{1/2}$. If a_0 is smooth, then a is a smooth function of $t^{1/2}$, x, y and θ.

Proof. Write $L^\pm = [\Box, M_{\phi^\pm}]$. Omitting the $\pm$, the operator is

$$\begin{aligned} L =& \left(\frac{2\hat{g}^{00}}{t}\phi_t + 2\sum_{j=1}^n g^{0j}\phi_{x^j}\right)\partial_t + \sum_{j=1}^n \left(2g^{0j}\phi_t + 2\sum_{i=1}^n g^{ij}\phi_{x^i}\right)\partial_{x^j} \\ &+ \frac{\hat{g}^{00}}{t}\phi_{tt} + 2\sum_{j=1}^n g^{0j}\phi_{tx^j} + \sum_{i,j=1}^n g^{ij}\phi_{x^ix^j} \\ &+ \left(-\frac{\hat{g}^{00}}{2t^2} + \frac{\hat{g}^{00}}{2t\gamma}\frac{\partial\gamma}{\partial t} + \frac{1}{t}\frac{\partial\hat{g}^{00}}{\partial t} + \sum_{j=1}^n\left(\frac{\partial g^{0j}}{\partial x^j} + \frac{g^{0j}}{2\gamma}\frac{\partial\gamma}{\partial x^j}\right)\right)\phi_t \\ &+ \sum_{j=1}^n\left(\frac{g^{0j}}{2t} + \frac{\partial g^{0j}}{\partial t} + \frac{\partial g^{ij}}{\partial x^i} + \frac{g^{0j}}{2\gamma}\frac{\partial\gamma}{\partial t} + \sum_{i=1}^n\left(\frac{g^{ij}}{2\gamma}\frac{\partial\gamma}{\partial x^i}\right)\right)\phi_{x^j}. \end{aligned}$$

Using our knowledge of $\phi^{\pm}$, write $\phi^{\pm} = (x-y)\cdot\theta + \psi_3^{\pm} t^{3/2} + \psi_4^{\pm} t^2 + t^{5/2}\tilde{\psi}^{\pm}$. Now, observe that

$$\begin{aligned}\frac{\hat{g}^{00}}{t}\phi_{tt}^{\pm} - \frac{\hat{g}^{00}}{2t^2}\phi_t^{\pm} + \sum_{j=1}^{n}\frac{1}{2t}g^{0j}\phi_{x^j}^{\pm} &= \Big(\frac{3}{4}-\frac{3}{4}\Big)\hat{g}^{00}\psi_3^{\pm}t^{-3/2} \\ &\quad + \Big(2\hat{g}^{00}\psi_4^{\pm} - \hat{g}^{00}\psi_4^{\pm} + \frac{1}{2}\sum_{j=1}^{n} g^{0j}|_{t=0}\theta_j\Big)t^{-1} + O(t^{-1/2}) \\ &= O(t^{-1/2})\end{aligned}$$

since $\hat{g}^{00}\psi_4^{\pm} = -\frac{1}{2}\sum_{j=1}^{n} g^{0j}|_{t=0}\theta_j$. This implies that the term that does not differentiate has singularity in t of order $t^{-1/2}$, and not the apparent order of $t^{-3/2}$. Also, ϕ_t is of order $t^{1/2}$, and $\lim_{t\searrow 0}\frac{2\hat{g}^{00}\phi_t}{t^{1/2}} \neq 0$. Thus, the coefficient of ∂_t has a singularity of order $t^{-1/2}$. Furthermore, the coefficient of ∂_{x^j} is a smooth function of $t^{1/2}$. Now that the order of the singularities of each have been determined, we may write, again neglecting the $\pm$,

$$L = \frac{1}{t^{1/2}}w_0\partial_t + \sum_{j=1}^{n} w_j\partial_{x^j} + \frac{1}{t^{1/2}}w$$

where w_0, w_j and w are smooth functions of $t^{1/2}$, x, y and θ, and $w_0|_{t=0} \neq 0$. We make the change of coordinates $t = s^2$. Then, $\partial_t = \frac{1}{2s}\partial_s$, and w_0, w_j and w are all smooth functions of s. As the operator does not differentiate in y or θ, we will treat them as parameters. In these coordinates, the operator is of the form

$$L = \frac{1}{2s^2}w_0\partial_s + \sum_{j=1}^{n} w_j\partial_{x^j} + \frac{1}{s}w.$$

We may now solve

$$\Big(\frac{1}{2s^2}w_0\partial_s + \sum_{j=1}^{n} w_j\partial_{x^j} + \frac{1}{s}w\Big)a = 0.$$

Multiply both sides by $\frac{2s^2}{w_0}$ (a smooth function for s near 0). Then the equation becomes

$$\Big(\partial_s + \sum_{j=1}^{n} s^2\tilde{w}_j\partial_{x^j} + s\tilde{w}\Big)a = 0.$$

The vector field $V = \partial_s + \sum_{j=1}^{n}\tilde{w}_j\partial_{x^j}$ is smooth, and $Vs = 1$. We may therefore find coordinates $\tilde{x}^j$, $1 \leq j \leq n$ so that $\{s, \tilde{x}^j\}$ are local coordinates and $V\tilde{x}^j = 0$. Moreover, we may choose them so that at $s = 0$, $\tilde{x}^j = x^j$. In these coordinates, $V = \partial_s$ and the equation is

$$(\partial_s + s\tilde{w})a = 0 \tag{5.2}$$

where $\tilde{w}$ is a smooth function of $s, \tilde{x}, \theta$. This can be solved by integration, giving $a = C(\tilde{x}, \theta)|_{s=0}e^{-\int s\tilde{w}\,ds}$ where the integral is the antiderivative that is 0 at $s = 0$. Clearly, the integral is smooth. Reverting back to the original coordinates, we have that $a = C(x, \theta)e^{-\int s\tilde{w}\,ds}$. This solution is unique up to the function $C(x, \theta)$.

If we ask that $a|_{s=0} = a_0(x, \theta)$, with a_0 smooth, then a is a smooth function of (s, x, θ). Moreover, since a satisfies $L(a) = 0$, by (5.2) we have $\partial_s a|_{s=0} = 0$. Therefore, a is of the form $a_0(x, \theta) + s^2\tilde{a}(s, x, \theta)$ with $\tilde{a}$ a smooth function of (s, x, θ). This is the unique solution of (5.2) with $a|_{s=0} = a_0(x, \theta)$.

Finally, reverting back to the original coordinates, we have $a = a_0(x, \theta) + t\tilde{a}(t, x, \theta)$ where $\tilde{a}$ is a smooth function of $t^{1/2}$. Moreover, this function is the unique solution of $L(a) = 0$ subject to the condition $a|_{t=0} = a_0$. $\square$

It should be noted that if the initial data for the homogeneous transport equation is homogeneous of degree k in θ, then so is the solution.

5.3 The Inhomogeneous Transport Equations

5.3.1 Preliminaries

In order to determine the precise nature of the solutions of the inhomogeneous transport equations we need an understanding of the singularities of $\square(t^{-\nu/2}\alpha)$, and some technical results on the nature of solutions of the inhomogeneous transport equations. Specifically, how the singularity in t of the right hand side affects the singularity of the solution. Also, we are interested in the order of homogeneity in θ because after all, we are thinking of the solutions as symbols.

Lemma 5.3. *If $a_0^{i,\pm}$ is a solution of the homogeneous transport equation with given nonzero initial data, then $\square(a_0^{i,\pm}) = t^{-2}\tilde{c}$ where $\tilde{c}$ is a smooth function of $t^{1/2}, x, \theta$, with $\tilde{c}|_{t=0} \neq 0$.*

Proof. We have that $a_0^{i,\pm}$ is of the form $a_0 + t\tilde{a}$. Simply applying $\square$ to $a_0 + t\tilde{a}$

and collecting terms according to order in t one has

$$\Box(a) = -\frac{1}{2t^2}\hat{g}^{00}|_{t=0}a + t^{-3/2}\tilde{b}$$

where $\tilde{b}$ is a smooth function of $t^{1/2}, x, \theta$. Therefore, $\Box(a_0^{i,\pm}) = t^{-2}\tilde{c}$ where $\tilde{c}$ is a smooth function of $t^{1/2}, x, \theta$. Moreover, $\tilde{c}|_{t=0} = a_0^{i,\pm} = a_0(x,\theta)$, the given initial data for the homogeneous transport equation. If this data is nonzero, then $\tilde{c}|_{t=0} \neq 0$. □

We will encounter more functions of this form. Therefore, it is useful to have an idea of how $\Box$ affects the order of singularities in t.

Lemma 5.4. *If $\tilde{a}$ is a smooth function of $t^{1/2}, x, \theta$, homogeneous of degree k in θ and $\nu \in \mathbb{N}$, then $\Box(t^{-\nu/2}\tilde{a})$ is of the form $t^{-\nu/2-3}\tilde{c}$ with $\tilde{c}$ homogeneous of degree k in θ. Furthermore, if $\tilde{a}|_{t=0} \neq 0$, then $\tilde{c}|_{t=0} \neq 0$.*

Proof. It is clear that $\Box(t^{-\nu/2}\tilde{a})$ is of the form $t^{\mu}\tilde{\alpha}$ with $\tilde{\alpha}$ a smooth function of $t^{1/2}, x, \theta$. Write $t^{-\nu/2}\tilde{a} = t^{-\nu/2}a_0(x,\theta) + O(t^{-\nu/2+1/2})$, where $a_0(x,\theta) = \tilde{a}|_{t=0}$. Then,

$$\Box(t^{-\nu/2}\tilde{a}) = \frac{\nu(\nu+3)}{4}\hat{g}^{00}|_{t=0}a_0(x,\theta)t^{-\nu/2-3} + O(t^{-(\nu+5)/2}).$$

Since $\nu \in \mathbb{N}$, $\nu \neq 0, -3$ and the coefficient $\frac{\nu(\nu+3)}{4} \neq 0$. If $\tilde{a}|_{t=0} \neq 0$, then $\frac{\nu(\nu+3)}{4}\hat{g}^{00}|_{t=0}a_0(x,\theta) \neq 0$ and the singularity in t is of order $t^{-\nu/2-3}$. Therefore, we may write $\Box(t^{-\nu/2}\tilde{a}) = t^{-\nu/2-3}\tilde{\alpha}$ with $\tilde{\alpha}$ a smooth function of $t^{1/2}, x, \theta$. Moreover, $\tilde{\alpha}|_{t=0} = \frac{\nu(\nu+3)}{4}\hat{g}^{00}|_{t=0}a_0(x,\theta) = \frac{\nu(\nu+3)}{4}\hat{g}^{00}|_{t=0}\tilde{a}|_{t=0}$. Thus, if $\tilde{a}|_{t=0} \neq 0$, then $\tilde{\alpha}|_{t=0} \neq 0$.

Finally, if $\tilde{a}$ is homogeneous in θ of degree k, then $t^{-\nu/2-3}\tilde{\alpha}(t,x,r\theta) = \Box(\tilde{a}(t,x,r\theta)) = \Box(r^k\tilde{a}(t,x,\theta)) = r^k\,\Box(\tilde{a}(t,x,\theta) = r^k t^{-\nu/2-3}\tilde{\alpha}(t,x,\theta)$. Thus, $\tilde{\alpha}(t,x,r\theta) = r^k\tilde{\alpha}(t,x,\theta)$ and $\tilde{\alpha}$ is homogeneous of degree k in θ. $\Box$

Now that we know how $\Box$ affects singularities in t and homogeneity in θ, we analyze solutions of the inhomogeneous transport equations. Again, we work in a coordinate patch.

Lemma 5.5. *Let $\alpha_{-(j-1)}$ be a smooth function of $t^{1/2}, x, \theta$, with $\alpha_{-(j-1)}|_{t=0} \neq 0$. Then, for any $\nu \in \mathbb{N}$, $\nu \neq 3$, there exists a unique solution of*

$$L(a_{-j}) = t^{-\nu/2}\alpha_{-(j-1)}$$

$$C_0(a_{-j}) = a_{-j,0}(x,\theta)$$

on the coordinate patch U, where $C_0(a_{-j})$ is the coefficient of t^0 in the expansion of a_{-j} in powers of $t^{1/2}$. The solution is of the form $a_{-j} = t^{-\nu/2+3/2}\tilde{a}_{-j}$ with $\tilde{a}_{-j}$ a smooth function of $t^{1/2}, x, \theta$.

Proof. As in Lemma 5.2, we change coordinates to $s = \sqrt{t}$ and write the transport equation as

$$\Big(\frac{1}{2s^2}w_0\partial_s + \sum_{j=1}^{n} w_j\partial_{x^j} + \frac{1}{s}w\Big)a_{-j} = s^{-\nu}\alpha_{-(j-1)}$$

where w_0, w_j, w and $\alpha_{-(j-1)}$ are smooth functions of s, x, θ. Multiplying both sides by $\frac{2s^2}{w_0}$ we have

$$\Big(\partial_s + s^2\sum_{j=1}^{n} \tilde{w}_j\partial_{x^j} + s\tilde{w}\Big)a_{-j} = s^{-\nu+2}\alpha_{-(j-1)}.$$

Again as in Lemma 5.2, we change coordinates: with $V = \partial_s + s^2 \sum_{j=1}^n \tilde{w}_j \partial_{x^j}$, choose $\tilde{x}^j$ so that $V\tilde{x}^j = 0$ and $\tilde{x}^j|_{s=0} = x^j$, giving coordinates $(s, \tilde{x})$. Then, in these coordinates the operator is

$$\Big(\partial_s + s\tilde{w}\Big) a_{-j} = s^{-\nu+2}\tilde{\alpha}_{-(j-1)}$$

with $\tilde{w}$ and $\tilde{\alpha}_{-(j-1)}$ smooth functions of $s, \tilde{x}, \theta$.

Observe that

$$\Big(\partial_s + s\tilde{w}\Big) e^{-\int \tilde{w}\, ds} f = e^{-\int \tilde{w}\, ds} \partial_s f.$$

With $a_{-j} = e^{-\int \tilde{w}\, ds} f$, the problem becomes

$$e^{-\int \tilde{w}\, ds} \partial_s f = s^{-\nu+2}\tilde{\alpha}_{-(j-1)}$$

or equivalently,

$$\partial_s f = s^{-\nu+2}\beta$$

where $\beta = \tilde{\alpha}_{-(j-1)} e^{\int \tilde{w}\, ds}$, a smooth function of $s, \tilde{x}, \theta$. This can be solved by integration: $f = \int s^{-\nu+2}\beta\, ds$. Since $\nu \neq 3$, the result of integration is of the form $s^{-\nu+3}\tilde{\beta} + C(\tilde{x}, \theta)$ with $\tilde{\beta}$ a smooth function of $s, \tilde{x}, \theta$, and $C(\tilde{x}, \theta)$ a constant of integration. This solution is unique up to the additive function C. Now, recalling the definition of f we have that

$$a_{-j} = e^{-\int \tilde{w}\, ds}\big(s^{-\nu+3}\tilde{\beta} + C(\tilde{x}, \theta)\big) = s^{-\nu+3}\tilde{a}_{-j} + C(\tilde{x}, \theta)$$

with $\tilde{a}_{-j}$ smooth in $s, \tilde{x}, \theta$ (and depending on the function C).

The coefficient of s^0 in the expansion of a_{-j} can be specified by an appropriate choice of $C(\tilde{x}, \theta)$. This is equivalent to adding a specific solution of the homogeneous equation.

Reverting back to the original coordinates, we have $a_{-j} = t^{-\nu/2+3/2}\tilde{a}_{-j}$ with a_{-j} a smooth function of $t^{1/2}, x, \theta$. □

Now we know the order of the singularity of the solution of the inhomogeneous transport equation in terms of the order of the singularity of the inhomogeneous part. As for the homogeneous transport equation, we seek information about homogeneity in θ.

Lemma 5.6. *Let $\alpha_{-(j-1)}$ be a smooth function of $t^{1/2}, x, \theta$, with α homogeneous in θ of degree ℓ and $\alpha_{-(j-1)}|_{t=0} \neq 0$. Let $a_{-j,0}(x, \theta)$ be smooth and homogeneous of degree $\ell - 1$ in θ. Then, the solution a_{-j} of*

$$L(a_{-j}) = t^{-\nu/2}\alpha_{-(j-1)}$$

$$C_0(a_{-j}) = a_{-j,0}(x, \theta)$$

is homogeneous in θ of degree $\ell - 1$.

Proof. By Lemma 5.5, a_{-j} is of the form $t^{-\nu/2+3/2}\tilde{a}_{-j}$ with $\tilde{a}_{-j}$ a smooth function of $t^{1/2}, x, \theta$. The coefficient of each term in L is homogeneous in θ of degree 1. The operator does not differentiate in θ. As the right hand side is assumed to be homogeneous in θ of degree ℓ, any particular solution must be homogeneous of degree $\ell - 1$, up to an additive solution of the homogeneous transport equation. The condition that $a_{-j,0}(x,\theta)$ be homogeneous of degree $\ell-1$ in θ gives that the homogeneous solution being added is also homogeneous of degree $\ell-1$ in θ. Hence, a_{-j} is homogeneous in θ of degree $\ell-1$. □

5.3.2 Solutions

We make the ansatz that the solution operator is of the form

$$u = \int e^{i\phi^+}(a^{0,+}u_0 + a^{1,+}u_1) + e^{i\phi^-}(a^{0,-}u_0 + a^{1,-}u_1)\,dy\,d\theta.$$

Furthermore, we want some form of the "initial conditions" for the transport equation that respect the conditions for the problem we are trying to solve:

$$\lim_{t\searrow 0} u = u_0 \text{ and } \lim_{t\searrow 0}(t^{-1/2})u_t = u_1.$$

Writing the functions $a^{i,\pm}$ as expansions $a^{i,\pm} = \sum_{j=0}^{\infty} a^{i,\pm}_{-j}$, where, as it turns out, $a^{i,\pm}_{-j}$ is eventually homogeneous in θ of degree $-j-i$, these conditions can formally be interpreted as

$$\sum_{j=0}^{\infty} C_0(a^{0,+}_{-j}) + C_0(a^{0,-}_{-j}) = 1$$

$$\sum_{j=0}^{\infty}\Big[(t^{-1/2}i\phi^+_t)|_{t=0}C_0(a^{0,+}_{-j}) + (t^{-1/2}i\phi^-_t)|_{t=0}C_0(a^{0,-}_{-j})$$
$$+C_0(t^{-1/2}\partial_t a^{0,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{0,-}_{-(j-1)})\Big] = 0$$

$$\sum_{j=0}^{\infty} C_0(a^{1,+}_{-j}) + C_0(a^{1,-}_{-j}) = 0$$

$$\sum_{j=0}^{\infty}\Bigg[(t^{-1/2}i\phi_t^+)|_{t=0}C_0(a_{-j}^{1,+})+(t^{-1/2}i\phi_t^-)|_{t=0}C_0(a_{-j}^{1,-})$$

$$+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,+})+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,-})\Bigg]=1.$$

We solve this system by asking that

$$C_0(a_0^{0,+})+C_0(a_0^{0,-})=1$$

$$C_0(a_0^{1,+})+C_0(a_0^{1,-})=0$$

$$(t^{-1/2}i\phi_t^+)|_{t=0}C_0(a_{-j}^{0,+})+(t^{-1/2}i\phi_t^-)|_{t=0}C_0(a_{-j}^{0,-})=0$$

$$(t^{-1/2}i\phi_t^+)|_{t=0}C_0(a_{-j}^{1,+})+(t^{-1/2}i\phi_t^-)|_{t=0}C_0(a_{-j}^{1,-})=1$$

and for $j\geq 1$

$$C_0(a_{-j}^{0,+})+C_0(a_{-j}^{0,-})=0$$

$$(t^{-1/2}i\phi_t^+)|_{t=0}C_0(a_{-j}^{0,+})+(t^{-1/2}i\phi_t^-)|_{t=0}C_0(a_{-j}^{0,-})$$

$$+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{0,+})+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{0,-})=0$$

$$C_0(a_{-j}^{1,+})+C_0(a_{-j}^{1,-})=0$$

$$(t^{-1/2}i\phi_t^+)|_{t=0}C_0(a_{-j}^{1,+})+(t^{-1/2}i\phi_t^-)|_{t=0}C_0(a_{-j}^{1,-})$$

$$+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,+})+C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,-})=0.$$

Definition. For notational convenience, let

$$\sqrt{-\sum_{i,j=1}^{n}g^{ij}\theta_i\theta_j}=|\theta|_g$$

and

$$\sqrt{-\sum_{i,j=1}^{n}g^{ij}|_{t=0}\theta_i\theta_j}=|\theta|_g^0$$

Using our knowledge of $\phi^{\pm}$, we have that $(t^{-1/2}i\phi_t^{\pm})|_{t=0} = \pm|\theta|_g/\sqrt{\hat{g}^{00}}|_{t=0}$. Thus, we can determine $C_0(a_0^{i,\pm})$. The equations are

$$C_0(a_0^{0,+}) + C_0(a_0^{0,-}) = 1$$

$$C_0(a_0^{1,+}) + C_0(a_0^{1,-}) = 0$$

$$i\frac{|\theta|_g}{\sqrt{\hat{g}^{00}}}\Big|_{t=0} C_0(a_0^{0,+}) - i\frac{|\theta|_g}{\sqrt{\hat{g}^{00}}}\Big|_{t=0} C_0(a_0^{0,-}) = 0$$

$$i\frac{|\theta|_g}{\sqrt{\hat{g}^{00}}}\Big|_{t=0} C_0(a_0^{1,+}) - i\frac{|\theta|_g}{\sqrt{\hat{g}^{00}}}\Big|_{t=0} C_0(a_0^{1,-}) = 1$$

from which we have the solution

$$C_0(a_0^{0,+}) = \frac{1}{2} \qquad C_0(a_0^{0,-}) = \frac{1}{2}$$

$$C_0(a_0^{1,+}) = \frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|_g^0} \qquad C_0(a_0^{1,-}) = -\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|_g^0}.$$

For $j \geq 1$ the system can be written

$$C_0(a_{-j}^{0,+}) + C_0(a_{-j}^{0,-}) = 0$$

$$C_0(a_{-j}^{1,+}) + C_0(a_{-j}^{1,-}) = 0$$

$$C_0(a_{-j}^{0,+}) - C_0(a_{-j}^{0,-}) = -\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{i|\theta|_g^0}\Big(C_0(t^{-1/2}\partial_t a_{-(j-1)}^{0,+}) + C_0(t^{-1/2}\partial_t a_{-(j-1)}^{0,-})\Big)$$

$$C_0(a_{-j}^{1,+}) - C_0(a_{-j}^{1,-}) = -\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{i|\theta|_g^0}\Big(C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,+}) + C_0(t^{-1/2}\partial_t a_{-(j-1)}^{1,-})\Big).$$

We can write the solutions for $C_0(a_{-j}^{i,\pm})$ in terms of $a_{-(j-1)}^{i,\pm}$. Explicitly, we ask that

$$C_0(a^{0,+}_{-j}) = -\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|^0_g}\Big(C_0(t^{-1/2}\partial_t a^{0,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{0,-}_{-(j-1)})\Big)$$

$$C_0(a^{0,-}_{-j}) = \frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|^0_g}\Big(C_0(t^{-1/2}\partial_t a^{0,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{0,-}_{-(j-1)})\Big)$$

$$C_0(a^{1,+}_{-j}) = -\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|^0_g}\Big(C_0(t^{-1/2}\partial_t a^{1,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{1,-}_{-(j-1)})\Big)$$

$$C_0(a^{1,-}_{-j}) = \frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|^0_g}\Big(C_0(t^{-1/2}\partial_t a^{1,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{1,-}_{-(j-1)})\Big)$$

At this point, we are ready to determine precisely the nature of the singularities in t and homogeneity in θ for the functions $a^{i,\pm}_{-j}$.

Lemma 5.7. *The solutions of the homogeneous transport equation on the coordinate patch U with the desired initial conditions are of the form $a^{0,\pm}_0 = a^{0,\pm}_{0,0} + t\tilde{a}^{0,\pm}_0$ with $\tilde{a}^{0,\pm}_0$ a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree 0, and $a^{0,\pm}_{0,0} = 1/2$. Also, $a^{1,\pm}_0 = a^{1,\pm}_{0,0} + t\tilde{a}^{1,\pm}_0$ with $\tilde{a}^{1,\pm}_0$ a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree -1, and $a^{1,\pm}_{0,0} = \pm\frac{1}{2i|\theta|^0_g}$.*

Proof. This is immediate from Lemma 5.2 together with the fact that the initial conditions are homogeneous in θ of the proper degree. □

Lemma 5.8. *For $j \geq 1$, $i = 0, 1$, there is a unique solution of*

$$iL(a^{i,\pm}_{-j}) = \Box(a^{i,\pm}_{-(j-1)})$$

$$C_0(a^{i,\pm}_{-j}) = \mp\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|^0_g}\Big(C_0(t^{-1/2}\partial_t a^{i,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{i,-}_{-(j-1)})\Big).$$

on the coordinate patch U. The solution is of the form $a^{i,\pm}_{-j} = t^{-3j/2+1}\tilde{a}^{i,\pm}_{-j}$ with $\tilde{a}^{i,\pm}_{-j}$ a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree $-j-i$.

Proof. The symbols $a_0^{i,\pm}$ are smooth and homogeneous in θ of degree $-i$. Therefore, $\Box(a_0^{i,\pm}) = t^{-2}\alpha_0^{i,\pm}$ where $\alpha_0^{i,\pm}$ are smooth functions of $t^{1/2}, x, \theta$ and homogeneous in θ of degree $-i$. Therefore, $C_0(t^{-1/2}\partial_t a_0^{i,+}) + C_0(t^{-1/2}\partial_t a_0^{i,-})$ is also homogeneous in θ of degree $-i$, and smooth in x, θ. This implies that $C_0(a_{-1}^{i,\pm})$ is a smooth function of x, θ and homogeneous in θ of degree $-i-1$. By Lemma 5.5, there is a unique solution of

$$\begin{aligned} iL(a_{-1}^{i,\pm}) &= t^{-2}\alpha_0^{i,\pm} \\ C_0(a_{-1}^{i,\pm}) &= \mp\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|_g^0}\Big(C_0(t^{-1/2}\partial_t a_0^{i,+}) + C_0(t^{-1/2}\partial_t a_0^{i,-})\Big) \end{aligned}$$

and the solution is of the form $a_{-1}^{i,\pm} = t^{-1/2}\tilde{a}_{-1}^{i,\pm} = t^{-3j/2+1}\tilde{a}_{-1}^{i,\pm}$ where $\tilde{a}_{-1}^{i,\pm}$ is a smooth function of $t^{1/2}, x, \theta$. Moreover, by Lemma 5.6, $\tilde{a}_{-1}^{i,\pm}$ is homogeneous in θ of degree $-i-1$. Therefore, the conclusion holds for $j = 1$.

Assume the conclusion holds for $j \le J$. Then, $a_{-J}^{i,\pm} = t^{-3J/2+1}\tilde{a}_{-J}^{i,\pm}$ and $\tilde{a}_{-J}^{i,\pm}$ is a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree $-J-i$. Therefore, $\mp\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|_g^0}\Big(C_0(t^{-1/2}\partial_t a_{-J}^{i,+}) + C_0(t^{-1/2}\partial_t a_{-J}^{i,-})\Big)$ is a smooth function of x, θ and is homogeneous in θ of degree $-J-i-1$. By Lemma 5.4, $\Box(a_{-J}^{i,\pm}) = t^{-3J/2-2}\tilde{\alpha}_{-J}^{i,\pm}$, where $\alpha_{-J}^{i,\pm}$ is a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree $-J-i-1$. By Lemma 5.5 and Lemma 5.6, there is a unique solution of

$$\begin{aligned} iL(a_{-(J+1)}^{i,\pm}) &= t^{-3J/2-2}\alpha_{-J}^{i,\pm} \\ C_0(a_{-1}^{i,\pm}) &= \mp\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{2i|\theta|_g^0}\Big(C_0(t^{-1/2}\partial_t a_0^{i,+}) + C_0(t^{-1/2}\partial_t a_0^{i,-})\Big) \end{aligned}$$

of the form $a_{-(J+1)}^{i,\pm} = t^{-3J/2-1/2}\tilde{a}_{-(J+1)}^{i,\pm} = t^{-3(J+1)/2+1}\tilde{a}_{-(J+1)}^{i,\pm}$ with $\tilde{a}_{-(J+1)}^{i,\pm}$ a smooth function of $t^{1/2}, x, \theta$, homogeneous of degree $-(J+1)-i$ in θ. Hence, by induction, the conclusion holds for all $j \geq 1$, and $i = 0, 1$. $\square$

Essentially, this says that $a_{-j}^{i,\pm}$ has a singularity in t at worst of order $t^{-3j/2+1}$, and is homogeneous in θ of degree $-j-i$. Moreover, we can arrange that $C_0(a_{-j}^{i,\pm})$ be the given functions, and with these conditions the solutions are unique.

CHAPTER 6

The Symbols

6.1 Review

Definition. Let X be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}$ and let ρ, δ be any real numbers. A *symbol of order* m *and type* (ρ, δ) *on* $X \times \mathbb{R}^N$ is a function $a \in C^\infty(X \times \mathbb{R}^N)$ such that for any K compactly contained in X and any multiindices α, β, there exists a C such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \leq C(1 + |\theta|)^m - |\beta|\rho + |\alpha|\delta \ \forall \ x \in K, \ \theta \in \mathbb{R}^N.$$

The set of symbols of order m and type (ρ, δ) on $X \times \mathbb{R}^N$ is denoted $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. Also, define

$$S^\infty_{\rho,\delta}(X \times \mathbb{R}^N) = \bigcup_m S^m_{\rho,\delta}(X \times \mathbb{R}^N), \qquad S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_m S^m_{\rho,\delta}(X \times \mathbb{R}^N).$$

The space $S^{-\infty}(X \times \mathbb{R}^N)$ is independent of ρ, δ.

With the topology generated by the seminorms, $S^m_{\rho,\delta}$ is a Fréchet space.

One of the most useful facts is one of summability, given in the following lemma. The proof is motivated by that of Borel's lemma.

Lemma 6.1. *Suppose a_j is a sequence of symbols in $S^{m_j}_{\rho,\delta}(U \times \mathbb{R}^N)$, where U is an open subset of $\mathbb{R}^n$ with $m_j \searrow -\infty$ as $j \to \infty$. Suppose further that K_ℓ is a sequence of compact subsets of U such that $K_\ell \subset K_{\ell+1}$ and $\bigcup K_\ell = U$. Let $p^m_{M,K}$ be the seminorms of type (ρ,δ). Finally, suppose that $p^{m_\ell}_{\ell,K_\ell}(a_j) \leq 2^{-j}$ for $0 \leq \ell < j$. Then, for any $j_0 \in \mathbb{N} \cup \{0\}$,*

$$\sum_{j=j_0}^{\infty} a_j \in S^{m_{j_0}}_{\rho,\delta}(U \times \mathbb{R}^n).$$

Proof. We show that the $p^{m_{j_0}}_{M,K}$ seminorms of the terms forms a converging sequence. Fix $M \geq 0$ and a compact set $K \subset U$ and consider the sum $\sum_{j=j^0}^{\infty} a_j$. Let ℓ be such that $\ell \geq M, j_0$ and $K \subset K_j$. Then, for $j > \ell$, $p^{m_{j_0}}_{M,K}(a_j) \leq p^{m_{j_0}}_{j,K_j}(a_j) \leq 2^{-j}$. For $j_0 \leq j \leq \ell$, $p^{m_{j_0}}_{M,K}(a_j) \leq p^{m_j}_{M,K}(a_j) < \infty$ since $m_j \geq m_{j_0}$. Therefore, $p^{m_{j_0}}_{M,K}(a_j)$ is a Cauchy sequence, and hence $p^{m_{j_0}}_{M,K}(\sum_{j=j^0}^{\infty} a_j)$ is finite, and the partial sum is in the desired space. □

We would like spaces that are a little more general. Fix $\rho, \delta, \eta, m \in \mathbb{R}$, and let I be a compact subset of $\mathbb{R}$ and $U \subset \mathbb{R}^n$ open. We would like to define $\tilde{S}^m_{\rho,\delta,\eta}(I \times U \times \mathbb{R}^n)$, so that this space is a Fréchet space. To do this, for any compact set $K \subset U$ and any $M \in \mathbb{N} \cup \{0\}$ we define seminorms

$$\varrho^m_{M,K}(f(t,x,\theta)) = \sup_{\substack{(t,x) \in I \times K \\ r+|\alpha|+|\beta| \leq M}} \left| \frac{\partial_t^r \partial_x^\alpha \partial_\theta^\beta f(t,x,\theta)}{(1+|\theta|)^{m+r\delta+\eta|\alpha|-\rho|\beta|}} \right|.$$

We define $S^m_{\rho,\delta,\eta}(I \times U \times \mathbb{R}^n)$ to be the set of functions $f(t,x,\theta) \in C^\infty(I \times U \times \mathbb{R}^N)$ for which $\varrho^m_{M,K}(f(t,x,\theta))$ is finite for every $M \in \mathbb{N} \cup \{0\}$ and every compact set $K \subset U$.

The proof that this is a Fréchet space is the same as for the regular symbols of type (ρ, δ). First, define the topology given by the seminorms for countably many well-chosen compact sets. Show that in this topology, it is complete with respect to each seminorm, making it a Fréchet space. Then, show that the topology generated by the seminorms for any compact set is equivalent to the constructed topology. Since it is a Fréchet space, the proof of Lemma 6.1 is valid for sums whose symbols are of type ρ, δ, η with a given compact set $I \subset \mathbb{R}$.

Finally, there is a useful notion of a sum of symbols being asymptotic to a given one. Suppose $a_j \in S^{m_j}_{\rho,\delta}(U \times \mathbb{R}^n)$ for some $U \subset \mathbb{R}^N$ open, and $m_j \searrow -\infty$ as $j \to \infty$. If $a \in S^{m_0}_{\rho,\delta}(U \times \mathbb{R}^n)$ is such that

$$a - \sum_{j<k} a_{-j} \in S^{m_k}_{\rho,\delta}(U \times \mathbb{R}^n)$$

we say that the sum of the a_j is asymptotic to a and write $a \sim \sum_{j=0}^\infty a_{-j}$. This a is unique modulo $S^{-\infty}$, and any rearrangement of the sum is also asymptotic to a. For a proof, see [5].

6.2 Amplitudes

We have from Lemma 5.8 that for $j \geq 1$ the solutions of the transport equations are of the form $a^{i,\pm}_{-j} = t^{-3j/2+1}\tilde{a}^{i,\pm}_{-j}$ with $\tilde{a}^{i,\pm}_{-j}$ a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree $-j-i$. The solutions of the homogeneous transport equation are of the form $a^{i,\pm}_0 = a^{i,\pm}_{0,0} + t\tilde{a}^{i,\pm}_0$ with $\tilde{a}^{i,\pm}_0$ a smooth function of $t^{1/2}, x, \theta$, homogeneous in θ of degree $-i$.

We seek actual symbols so we can add them asymptotically to get our solution operator. One of the issues is the singularities in t. Another issue is the singularities of $t^{-1/2}\partial_t a^{i,\pm}_{-j}$ that appear in the initial conditions. The new singularities in the terms involving the derivative come from terms of order $t^{1/2}$ or t^1 in the expansion of the functions $a^{i,\pm}$. We will isolate these terms that give rise to singularities, and show there is a uniform bound so that the asymptotic sums converge.

For $j \geq 1$, the functions $a^{i,\pm}_{-j}$ are of the form $t^{-3j/2+1}\tilde{a}^{i,\pm}_{-j}$ with $\tilde{a}^{i,\pm}_{-j}$ a smooth function of $t^{1/2}$. Therefore, we may write an expansion of the form

$$\begin{aligned} a^{i,\pm}_{-j} =& \left[\sum_{k=-3j+2}^{-1} \left(a^{i,\pm}_{-j,-k} t^{-k/2} \right) + a^{i,\pm}_{-j,1} t^{1/2} + a_{-j,2} t \right] + a^{i,\pm}_{-j,0} + \left[t^{3/2} \tilde{\alpha}^{i,\pm}_{-j} \right] \\ =& a^{i,\pm}_{-j,S} + a^{i,\pm}_{-j,0} + a^{i,\pm}_{-j,R} \end{aligned}$$

where $a^{i,\pm}_{-j,\ell}$ are smooth functions of (x,θ), $\alpha^{i,\pm}_{-j}$ is a smooth function of $t^{1/2}, x, \theta$ and all are homogeneous in θ of degree $-j-i$.

The function $a^{i,\pm}_{-j,S}$ will be referred to as the *singular part of* $a^{i,\pm}_{-j}$, and the function $a^{i,\pm}_{-j,R}$ will be referred to as the *regular part of* $a^{i,\pm}_{-j}$. Likewise, for $a^{i,\pm}_{0}$, we may write an expansion of the form

$$a^{i,\pm}_0 = ta^{i,\pm}_{0,2} + a^{i,\pm}_{0,0} + \left[t^{3/2}\tilde{\alpha}^{i,\pm}_0 \right]$$

where $a^{i,\pm}_{0,2}$ and $a^{i,\pm}_{0,0}$ are smooth functions of (x, θ) homogeneous in θ of degree $-i$, and $\alpha^{i,\pm}_0$ is a smooth function of $t^{1/2}, x, \theta$, homogeneous of degree $-i$ in θ. Denote $a^{i,\pm}_{0,2}t$ as $a^{i,\pm}_{0,S}$ and $a^{i,\pm}_{0,R} = t^{3/2}\tilde{\alpha}^{i,\pm}_0$. Again, we will call $a^{i,\pm}_{0,R}$ the regular part, and $a^{i,\pm}_{0,S}$ the singular part of $a^{i,\pm}_0$.

The order of the singularity in t for each $a^{i,\pm}_{-j,S}$ is $t^{-3j/2+1}$ for $j \geq 0$, $i = 0, 1$. For convenience, write $a^{i,\pm}_{-j,S} = t^{-3j/2+1}\tilde{a}^{i,\pm}_{-j,S}$. Then, $\tilde{a}^{i,\pm}_{-j,S}$ is a smooth function of $t^{1/2}, x, \theta$ on $[0, \eta) \times U \times \mathbb{R}^n$ and homogeneous in θ of degree $-j - i$.

We will exploit the fact that the t singularity of these amplitudes increases in order linearly, while the order in θ decreases linearly. In order for the sum of the amplitudes to converge in a suitable symbol space, we need the j^{th} symbol to be small when regarded as a symbol in a space of higher order. This is achieved with the aid of a cutoff function. For the singular part, we choose a cutoff function that is 0 for $t = 0$, while still allowing for decrease in the order of θ. With the same cutoff function, but having an argument independent of t, we cut off the regular part so that we can add the symbols. We now define these symbols.

Definition. Let $\chi \in C^\infty_c([0, \infty))$ be so that $\chi(r) = 0$ if $r \leq 1$ and $\chi(r) = 1$ if $r \geq 2$. Furthermore, let $0 < \varepsilon_j \leq 1$, $\delta > 0$, all to be determined later. Change coordinates with $s = \sqrt{t}$. For $j \geq 1$, $i = 0, 1$ define

$$A^{i,\pm}_{-j} = s^{-3j+2}\tilde{a}^{i,\pm}_{-j,S}\chi(\varepsilon_j s|\theta|^\delta)\chi(\varepsilon_j|\theta|) + a^{i,\pm}_{-j,0}\chi(\varepsilon_j|\theta|) + a^{i,\pm}_{-j,R}\chi(\varepsilon_{j+1}|\theta|)$$

and for $j = 0$ we define

$$A^{i,\pm}_0 = a^{i,\pm}_{0,S}\chi(\varepsilon_0 s|\theta|^\delta)\chi(\varepsilon_0|\theta|) + a^{i,\pm}_{0,0} + a^{i,\pm}_{0,R}\chi(\varepsilon_1|\theta|).$$

We write

$$A_{-j,S} = s^{-3j+2}\tilde{a}^{i,\pm}_{-j,S}\chi(\varepsilon_j s|\theta|^\delta)\chi(\varepsilon_j|\theta|)$$

and

$$A_{-j,R} = a^{i,\pm}_{-j,0}\chi(\varepsilon_j|\theta|) + a^{i,\pm}_{-j,R}\chi(\varepsilon_{j+1}|\theta|)$$

and call $A_{-j,S}$ the *singular part* and $A_{-j,R}$ the *regular part* of the function $A^{i,\pm}_{-j}$.

Remark. The choice of using different ε for the regular and singular parts is only for the purpose of making some later computations easier. Taking the same ε_j in the argument of χ will also work.

Note that with a change of coordinates of the form $s = rs$ for some $r > 0$, we may assume that A_{-j} is defined for $s \in [0, 1]$ and smooth on $[0, 1] \times U \times \mathbb{R}^n$. We make this assumption from now on.

To simplify notation, we will work in coordinates with $|\theta| = \tau$. This affords relatively simple bounds on functions of the nature we will encounter. In order to facilitate this change of coordinates, we have a lemma.

Lemma 6.2. *For $\theta \neq 0$, the change coordinates $\theta \mapsto (\omega, \tau)$ with $(\omega, \tau) \in S^{n-1} \times (0, \infty)$ where $\omega = \theta/|\theta|$ and $\tau = |\theta|$ is smooth. Moreover, for any multiindex β*

$$\partial_\theta^\beta = \sum_{0<|\gamma|+r\leq|\beta|} g_{\gamma,\beta,r}(\omega)\tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma$$

where each $g_{\gamma,\beta,r}(\omega)$ is smooth for $\omega \neq 0$.

Proof. It is clear that the change of coordinates is smooth for $\theta \neq 0$. For the derivative, note that

$$\partial_{\theta_m} = \omega_m\partial_\tau + \frac{1}{\tau}\sum_{\ell=1}^{n}(\delta_{\ell,m} - \omega_\ell\omega_m)\partial_{\omega_\ell}.$$

where $\delta_{\ell,m}$ is the Kroneker delta. This is of the form

$$\sum_{0<|\gamma|+r\leq|\beta|} g_{\gamma,r}(\omega)\tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma$$

where $|\beta| = 1$ and $g_{\gamma,r}$ is smooth. Suppose that

$$\partial_\theta^\beta = \sum_{0<|\gamma|+r\leq|\beta|} g_{\gamma,\beta,r}(\omega)\tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma$$

for $0 < |\beta| < N$. Fix β with $|\beta| = N$. Then, for some m and $\tilde{\beta}$ with $|\tilde{\beta}| = N - 1$, $\partial_\theta^\beta = \partial_{\theta_m}\partial_\theta^{\tilde{\beta}}$. By assumption, this is of the form

$$\partial_{\theta_m} \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\tilde{\beta}|}\partial_\tau^r\partial_\omega^\gamma.$$

Having an explicit formula for ∂_{θ_m} in terms of τ and ω, this is

$$\left(\omega_m\partial_\tau + \frac{1}{\tau}\sum_{\ell=1}^{n}(\delta_{\ell,m} - \omega_\ell\omega_m)\partial_{\omega_\ell}\right)\left(\sum_{0<|\gamma|+r\leq|\tilde{\beta}|} g_{\gamma,\beta,r}(\omega)\tau^{r-|\tilde{\beta}|}\partial_\tau^r\partial_\omega^\gamma\right)$$

Expanding gives

$$\omega_m \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \Big((r-|\tilde{\beta}|) g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\tilde{\beta}|-1}\partial_\tau^r\partial_\omega^\gamma + g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\tilde{\beta}|}\partial_\tau^{r+1}\partial_\omega^\gamma \Big)$$
$$+ \sum_{\ell=1}^n \left((\delta_{\ell,m}-\omega_\ell\omega_m) \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \Big(\partial_{\omega_\ell}(g_{\gamma,\tilde{\beta},r}(\omega))\tau^{r-|\tilde{\beta}|-1}\partial_\tau^r\partial_\omega^\gamma \right.$$
$$\left. + g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\tilde{\beta}|-1}\partial_\tau^r\partial_\omega^\gamma\partial_{\omega_\ell} \Big) \right)$$

which can be written as

$$= \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \Bigg[\omega_m(r-|\tilde{\beta}|) g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-(|\tilde{\beta}|+1)}\partial_\tau^r\partial_\omega^\gamma$$
$$+ \omega_m g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r+1-(|\tilde{\beta}|+1)}\partial_\tau^{r+1}\partial_\omega^\gamma$$
$$+ \sum_{\ell=1}^n (\delta_{\ell,m}-\omega_\ell\omega_m)\partial_{\omega_\ell}(g_{\gamma,\tilde{\beta},r}(\omega))\tau^{r-(|\tilde{\beta}|+1)}\partial_\tau^r\partial_\omega^\gamma$$
$$+ \sum_{\ell=1}^n (\delta_{\ell,m}-\omega_\ell\omega_m) g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-(|\tilde{\beta}|+1)}\partial_\tau^r\partial_\omega^\gamma\partial_{\omega_\ell} \Bigg]$$

$$= \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \Big(\omega_m(r-|\tilde{\beta}|) g_{\gamma,\tilde{\beta},r}(\omega)$$
$$+ \sum_{\ell=1}^n (\delta_{\ell,m}-\omega_\ell\omega_m)\partial_{\omega_\ell}(g_{\gamma,\tilde{\beta},r}(\omega)) \Big) \tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma$$
$$+ \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \omega_m g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r+1-|\beta|}\partial_\tau^{r+1}\partial_\omega^\gamma$$
$$+ \sum_{0<|\gamma|+r\leq|\tilde{\beta}|} \sum_{\ell=1}^n (\delta_{\ell,m}-\omega_\ell\omega_m) g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma\partial_{\omega_\ell}$$

The function $g^1_{\gamma,\beta,r}(\omega) = \omega_m(r-|\tilde{\beta}|)g_{\gamma,\tilde{\beta},r}(\omega) + \sum_{\ell=1}^n(\delta_{\ell,m}-\omega_\ell\omega_m)\partial_{\omega_\ell}(g_{\gamma,\tilde{\beta},r}(\omega))$ is smooth. Let $g^1_{\gamma,\beta,r} = 0$ if $r+|\gamma| = \beta$. Then, the first sum can be written as

$$\sum_{0<|\gamma|+r\leq|\beta|} g^1_{\gamma,\beta,r}(\omega)\tau^{r-|\beta|}\partial_\tau^r\partial_\omega^\gamma$$

For the second sum, put $g^2_{\gamma,\beta,r+1}(\omega) = \omega_m g_{\gamma,\tilde{\beta},r}(\omega)$ and let $g^2_{\gamma,\beta,0} = 0$. After reindexing, the second sum becomes

$$\sum_{0<|\gamma|+r\leq|\beta|} g^2_{\gamma,\beta,r}(\omega)\tau^{r-|\beta|}\partial^r_\tau\partial^\gamma_\omega$$

For the last sum, put $g^3_{\tilde{\gamma},\beta,r} = 0$ if $\tilde{\gamma} = 0$, and we can write

$$\begin{aligned}&\sum_{0<|\gamma|+r\leq|\tilde{\beta}|}\sum_{\ell=1}^{n}(\delta_{\ell,m} - \omega_\ell\omega_m)g_{\gamma,\tilde{\beta},r}(\omega)\tau^{r-|\beta|}\partial^r_\tau\partial^\gamma_\omega\partial_{\omega_\ell}\\ &= \sum_{0<|\tilde{\gamma}|+r\leq|\beta|} g^3_{\tilde{\gamma},\beta,r}(\omega)\tau^{r-|\beta|}\partial^r_\tau\partial^{\tilde{\gamma}}_\omega\end{aligned}$$

where $g^3_{\tilde{\gamma},\beta,r}(\omega)$ is smooth.

Finally, writing γ for the index in the last sum and setting $g_{\gamma,\beta,r} = g^1_{\gamma,\beta,r} + g^2_{\gamma,\beta,r} + g^3_{\gamma,\beta,r}$ we have

$$\partial^\beta_\theta = \sum_{0<|\gamma|+r\leq|\beta|} g_{\gamma,\beta,r}(\omega)\tau^{r-|\beta|}\partial^r_\tau\partial^\gamma_\omega.$$

Hence, the formula holds for all β. □

The task of determining the symbol spaces and convergence of the asymptotic sums will rely on determining suitable bounds on the functions $A^{i,\pm}_{-j}$. Since our initial conditions include the operator $s^{-2}\partial_s$, we consider functions slightly more general than the $A^{i,\pm}_{-j}$ to allow for the extra negative powers of s. The bounds reflect the space the symbols are in, and that the symbols are small in the seminorms of symbol spaces of higher order and the same type. We begin with some simple bounds on the cutoffs.

Lemma 6.3. *If $\chi(r)$ is a function supported on $r \geq 1$ and constant for $r \geq 2$, then for $s \in [0,1]$, $0 < \varepsilon \leq 1$, $\tau \geq 0$, $\delta > 0$ and $m > 0$, the function $\partial_\tau^m \chi(\varepsilon s \tau^\delta)$ is smooth in s, τ and is of the form $\tau^{-m}\chi_m(\varepsilon s \tau^\delta)$ where $\chi_m(r)$ is supported in $1 \leq r \leq 2$.*

Proof. We use induction. The first derivative

$$\partial_\tau \chi(\varepsilon s \tau^\delta) = \chi'(\varepsilon s \tau^\delta)\delta \varepsilon s \tau^{-1+\delta} = \tau^{-1}\Big[\delta \chi'(\varepsilon s \tau^\delta)\varepsilon s \tau^\delta\Big].$$

The expression in brackets is a smooth function of $\varepsilon s \tau^\delta$. Moreover, it is supported on $1 \leq \varepsilon s \tau^\delta \leq 2$ owing to the fact that $\chi(r)$ is constant on $r < 1$ and $r > 2$. Hence, with $\chi_1(\varepsilon s \tau^\delta) = \delta \chi'(\varepsilon s \tau^\delta)\varepsilon s \tau^\delta$, the result holds.

Assume the result holds for ∂_τ^{m-1} for some $m > 1$. Then, $\partial_\tau^m \chi(\varepsilon s \tau^\delta) = \partial_\tau \partial_\tau^{m-1}\chi(\varepsilon s \tau^\delta) = \partial_\tau(\tau^{-(m-1)}\chi_{m-1}(\varepsilon s \tau^\delta))$ with $\chi_{m-1}(r)$ supported in $1 \leq r \leq 2$. Now, we have

$$\begin{aligned}
\partial_\tau^m \chi(\varepsilon s \tau^\delta) &= \partial_\tau \tau^{-(m-1)}\chi_{m-1}(\varepsilon s \tau^\delta) \\
&= -(m-1)\tau^{-m}\chi_{m-1}(\varepsilon s \tau^\delta) + \tau^{-(m-1)}\chi'_{m-1}(\varepsilon s \tau^\delta)\delta \varepsilon s \tau^{\delta-1} \\
&= \tau^{-m}\Big[\delta \chi'_{m-1}(\varepsilon s \tau^\delta)\varepsilon s \tau^\delta - (m-1)\chi_{m-1}(\varepsilon s \tau^\delta)\Big].
\end{aligned}$$

The expression in brackets is a smooth function of $\varepsilon s \tau^\delta$, and is supported in $1 \leq \varepsilon s \tau^\delta \leq 2$. Hence, the result holds for all $m \geq 1$. □

We now bound derivatives of the cutoff functions.

Lemma 6.4. *Suppose χ is as in Lemma 6.3, and that $m \in \mathbb{N} \cup \{0\}$ and $0 < \varepsilon \leq 1$ are given. Then, there is a constant C independent of ε for which*

$$\left|\partial_\tau^m \chi(\varepsilon\tau)\right| \leq C\tau^{-m}.$$

Moreover, for any nonnegative integer k and $\delta > 0$, there is a constant $\tilde{C}$ independent of s, ε for which

$$\left|\partial_s^k \partial_\tau^m \Big(\chi(s\varepsilon\tau^\delta)\chi(\varepsilon\tau)\Big)\right| \leq \tilde{C} s^{-k}\tau^{-m}.$$

Proof. For the first inequality, simply note that $\partial_\tau^m \chi(\varepsilon\tau) = \varepsilon^m \chi^{(m)}(\varepsilon\tau) = \tau^{-m}(\varepsilon\tau)^m \chi^{(m)}(\varepsilon\tau)$. If $m = 0$, we have not differentiated and the inequality is immediate due to the fact that χ is bounded. If we have differentiated, then $\chi^{(m)}(r)$ is smooth and has support in $1 \leq r \leq 2$. Therefore, $\chi^{(m)}(r)$ is bounded, and $(\varepsilon\tau)^m \leq 2^m$. With $C = 2^m \sup|\chi^{(m)}(r)|$, we have the first inequality.

By the Leibnitz rule,

$$\partial_\tau^m \Big(\chi(s\varepsilon\tau^\delta)\chi(\varepsilon\tau)\Big) = \sum_{r=0}^{m} \binom{m}{r} \partial_\tau^r \Big(\chi(s\varepsilon\tau^\delta)\Big)\Big(\partial_\tau^{m-r}\chi(\varepsilon\tau)\Big).$$

By Lemma 6.3, we can write

$$\partial_\tau^r \chi(\varepsilon s\tau^\delta) = \tau^{-r}\chi_r(\varepsilon s\tau^\delta)$$

where χ_r is smooth. Therefore, the expression to bound is the absolute value of

$$\sum_{r=0}^{m} \binom{m}{r} \tau^{-r} \partial_s^k \chi_r(\varepsilon s\tau^\delta)\partial_\tau^{m-r}\chi(\varepsilon\tau).$$

It suffices to bound the absolute value of the summands, since the sum is finite. Clearly,

$$\partial_\tau^{m-r}\chi(\varepsilon\tau) = \varepsilon^{m-r}\chi^{(m-r)}(\varepsilon\tau) = \tau^{r-m}(\varepsilon\tau)^{m-r}\chi^{(m-r)}(\varepsilon\tau).$$

Any derivative of χ is compactly supported, so that $(\varepsilon\tau)^{m-r}\chi^{(m-r)}(\varepsilon\tau)$ is bounded for any $0 \leq r \leq m$. This gives the bound

$$|\tau^{-r}\partial_\tau^{m-r}\chi(\varepsilon\tau)| \leq C\tau^{-m}$$

Furthermore, $\chi(\varepsilon s\tau^\delta)$ is supported for $\varepsilon s\tau^\delta \geq 1$. Therefore, $\chi(\varepsilon s\tau^\delta)$ vanishes to infinite order in s at $s = 0$. For $s \neq 0$, we may write

$$(\varepsilon\tau^\delta)^k\chi_r^{(k)}(\varepsilon s\tau^\delta) = s^{-k}(\varepsilon s\tau^\delta)^k\chi_r^{(k)}(\varepsilon s\tau^\delta).$$

If $k = 0$, we clearly have the bound, since $\chi_r(\varepsilon s\tau^\delta)$ is smooth and constant for its argument greater than 2. Hence, if $k = 0$, we have

$$\left|\partial_\tau^m\Big(\chi(s\varepsilon\tau^\delta)\chi(\varepsilon\tau)\Big)\right| \leq \tilde{C}\tau^{-m}.$$

If $k > 0$, then we have differentiated χ_r. Since it is constant when its argument is less than 1 or greater than 2, $\chi_r^{(k)}(\varepsilon s\tau^\delta)$ is supported in $1 \leq \varepsilon s\tau^\delta \leq 2$. Thus, $|(\varepsilon s\tau^\delta)^k| \leq 2^k$, and $|\chi_r^{(k)}(\varepsilon s\tau^\delta)|$ is a bounded function as it is smooth and compactly supported. Hence, there is a constant C_1, independent of ε, s such that

$$|\tau^{-r}s^{-k}(\varepsilon s\tau^\delta)^k\chi_r^{(k)}(\varepsilon s\tau^\delta)| \leq C_1 s^{-k}\tau^{-m}.$$

$\square$

These allow us to sufficiently manage the derivatives of the cutoff functions. Now, we derive bounds for functions reminiscent of the regular and singular parts of our amplitudes. These will be the key to summability of the symbols.

Lemma 6.5. *Let $U \subset \mathbb{R}^n$ be open and let $f(s,x,\theta) : [0,1] \times U \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function of s, x, θ, homogeneous in θ of degree d. Fix $\mu, \nu \geq 0$, $\delta > 0$, $0 < \varepsilon \leq 1$. Let $\chi : [0,\infty) \to \mathbb{R}$ be as in Lemma 6.3. For any compact set $K \subset U$ and any $M \in \mathbb{N} \cup \{0\}$, there are constants C and C', both independent of ε and s, such that*

$$\sup_{\substack{(s,x)\in[0,1]\times K \\ r+|\alpha|+|\beta|\leq M}} \left| \frac{\partial_s^r \partial_x^\alpha \partial_\theta^\beta \Big(s^{-\nu} f(s,x,\theta) \chi(\varepsilon s |\theta|^\delta) \chi(\varepsilon|\theta|) \Big)}{(1+|\theta|)^{\mu+d+\delta\nu-|\beta|+r\delta}} \right| \leq C\varepsilon^\mu \tag{6.1}$$

and

$$\sup_{\substack{(s,x)\in[0,1]\times K \\ r+|\alpha|+|\beta|\leq M}} \left| \frac{\partial_s^r \partial_x^\alpha \partial_\theta^\beta \Big(f(s,x,\theta) \chi(\varepsilon|\theta|) \Big)}{(1+|\theta|)^{\mu+d-|\beta|}} \right| \leq C'\varepsilon^\mu \tag{6.2}$$

Moreover, if we restrict s to a compact set $\tilde{K} \subset (0,1)$, then we have the bound

$$\sup_{\substack{(s,x)\in\tilde{K}\times K \\ r+|\alpha|+|\beta|\leq M}} \left| \frac{\partial_s^r \partial_x^\alpha \partial_\theta^\beta \Big(s^{-\nu} f(s,x,\theta) \chi(\varepsilon s |\theta|^\delta) \chi(\varepsilon|\theta|) \Big)}{(1+|\theta|)^{\mu+d-|\beta|}} \right| \leq \tilde{C}\varepsilon^\mu \tag{6.3}$$

where $\tilde{C}$ is independent of ε.

Proof. The expression to bound is supported in $\varepsilon|\theta| \geq 1$. Since $\varepsilon \leq 1$ it follows that $|\theta| \geq 1$. Therefore, we may change coordinates $\theta \mapsto (\omega,\tau)$, as in Lemma 6.2. Moreover, by Lemma 6.2, with this change of coordinates

$$\partial_s^r \partial_x^\alpha \partial_\theta^\beta = \sum_{0<|\gamma|+q\leq|\beta|} g_{\gamma,\beta,q}(\omega) \tau^{q-|\beta|} \partial_s^r \partial_x^\alpha \partial_\omega^\gamma \partial_\tau^q.$$

With this change of coordinates, f is only defined for $\omega \in S^{n-1}$. Extend f as constant radially in ω so that it is defined in a neighborhood $0 \notin \tilde{U}$ of S^{n-1}. Furthermore, f is homogeneous of degree d in τ.

Now, the supremum for $x \in K$ and $\theta \in \mathbb{R}^n$ is equivalent to that for $(x, \omega) \in K \times S^{n-1}$ and $\tau \in [0, \infty)$. The function $\chi(\varepsilon\tau)$ vanishes to infinite order at $\tau = 0$. Therefore, it is sufficient to bound for $\tau \in (0, \infty)$. At this point, we must bound functions of the form

$$\sup_{\substack{(s,x,\omega)\in K_1\times K\times S^{n-1}\\ r+|\alpha|+|\beta|\leq M}} \left| \sum_{0<|\gamma|+q\leq|\beta|} g_{\gamma,\beta,q}(\omega)\tau^{q-|\beta|} \frac{\partial_s^r \partial_x^\alpha \partial_\omega^\gamma \partial_\tau^q h(s,x,\omega,\tau)}{(1+\tau)^{\mu+\eta}} \right| \leq C\varepsilon^\mu \quad (6.4)$$

where K_1 is either $[0, 1]$ or a compact subset of $(0, 1)$, and η is the appropriate power in the denominator. Since $0 < |\gamma| + q \leq |\beta|$, it is sufficient to obtain the bound for the supremum over $r + |\alpha| + |\gamma| + q \leq M$. Moreover, since the sum is finite, it is sufficient to have a bound of the desired form for the absolute value of each summand. That is, it is sufficient to bound

$$\sup_{\substack{(s,x,\omega)\in K_1\times K\times S^{n-1}\\ r+|\alpha|+|\gamma|+q\leq M}} \left| g_{\gamma,\beta,q}(\omega)\tau^{q-|\beta|} \frac{\partial_s^r \partial_x^\alpha \partial_\omega^\gamma \partial_\tau^q h(s,x,\omega,\tau)}{(1+\tau)^{\mu+\eta}} \right|$$

with $|\gamma| + q \leq |\beta|$. Furthermore, since each $g_{\gamma,\beta,q}(\omega)$ is smooth, there is a constant C for which $g_{\gamma,\beta,q}(\omega) \leq C$ for $\omega \in S^{n-1}$ and $|\gamma| + q \leq |\beta|$ for all β with $|\beta| \leq M$. Finally, writing $\tilde{x} = (x, \omega)$ and $\tilde{\alpha}$ for the multiindex obtained by appending γ to α gives $g(s, x, \omega, \tau) = g(s, \tilde{x}, \tau)$ and $\partial_x^\alpha \partial_\omega^\gamma = \partial_{\tilde{x}}^{\tilde{\alpha}}$. Therefore, having the bound for (6.4), is equivalent to having the bound

$$\sup_{\substack{(s,x,\omega)\in K_1\times K\times S^{n-1}\\ r+|\tilde{\alpha}|+q\leq M}} \left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q h(s,\tilde{x},\tau)}{(1+\tau)^{\mu+\eta}} \right| \leq C\varepsilon^\mu.$$

Finally, the last simplification comes from the observation that each denominator has a factor of $(1+\tau)^\mu$. Since the expression to bound is supported on $\varepsilon\tau \geq 1$ and $\varepsilon \leq 1$, we have $(1+\tau)^{-\mu} \leq \tau^{-\mu} \leq \varepsilon^\mu$ on the support. Therefore,

$$\sup_{\substack{(s,x,\omega)\in K_1\times K\times S^{n-1}\\ r+|\tilde{\alpha}|+q\leq M}} \left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q h(s,\tilde{x},\tau)}{(1+\tau)^{\mu+\eta}} \right| \leq \varepsilon^\mu \sup_{\substack{(s,x,\omega)\in K_1\times K\times S^{n-1}\\ r+|\tilde{\alpha}|+q\leq M}} \left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q h(s,\tilde{x},\tau)}{(1+\tau)^{\eta}} \right|$$

so it suffices to show that

$$\left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q h(s,\tilde{x},\tau)}{(1+\tau)^{\eta}} \right| \tag{6.5}$$

is bounded for $\tilde{x}$ in $K \times S^{n-1}$, $s \in K_1$, and $r + |\tilde{\alpha}| + q \leq M$.

For the first inequality, we must show that

$$\left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q s^{-\nu} f(s,\tilde{x},\tau)\chi(\varepsilon s\tau^\delta)\chi(\varepsilon\tau)}{(1+\tau)^{d+\delta\nu-|\beta|+r\delta}} \right|$$

is bounded for $s \in [0,1]$, $\tilde{x} \in K \times S^{n-1}$.

Since $1 \leq \tau$, for any $\eta \in \mathbb{R}$ there are positive constants c_η and C_η such that $c_\eta\tau^\eta \leq (1+\tau)^\eta \leq C_\eta\tau^\eta$. Therefore, we may replace the $(1+\tau)$ in the denominator with τ.

Furthermore, we can omit the $\tilde{x}$ derivatives. Since f is smooth and homogeneous in τ of degree d, so is $\partial_{\tilde{x}}^{\tilde{\alpha}} f$. The only dependence on ω is in f, and

$\partial_{\tilde{x}}^{\tilde{\alpha}} f$ has the same properties as f. Therefore, it is sufficient to show the bound for $\tilde{\alpha} = 0$.

These observations together give that we need only show that

$$\left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_\tau^q s^{-\nu} f(s, \tilde{x}, \tau) \chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau)}{\tau^{d+\delta\nu-|\beta|+r\delta}} \right|$$

is bounded for $s \in [0, 1]$, $\tilde{x} \in K \times S^{n-1}$. By Leibnitz rule, and cancelling the $\tau^{-\beta}$ we have

$$\left| \sum_{p=0}^{r} \sum_{\ell=0}^{q} \binom{r}{p} \binom{q}{\ell} \tau^{q-d-\delta\nu-r\delta} \partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s, \tilde{x}, \tau) \Big) \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big) \right|.$$

Since this is a finite sum, it is sufficient to bound the summands.

By Lemma 6.4, there is a constant C_1 for which

$$\left| \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big) \right| \leq C_1 s^{-p} \tau^{-\ell}.$$

Furthermore, for $s > 0$

$$\begin{aligned} \partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s, \tilde{x}, \tau) \Big) &= \sum_{k=0}^{r-p} \partial_s^k (s^{-\nu}) \partial_\tau^{q-\ell} \partial_s^{r-p-k} f(s, \tilde{x}, \tau) \\ &= \sum_{k=0}^{r-p} C_{k,\nu} s^{-\nu-k} \partial_\tau^{q-\ell} \partial_s^{r-p-k} f(s, \tilde{x}, \tau) \end{aligned}$$

where $C_{k,\nu}$ is a combinatorial coefficient (and 0 if $\nu = 0$). We have $s \leq 1$ and $k \leq r - p$, giving $s^{-\nu-k} \leq s^{-\nu-r+p}$. Also, f is smooth and homogeneous of degree d in τ, so $\partial_s^{r-p-k} f(s, \tilde{x}, \tau)$ also has these properties. Therefore, $\partial_\tau^{q-\ell} \partial_s^{r-p-k} f(s, \tilde{x}, \tau)$ is smooth and homogeneous in τ of degree $d - q + \ell$. Thus, for $\tilde{x}$ in a compact set, $0 < s \leq 1$, and $\tau \geq 1$ since we are only interested in the bound on the support of the expression, there is a constant C_2 for which

$$|C_{k,\nu} s^{-\nu-k} \partial_\tau^{q-\ell} \partial_s^{r-p-k} f(s,\tilde{x},\tau)| \le C_2 s^{-\nu-r+p} \tau^{d-q+\ell}$$

Hence, we have the bound

$$\left| \partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s,\tilde{x},\tau) \Big) \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big) \right|$$

$$\le C s^{-\nu-r+p} \tau^{d-q+\ell} s^{-p} \tau^{-\ell} = C s^{-\nu-r} \tau^{d-q}$$

from which it follows that

$$\tau^{q-d-\delta\nu-r\delta} \left| \partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s,\tilde{x},\tau) \Big) \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big) \right|$$

$$\le C \tau^{q-d-\delta\nu-r\delta} s^{-\nu-r} \tau^{d-q} = C (s\tau^\delta)^{-\nu-r}$$

Finally, the expression is supported in $\varepsilon s \tau^\delta \ge 1$. Therefore, $s\tau^\delta \ge \varepsilon^{-1} \ge 1$. Hence, $(s\tau^\delta)^{-\nu-r} \le 1$, proving the inequality in (6.1).

Next, we prove the bound in (6.3). With the same argument as above, it suffices to prove that

$$\left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_\tau^q s^{-\nu} f(s,\tilde{x},\tau) \chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau)}{\tau^{d-|\beta|}} \right|$$

is bounded for s in a compact subset of $(0,1)$. By Leibnitz and rearranging terms, we may write this as

$$\left| \sum_{p=0}^{r} \sum_{\ell=0}^{q} \binom{r}{p} \binom{q}{\ell} \frac{\partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s,\tilde{x},\tau) \Big) \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big)}{\tau^{d-q}} \right|.$$

By (6.2), there is a constant C for which

$$\left| \partial_s^{r-p} \Big(s^{-\nu} \partial_\tau^{q-\ell} f(s,\tilde{x},\tau) \Big) \partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta) \chi(\varepsilon \tau) \Big) \right| \le C s^{-\nu-r} \tau^{d-q}$$

for $s \in (0,1)$ and $\tilde{x}$ in a given compact set. Since s is in a compact subset of $(0,1)$, there is a constant c for which $s^{-\nu-r} \le c$. Therefore,

$$\left| \sum_{p=0}^{r} \sum_{\ell=0}^{q} \binom{r}{p} \binom{q}{\ell} \frac{\partial_s^{r-p}\Big(s^{-\nu}\partial_\tau^{q-\ell} f(s,\tilde{x},\tau)\Big)\partial_s^p \partial_\tau^\ell \Big(\chi(\varepsilon s \tau^\delta)\chi(\varepsilon\tau)\Big)}{\tau^{d-q}} \right| \le \tilde{C}\frac{\tau^{d-q}}{\tau^{d-q}}.$$

Hence, the inequality in (6.3) holds.

Finally, we prove the inequality in (6.2). As in (6.5) it is sufficient to show that

$$\left| \tau^{q-|\beta|} \frac{\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_\tau^q f(s,\tilde{x},\tau)\chi(\varepsilon\tau)}{(1+\tau)^{d-|\beta|}} \right|$$

is bounded for $\tilde{x}$ in $K \times S^{n-1}$, and $s \in [0,1]$. As before, bounding this expression is equivalent to bounding the expression with $1+\tau$ in the denominator replaced with τ. Furthermore, f is smooth and homogeneous in τ of degree d. Thus, $\partial_s^r \partial_{\tilde{x}}^{\tilde{\alpha}} f$ is also smooth and homogeneous in τ of degree d. Furthermore, the only dependence on s and $\tilde{x}$ are in f. Thus, it is sufficient to prove the bound for $r = 0$ and $\tilde{\alpha}$ the 0 multiindex, i.e. that

$$\left| \frac{\partial_\tau^q f(s,\tilde{x},\tau)\chi(\varepsilon\tau)}{\tau^{d-q}} \right|$$

is bounded for $(s,\tilde{x}) \in [0,1] \times K \times S^{n-1}$. By Leibnitz, this is

$$\left| \sum_{r=0}^{q} \binom{q}{r} \frac{\Big(\partial_\tau^{q-r} f(s,\partial x,\tau)\Big)\partial_\tau^r \chi(\varepsilon\tau)}{\tau^{d-q}} \right|.$$

By Lemma 6.4, there is a constant C_q such that $\partial_\tau^r \chi(\varepsilon|\theta|) \le C_q \tau^{-r}$ for all $0 \le r \le q$. Furthermore, f is homogeneous in τ of degree d so $\partial_\tau^{q-r} f(s,x,\theta)$ is homogeneous in τ of degree $d-q+r$. Therefore, there is a constant C for which $|\partial_\tau^{q-r}\Big(f(s,\partial x,\tau)\Big)\partial_\tau^r\chi(\varepsilon\tau)| \leq C\tau^{d-q+r}\tau^{-r} = C\tau^{d-q}$. Thus, each term in the sum is bounded, and hence the sum itself is bounded, giving the final bound. □

These bounds are extremely flexible. At this point we recall that for $j \geq 1$, $A^{i,\pm}_{-j,S} = s^{-3j+2}\tilde{a}^{i,\pm}_{-j,S}\chi(\varepsilon_j s|\theta|^\delta)\chi(\varepsilon_j|\theta|)$ where $\tilde{a}^{i,\pm}_{-j,S}$ is a smooth function of s, x, θ, homogeneous in θ of degree $-j-i$. Also, $A_{-j,R} = a_{-j,R}\chi(\varepsilon_j|\theta|)$ where $a_{-j,R}$ is a smooth function of s, x, θ, homogeneous in θ of degree $-j-i$. Using the bounds we have just derived, we prove a corollary that is immediately applicable to our amplitudes.

Corollary 6.6. *Suppose $U \subset \mathbb{R}^n$ is open and $f(s,x,\theta) : [0,1]\times U\times\mathbb{R}^n \to \mathbb{R}$ is smooth and homogeneous in θ of degree d. Let $0 < \varepsilon \leq 1$, $\delta > 0$ and $\nu \geq 0$ be given. Then,*

$$s^{-\nu}f(s,x,\theta)(\varepsilon s|\theta|^\delta)\chi(\varepsilon|\theta|) \in \tilde{S}^{d+\delta\nu}_{(1,\delta,0)}([0,1]\times U\times\mathbb{R}^n)$$

$$s^{-\nu}f(s,x,\theta)(\varepsilon s|\theta|^\delta)\chi(\varepsilon|\theta|) \in S^{d}_{(1,0)}((0,1)\times U\times\mathbb{R}^n)$$

$$f(s,x,\theta)\chi(\varepsilon|\theta|) \in \tilde{S}^{d}_{(1,0,0)}([0,1]\times U\times\mathbb{R}^n)$$

Proof. The conditions in Lemma 6.5 are satisfied. With $\mu = 0$, the bounds in Lemma 6.5 are precisely the seminorms for the functions above in their respective symbol spaces. □

At this point, it is a matter of notation to state the symbol spaces that contain the regular and singular parts of the $A^{i,\pm}_{-j}$.

Corollary 6.7. *For $j \geq 1$, with any choice of $0 < \varepsilon_j \leq 1$, and any fixed $\delta > 0$,*

$$A^{i,\pm}_{-j,S} \in \tilde{S}^{(3\delta-1)j-i-2\delta}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n)$$

$$A^{i,\pm}_{-j,S} \in S^{-j-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

$$A^{i,\pm}_{-j,R} \in \tilde{S}^{-j-i}_{1,0,0}([0,1] \times U \times \mathbb{R}^n)$$

and

$$A^{i,\pm}_{0,S} \in \tilde{S}^{-i}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n)$$

$$A^{i,\pm}_{0,S} \in S^{-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

$$A^{i,\pm}_{0,R} \in \tilde{S}^{-i}_{1,0,0}([0,1] \times U \times \mathbb{R}^n).$$

Hence, for $j \geq 1$

$$A^{i,\pm}_{-j} \in \tilde{S}^{(3\delta-1)j-i-2\delta}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap S^{-j-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

and

$$A^{i,\pm}_{0} \in \tilde{S}^{-i}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap \in S^{-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

Proof. By construction, for $j \geq 1$, $A^{i,\pm}_{-j,S} = s^{-3j+2}\tilde{a}^{i,\pm}_{-j,S}\chi(\varepsilon_j s|\theta|^\delta)\chi(\varepsilon_j|\theta|)$ where $\tilde{a}^{i,\pm}_{-j,S}$ is a smooth function of s, x, θ, homogeneous in θ of degree $-j-i$. Therefore, by Corollary 6.6, with $\nu = 3j-2$ and $d = -j-i$, the singular parts are in the stated space.

Also for $j \geq 1$, $A^{i,\pm}_{-j,R} = a^{i,\pm}_{-j,0}\chi(\varepsilon_j|\theta|) + a^{i,\pm}_{-j,R}\chi(\varepsilon_{j+1}|\theta|)$ where $a^{i,\pm}_{-j,0}$ and $a^{i,\pm}_{-j,R}$ are smooth functions of s, x, θ, homogeneous in θ of degree $-j-i$. Therefore, by Corollary 6.6 with $d = -j-i$, the regular part is in the space claimed.

By definition, $A_{0,S}^{i,\pm} = \tilde{a}_{-j,S}^{i,\pm}\chi(\varepsilon_0 s|\theta|^\delta)\chi(\varepsilon_0|\theta|)$, with $\tilde{a}_{-j,S}^{i,\pm}$ smooth and homogeneous in θ of degree $-i$. By Corollary 6.6 with $\nu = 0$ and $d = -i$, the singular part of $A_0^{i,\pm}$ is in the appropriate space.

Finally, $A_{0,R}^{i,\pm}$ is a smooth function of s, x, θ on $[0,1] \times U \times \mathbb{R}^n$ and homogeneous in θ of degree $-i$. Hence, $A_{0,R}^{i,\pm} \in \tilde{S}_{1,0,0}^{-i}([0,1] \times U \times \mathbb{R}^n)$.

The spaces of the $A_{-j}^{i,\pm}$ come from the fact that

$$\tilde{S}_{1,0,0}^{-j-i}([0,1] \times U \times \mathbb{R}^n) \subset \tilde{S}_{1,\delta,0}^{(3\delta-1)j-i-2\delta}([0,1] \times U \times \mathbb{R}^n),$$

and $\tilde{S}_{1,0,0}^{-i}([0,1] \times U \times \mathbb{R}^n) \subset \tilde{S}_{1,\delta,0}^{-i}([0,1] \times U \times \mathbb{R}^n)$. $\square$

Notably, this tells us that on the interior (that is, $s > 0$) using the coordinate s, we have the classical symbol spaces. Moreover, the change of coordinates $s = t^{1/2}$ is smooth on the interior, so that in fact the singular amplitudes, when viewed as symbols on the interior in the t, x, θ coordinates behave as the classical symbols. The problem lies in the "restriction to the boundary." The singularities in s can be managed, but at the expense of decay in θ. Fortunately, as we will see in the next chapter, the trade-off is nice enough to allow us to add the symbols.

These symbols are not the only things we need. We also have initial conditions involving $s^{-2}\partial_s A_{-j}^{i,\pm}$.

Lemma 6.8. *With the same assumptions as in Lemma 6.6,*

$$s^{-2}\partial_s s^{-\nu} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|) \in \tilde{S}^{d+\delta(\nu+3)}_{(1,\delta,0)}([0,1]\times U\times\mathbb{R}^n)$$

$$s^{-2}\partial_s s^{-\nu} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|) \in S^{d}_{(1,0)}((0,1)\times U\times\mathbb{R}^n)$$

Proof. Note that $s^{-2}\partial_s g = \partial_s(s^{-2}g) - 2s^{-3}g$. We look at the two terms separately. First,

$$\partial_s(s^{-2}s^{-\nu} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|)) = \partial_s(s^{-(\nu+2)} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|)).$$

The function

$$s^{-(\nu+2)} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|)$$
$$\in \tilde{S}^{d+\delta(\nu+2)}_{(1,\delta,0)}([0,1]\times U\times\mathbb{R}^n) \cap S^{d}_{(1,0)}((0,1)\times U\times\mathbb{R}^n).$$

The operator $\partial_s : S^m_{1,\delta,0} \to S^{m+\delta}_{1,\delta,0}$ and $\partial_s : S^m_{1,0} \to S^m_{1,0}$, for any m, and this is independent of the domain. Therefore,

$$\partial_s(s^{-2}s^{-\nu} f(s,x,\theta)(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|))$$
$$\in \tilde{S}^{d+\delta(\nu+3)}_{(1,\delta,0)}([0,1]\times U\times\mathbb{R}^n) \cap S^{d}_{(1,0)}((0,1)\times U\times\mathbb{R}^n).$$

For the second term, we simply have

$$s^{-(\nu+3)}(-2f(s,x,\theta))(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|).$$

The function $-2f$ is smooth, and homogeneous in θ of degree d. By Lemma 6.6,

$$s^{-(\nu+3)}(-2f(s,x,\theta))(\varepsilon s|\theta|^{\delta})\chi(\varepsilon|\theta|)$$
$$\in \tilde{S}^{d+\delta(\nu+3)}_{(1,\delta,0)}([0,1]\times U\times\mathbb{R}^n) \cap S^{d}_{(1,0)}((0,1)\times U\times\mathbb{R}^n).$$

Hence, the sum is in the desired space. □

Corollary 6.9. *For $j \geq 1$,*

$$s^{-2}\partial_s A^{i,\pm}_{-j} \in \tilde{S}^{(3\delta-1)j-i+\delta}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap S^{-j-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

and

$$A^{i,\pm}_0 \in \tilde{S}^{-i}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap \in S^{-i}_{1,0}((0,1) \times U \times \mathbb{R}^n).$$

Proof. The regular parts of these symbols are of the form

$$\tilde{a}_{-j,0}(x,\theta) + s^3 a^{i,\pm}_{-j,R}(s,x,\theta)\chi(\tilde{\varepsilon}|\theta|)$$

where $a^{i,\pm}_{-j,R}(s,x,\theta)$ are smooth and homogeneous in θ of degree $-j-i$. Therefore, $s^{-2}\partial_s A^{i,\pm}_{-j,R}(s,x,\theta) = (3a^{i,\pm}_{-j,R} + s\partial_s a^{i,\pm}_{-j,R})\chi(\varepsilon|\theta|)$, which is smooth, and $3a^{i,\pm}_{-j,R} + s\partial_s a^{i,\pm}_{-j,R}$ is homogeneous in θ of the same degree. Therefore, the operator $s^{-2}\partial_s$ does not change the smoothness or the homogeneity of the regular parts. Therefore, for $j \geq 0$,

$$s^{-2}\partial_s A^{i,\pm}_{-j,R} \in \tilde{S}^{-j-i}_{1,0,0}([0,1] \times U \times \mathbb{R}^n).$$

The issue is what happens with the singular part. By Corollary 6.7 and Lemma 6.8,

$$s^{-2}\partial_s A^{i,\pm}_{-j} \in \tilde{S}^{(3\delta-1)j-i+\delta}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap S^{-j-i}_{1,0}((0,1) \times U \times \mathbb{R}^n).$$

Since

$$\tilde{S}^{-j-i}_{1,0,0}([0,1] \times U \times \mathbb{R}^n)$$
$$\subset \tilde{S}^{(3\delta-1)j-i+\delta}_{1,\delta,0}([0,1] \times U \times \mathbb{R}^n) \cap S^{-j-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

the result follows. □

6.3 Asymptotics

We seek a sequence of symbols $\{A^{i,\pm}_{-j}\}_{j=0}^{\infty}$ that are in symbols spaces whose orders decrease to $-\infty$, and whose sum converges in the symbols space of $A^{i,\pm}_0$. The functions $A^{i,\pm}_{-j}$ are of order $(3\delta-1)j-i$ if we want to include $s=0$. Since $i=0$ or 1, in order for this to decrease, we fix $0<\delta<1/3$. For i,δ fixed, the sequence $m_{j,i,\delta}$ decreases to $-\infty$ as $j\to\infty$. Now that the symbols are in spaces that decrease in order, the issue is convergence of the sum. The proof is an adaptation of a proof due to Hörmander (see [5]), which is similar in spirit to that of Borel's lemma.

The task is to choose the ε_j in the definition of the $A^{i,\pm}_{-j}$ so that the partial sums are Cauchy sequences in each seminorm. In order to do this, we exploit the bounds we obtained, and impose finitely many conditions on each ε_j.

Fix $\varepsilon_0=1$. Let $\tilde{K}_\ell$ be compact subsets of U such that $\tilde{K}_\ell\subset\overset{\circ}{\tilde{K}}_{\ell+1}$ and $\bigcup\tilde{K}_\ell=U$. Also, let $I_k=[\frac{1}{k+1},1-\frac{1}{k+1}]$ and put $K_\ell=I_k\times\tilde{K}_\ell$. Let p^m_ℓ denote the seminorm p^m_{ℓ,K_ℓ} on $S^m_{1,0}((0,1)\times U\times\mathbb{R}^n)$, and let let ϱ^m_ℓ be the $\varrho^m_{\ell,\tilde{K}_\ell}$ seminorm on $\tilde{S}^m_{1,\delta,0}([0,1]\times U\times\mathbb{R}^n)$.

By definition, $A^{i,\pm}_{-j,R}=a^{i,\pm}_{-j,0}\chi(\varepsilon_j|\theta|)+a^{i,\pm}_{-j,R}\chi(\varepsilon_{j-1}|\theta|)$. Write $\tilde{A}^{i,\pm}_{-j,R}=a^{i,\pm}_{-j,R}\chi(\varepsilon_{j-1}|\theta|)$ and $\tilde{A}^{i,\pm}_{-j,0}=a^{i,\pm}_{-j,0}\chi(\varepsilon_j|\theta|)$. For $j\geq1$ we would like to choose

ε_j so that

$$\text{for } 0 \leq \ell < j$$

$$\varrho_\ell^{(3\delta-1)\ell-i-2\delta}(A^{i,\pm}_{-j,S}) \leq 2^{-j}, \qquad p_\ell^{-\ell-i}(A^{i,\pm}_{-j,S}) \leq 2^{-j}$$

$$\varrho_\ell^{(3\delta-1)\ell-i+\delta}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq 2^{-j}, \qquad p_\ell^{-\ell-i}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq 2^{-j}$$

$$\varrho_\ell^{-\ell-i}(\tilde{A}_{-j,0}) \leq 2^{-j}, \qquad p_\ell^{-\ell-i}(\tilde{A}^{i,\pm}_{-j,0}) \leq 2^{-j} \tag{6.6}$$

$$\text{and for } 0 \leq \ell < j-1$$

$$\varrho_\ell^{-\ell-i}(\tilde{A}_{-(j-1),R}) \leq 2^{-j}, \qquad p_\ell^{-\ell-i}(\tilde{A}^{i,\pm}_{-(j-1),R}) \leq 2^{-j}$$

$$\varrho_\ell^{-\ell-i}(s^{-2}\partial_s \tilde{A}^{i,\pm}_{-(j-1),R}) \leq 2^{-j}, \qquad p_\ell^{-\ell-i}(s^{-2}\partial_s \tilde{A}^{i,\pm}_{-(j-1),R}) \leq 2^{-j}$$

for $i = 0, 1$. Note the shift in the index for $A^{i,\pm}_{-(j-1),R}$. This is to give a condition on ε_j.

Write $\mu_{j,\ell} = (1-3\delta)(j-\ell)$, so that

$$(3\delta-1)\ell = (3\delta-1)j + \mu_{j,\ell}.$$

Noting the structure of these symbols and the definition of the seminorms, by Lemma 6.5, we have bounds

$$\text{for } 0 \leq \ell < j$$

$$\varrho_\ell^{(3\delta-1)\ell-i-2\delta}(A^{i,\pm}_{-j,S}) \leq \varrho_\ell^{(3\delta-1)j-i-2\delta+\mu_{j,\ell}}(A^{i,\pm}_{-j,S}) \leq C_1^{i,\pm}\varepsilon_j^{\mu_{j,\ell}}$$

$$p_\ell^{-\ell-i}(A^{i,\pm}_{-j,S}) \leq p_\ell^{-j-i+(j-\ell)}(A^{i,\pm}_{-j,S}) \leq C_2^{i,\pm}\varepsilon_j^{j-\ell}$$

$$\varrho_\ell^{(3\delta-1)\ell-i+\delta}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq \varrho_\ell^{(3\delta-1)j-i+\delta+\mu_{j,\ell}}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq C_3^{i,\pm}\varepsilon_j^{\mu_{j,\ell}}$$

$$p_\ell^{-\ell-i}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq p_\ell^{-j-i+(j-\ell)}(s^{-2}\partial_s A^{i,\pm}_{-j,S}) \leq C_4^{i,\pm}\varepsilon_j^{j-\ell}$$

$$\varrho_\ell^{-\ell-i}(\tilde{A}_{-j,0}) \leq \varrho_\ell^{-j-i+(j-\ell)}(\tilde{A}_{-j,0}) \leq C_5^{i,\pm}\varepsilon_j^{j-\ell}$$

$$p_\ell^{-\ell-i}(\tilde{A}_{-j,0}^{i,\pm}) \leq p_\ell^{-j-i+(j-\ell)}(\tilde{A}_{-j,0}^{i,\pm}) \leq C_6^{i,\pm}\varepsilon_j^{j-\ell}$$

and for $0 \leq \ell < j-1$

$$\varrho_\ell^{-\ell-i}(\tilde{A}_{-(j-1),R}) \leq \varrho_\ell^{-(j-1)-i+(j-\ell)}(\tilde{A}_{-(j-1),R}) \leq C_7^{i,\pm}\varepsilon_j^{j-1-\ell}$$

$$p_\ell^{-\ell-i}(\tilde{A}_{-(j-1),R}^{i,\pm}) \leq p_\ell^{-(j-1)-i+(j-\ell)}(\tilde{A}_{-(j-1),R}^{i,\pm}) \leq C_8^{i,\pm}\varepsilon_j^{j-1-\ell}$$

$$\varrho_\ell^{-\ell-i}(s^{-2}\partial_s\tilde{A}_{-(j-1),R}^{i,\pm}) \leq \varrho_\ell^{-(j-1)-i+(j-\ell)}(s^{-2}\partial_s\tilde{A}_{-(j-1),R}^{i,\pm}) \leq C_9^{i,\pm}\varepsilon_j^{j-1-\ell}$$

$$p_\ell^{-\ell-i}(s^{-2}\partial_s\tilde{A}_{-(j-1),R}^{i,\pm}) \leq p_\ell^{-(j-1)-i+(j-\ell)}(s^{-2}\partial_s\tilde{A}_{-(j-1),R}^{i,\pm}) \leq C_{10}^{i,\pm}\varepsilon_j^{j-1-\ell}$$

where each $C_k^{i,\pm}$ is independent of ε_j. Thus, we can choose the ε_j to satisfy the conditions sought in (6.6).

Fix a choice of ε_j, $0 < \delta < 1/3$, and a cutoff function χ with the desired properties so that the $A_{-j}^{i,\pm}$ are fully defined. Then, we can add the symbols, as described in the following lemma.

Lemma 6.10. *For any $J \geq 0$,*

$$\sum_{j=J}^{\infty} A_{-j,S}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)J-i-2\delta}([0,1]\times U\times\mathbb{R}^n) \cap S_{1,0}^{-J-i-2\delta}((0,1)\times U\times\mathbb{R}^n)$$

$$\sum_{j=J}^{\infty} s^{-2}\partial_s A_{-j,S}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)J-i+\delta}([0,1]\times U\times\mathbb{R}^n) \cap S_{1,0}^{-J-i-2\delta}((0,1)\times U\times\mathbb{R}^n)$$

$$\sum_{j=J}^{\infty} A_{-j,R}^{i,\pm} \in \tilde{S}_{1,0,0}^{-J-i}([0,1]\times U\times\mathbb{R}^n)$$

$$\sum_{j=J}^{\infty} s^{-2}\partial_s A_{-j,R}^{i,\pm} \in \tilde{S}_{1,0,0}^{-J-i}([0,1]\times U\times\mathbb{R}^n).$$

Hence,

$$\sum_{j=J}^{\infty} A_{-j}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)J-i-2\delta}([0,1]\times U\times\mathbb{R}^n)\cap S_{1,0}^{-J-i}((0,1)\times U\times\mathbb{R}^n)$$

and

$$\sum_{j=J}^{\infty} s^{-2}\partial_s A_{-j}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)J-i+\delta}([0,1]\times U\times\mathbb{R}^n)\cap S_{1,0}^{-J-i}((0,1)\times U\times\mathbb{R}^n).$$

Proof. By Corollary 6.7,

$$A_{-j,S}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)j-i-2\delta}([0,1]\times U\times\mathbb{R}^n)\cap S_{1,0}^{-j-i}((0,1)\times U\times\mathbb{R}^n).$$

Also, $A_{-j,R}^{i,\pm} \in \tilde{S}_{1,0}^{-j-i}([0,1]\times U\times\mathbb{R}^n)$. Furthermore, by Corollary 6.9,

$$s^{-2}\partial_s A_{-j,R}^{i,\pm} \in \tilde{S}_{1,0,0}^{-j-i}([0,1]\times U\times\mathbb{R}^n)$$

and

$$s^{-2}\partial_s A_{-j,S}^{i,\pm} \in \tilde{S}_{1,\delta,0}^{(3\delta-1)j-i+\delta}([0,1]\times U\times\mathbb{R}^n)\cap S_{1,0}^{-j-i}((0,1)\times U\times\mathbb{R}^n).$$

The order of the spaces $(3\delta-1)j-i-2\delta \to -\infty$ and $-j-i\to-\infty$ as $j\to\infty$. Moreover, with our choice of ε_j, we have the bounds in (6.6).

By Lemma 6.1, the sums are in the stated spaces. □

Definition. The amplitudes

$$a^{i,\pm}(s,x,\theta) \in \tilde{S}_{1,\delta,0}^{-i-2\delta}([0,1]\times U\times\mathbb{R}^n)\cap S_{1,0}^{-i}((0,1)\times U\times\mathbb{R}^n)$$

are given by

$$a^{i,\pm}(s,x,\theta) = \sum_{j=0}^{\infty} A_{-j}^{i,\pm}(s,x,\theta). \tag{6.7}$$

Lemma 6.11. *Each* $a^{i,\pm}(t,x,\theta)$ *and* $t^{-1/2}\partial_t a^{i,\pm}$ *are continuous in* t *down to* $t=0$.

Proof. Since

$$a^{i,\pm}(s,x,\theta) \in \tilde{S}^{-i-2\delta}_{1,\delta,0}([0,1]\times U\times\mathbb{R}^n)$$

and

$$t^{-1/2}\partial_t a^{i,\pm} = s^{-2}\partial_s a^{i,\pm} \in \tilde{S}^{-i+\delta}_{1,\delta,0}([0,1]\times U\times\mathbb{R}^n),$$

they are smooth in s on $[0,1]$. The change of variables $s=t^{1/2}$ is continuous at $t=0$. Therefore, the functions are continuous down to $t=0$. □

Moreover, the functions $a^{i,\pm}$ are related to the $a^{i,\pm}_{-j}$.

Lemma 6.12. *On* $(0,1)\times U\times\mathbb{R}^n$,

$$a^{i,\pm} \sim \sum_{j=0}^{\infty} a^{i,\pm}_{-j}.$$

Proof. First, note that the $a^{i,\pm}_{-j}$ are of the form $s^{-\nu}\tilde{a}^{i,\pm}_{-j}$ where $\tilde{a}^{i,\pm}_{-j}$ is smooth and homogeneous in θ of degree $-j-i$. Therefore, on any compact subset of $(0,1)\times U$, the function $a^{i,\pm}_{-j}$ is smooth and homogeneous in θ of degree $-j-i$. Hence, $a^{i,\pm}_{-j} \in S^{-j-i}_{1,0}((0,1)\times U\times\mathbb{R}^n)$. For $i=0,1$, $-j-i \searrow -\infty$ as $j\to\infty$. Therefore, it suffices to show that

$$a^{i,\pm} - \sum_{j=0}^{k-1} a^{i,\pm}_{-j} \in S^{-k-i}_{1,0}((0,1)\times U\times\mathbb{R}^n).$$

Observe that for $j \geq 1$,

$$\begin{aligned} A^{i,\pm}_{-j} - a^{i,\pm}_{-j} =& s^{-3j+2}\tilde{a}^{i,\pm}_{-j,S}(\chi(\varepsilon_j s|\theta|^\delta) - 1) + a^{i,\pm}_{-j,0}(\chi(\varepsilon_j|\theta|) - 1) \\ &+ a^{i,\pm}_{-j,R}(\chi(\varepsilon_{j+1}|\theta|) - 1). \end{aligned}$$

If s is in a fixed compact subset of $(0,1)$, then $\varepsilon_j s$ is in a compact set. Specifically, there is an $\eta_j > 0$ such that $\varepsilon_j s \geq \eta_j$. Thus, $\chi(\varepsilon_j s|\theta|^\delta) = 1$ for $|\theta| \geq \eta_j^{-\delta}$, so that $\chi(\varepsilon_j s|\theta|^\delta) - 1$ is compactly supported. Clearly, $\chi(\varepsilon_j|\theta|) - 1$ and $\chi(\varepsilon_{j+1}|\theta|) - 1$ are compactly supported in θ. Therefore, for (s,x) in a compact subset of $(0,1) \times U$, the function $A^{i,\pm}_{-j} - a^{i,\pm}_{-j}$ is smooth and compactly supported in (s,x,θ). Hence, $A^{i,\pm}_{-j} - a^{i,\pm}_{-j} \in S^{-\infty}((0,1) \times U \times \mathbb{R}^n)$ for every $j \geq 1$.

For $j = 0$, the only modification is that $A_{0,R} = a^{i,\pm}_{0,0} + a^{i,\pm}_{-j,R}\chi(\varepsilon_1|\theta|)$ so the difference is $A^{i,\pm}_0 - a^{i,\pm}_0 = a^{i,\pm}_{0,S}(\chi(\varepsilon_0 s|\theta|^\delta) - 1) + a^{i,\pm}_{-j,R}(\chi(\varepsilon_1|\theta|) - 1)$. As above, this is also in $S^{-\infty}((0,1) \times U \times \mathbb{R}^n)$.

Now, we have

$$\sum_{j=0}^{\infty} A^{i,\pm}_{-j} - \sum_{j=0}^{k-1} a^{i,\pm}_{-j} = \sum_{j=k}^{\infty} A^{i,\pm}_{-j} \mod S^{-\infty}((0,1) \times U \times \mathbb{R}^n).$$

By construction, $\sum_{j=k}^{\infty} A^{i,\pm}_{-j} \in S^{-k-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$. Hence,

$$a^{i,\pm} - \sum_{j=0}^{k-1} a^{i,\pm}_{-j} \in S^{-k-i}_{1,0}((0,1) \times U \times \mathbb{R}^n)$$

and $a^{i,\pm} \sim \sum_j a^{i,\pm}_{-j}$. $\square$

Now that we have a complete definition of our amplitudes, and how they relate to the solutions of the transport equations, we state some properties relating to the initial conditions. First, note that the singular parts $A^{i,\pm}_{-j,S}$ have a factor of $\chi(\varepsilon_j s|\theta|^\delta)$, which vanishes to infinite order at $s = 0$. Therefore, $A^{i,\pm}_{-j}(0,x,\theta) = A^{i,\pm}_{-j,R}(0,x,\theta)$. But, for $j \geq 1$ the regular part is simply $a^{i,\pm}_{-j,R}\chi(\varepsilon_j|\theta|)$, so that $A^{i,\pm}_{-j}(0,x,\theta) = a^{i,\pm}_{-j,R}(0,x,\theta)\chi(\varepsilon_j|\theta|)$. Finally, $a^{i,\pm}_{-j,R}(0,x,\theta) = C_0(a^{i,\pm}_{-j})\chi(\varepsilon_j|\theta|)$, giving

$$A^{i,\pm}_{-j}(0,x,\theta) = C_0(a^{i,\pm}_{-j})\chi(\varepsilon_j|\theta|).$$

Furthermore, $A^{i,\pm}_{0,R} = a^{i,\pm}_{0,R}$, so that $A^{i,\pm}_{0,R}(0,x,\theta) = C_0(a^{i,\pm}_0)$. By construction, for $j \geq 1$, $C_0(a^{i,+}_{-j}) + C_0(a^{i,-}_{-j}) = 0$, and for $j = 0$ we have the conditions $C_0(a^{0,+}_0) + C_0(a^{0,-}_0) = 1$ and $C_0(a^{1,+}_0) + C_0(a^{1,-}_0) = 0$. Hence, these initial conditions translate to the conditions

$$\begin{aligned}
&A^{0,+}_0(0,x,\theta) + A^{0,-}_0(0,x,\theta) = 1\\
&A^{1,+}_0(0,x,\theta) + A^{1,-}_0(0,x,\theta) = 0\\
&A^{i,+}_{-j}(0,x,\theta) + A^{i,-}_{-j}(0,x,\theta) = \Big(C_0(a^{i,+}_{-j}) + C_0(a^{i,-}_{-j})\Big)\chi(\varepsilon_j|\theta|) = 0.
\end{aligned}$$

Hence, by the definition of the symbols $a^{i,\pm}$, we have

$$\begin{aligned}
a^{0,+}(0,x,\theta) + a^{0,-}(0,x,\theta) = 1\\
a^{1,+}(0,x,\theta) + a^{1,-}(0,x,\theta) = 0.
\end{aligned} \tag{6.8}$$

We now seek the analogue of the remaining initial conditions. Specifically, we will show that

$$s^{-2}\partial_s[e^{i\phi^+}a^{0,+} + e^{i\phi^-}a^{0,-}]|_{s=0} = 0$$

$$s^{-2}\partial_s[e^{i\phi^+}a^{1,+} + e^{i\phi^-}a^{1,-}]|_{s=0} = e^{i(x-y)\cdot\theta}.$$

Note that

$$s^{-2}\partial_s\phi^{\pm}|_{s=0} = t^{-1/2}\partial_t\phi^{\pm}|_{t=0} = \pm i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}$$

$$s^{-2}\partial_s a^{i,\pm}|_{s=0} = \sum_{j=0}^{\infty} C_0(t^{-1/2}\partial_t a^{i,\pm}_{-j})\chi(\varepsilon_{j+1}|\theta|)$$

$$a^{i,\pm}|_{s=0} = C_0(a^{i,\pm}_0) + \sum_{j=1}^{\infty} C_0(a^{i,\pm}_{-j})\chi(\varepsilon_j|\theta|)$$

$$e^{i\phi^{\pm}}|_{s=0} = e^{i(x-y)\cdot\theta}.$$

Now,

$$\begin{aligned}&s^{-2}\partial_s[e^{i\phi^+}a^{i,+} + e^{i\phi^-}a^{i,-}]|_{s=0}\\ &= e^{i(x-y)\cdot\theta}\Big[is^{-2}\partial_s\phi^+|_{s=0}(a^{i,+}|_{s=0} - a^{i,-}|_{s=0}) + s^{-2}\partial_s(a^{i,+} + a^{i,-})|_{s=0}\Big].\end{aligned} \tag{6.9}$$

Neglecting the factor of $e^{i(x-y)\cdot\theta}$, this translates to

$$\begin{aligned}&i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a^{i,+}_0) - C_0(a^{i,-}_0)\big) + \sum_{j=1}^{\infty}\chi(\varepsilon_j|\theta|)\Big[\\ &i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a^{i,+}_{-j}) - C_0(a^{i,-}_{-j})\big) + \big(C_0(t^{-1/2}\partial_t a^{i,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{i,-}_{-(j-1)})\big)\Big]\end{aligned} \tag{6.10}$$

after noting the shift in the index of the ε in the construction. By construction,

$$i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a^{i,+}_{-j}) - C_0(a^{i,-}_{-j})\big) + C_0(t^{-1/2}\partial_t a^{i,+}_{-(j-1)}) + C_0(t^{-1/2}\partial_t a^{i,-}_{-(j-1)}) = 0$$

for $j \geq 1$ and $i = 0, 1$. Therefore,

$$\begin{aligned}&s^{-2}\partial_s\phi^+|_{s=0}(a^{i,+}|_{s=0} - a^{i,-}|_{s=0}) + s^{-2}\partial_s(a^{i,+} + a^{i,-})|_{s=0}\\ &= i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a^{i,+}_0) - C_0(a^{i,-}_0)\big).\end{aligned}$$

Again, by construction, we have

$$i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a_0^{0,+}) - C_0(a_0^{0,-})\big) = 0 \text{ and } i\frac{\sqrt{\hat{g}^{00}|_{t=0}}}{|\theta|_g^0}\big(C_0(a_0^{1,+}) - C_0(a_0^{1,-})\big) = 1.$$

Hence, we have the conditions

$$s^{-2}\partial_s[e^{i\phi^+}a^{0,+} + e^{i\phi^-}a^{0,-}]|_{s=0} = 0$$

$$s^{-2}\partial_s[e^{i\phi^+}a^{1,+} + e^{i\phi^-}a^{1,-}]|_{s=0} = e^{i(x-y)\cdot\theta}.$$

Writing in the t coordinate, we have

$$\begin{aligned} &\lim_{t\searrow 0} t^{-1/2}\partial_t\big(e^{i\phi^+}a^{0,+} + e^{i\phi^-}a^{0,-}\big) = 0 \\ &\lim_{t\searrow 0} t^{-1/2}\partial_t\big(e^{i\phi^+}a^{1,+} + e^{i\phi^-}a^{1,-}\big) = e^{i(x-y)\cdot\theta}. \end{aligned} \tag{6.11}$$

CHAPTER 7

The Solution Operator

We now assemble the phase functions and amplitudes into a Fourier Integral Operator, following Hörmander and Duistermaat (see [2, 5]).

Let $U \subset \mathcal{M}$ be a neighborhood of $p_0 \in \partial\mathcal{M}$ in which we have coordinates (t, x), phase functions $\phi^{\pm}(t, x, y, \theta)$ and $a^{i,\pm}_{-j}$ solutions of the transport equations as defined in the previous chapters. Then in particular, we have the asymptotic expansions of the symbols giving $a^{i,\pm}$ for $(t, x) \in U$.

Define the operator

$$E : C_c^{\infty}(\partial\mathcal{M}) \times C_c^{\infty}(\partial\mathcal{M}) \to C^{\infty}(U \cap \mathring{\mathcal{M}})$$

by

$$E \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} (t, x) = \frac{1}{(2\pi)^n} \iint \begin{bmatrix} e^{i\phi^+} & e^{i\phi^-} \end{bmatrix} \begin{bmatrix} a^{0,+} & a^{1,+} \\ a^{0,-} & a^{1,-} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} dy\, d\theta \qquad (7.1)$$

where this is to be interpreted as an oscillatory integral, see [5].

Lemma 7.1. *The operator E extends by continuity to an operator*

$$E : C_c^{-\infty}(\partial\mathcal{M}) \times C_c^{-\infty}(\partial\mathcal{M}) \to C^{-\infty}(U \cap \overset{\circ}{\mathcal{M}})$$

Proof. Let ϕ be one of our phase functions and a one of our amplitudes. Let

$$I(\phi, a)(t, x, y) = \frac{1}{(2\pi)^n} \int e^{i\phi(t,x,y,\theta)} a(t, x, y, \theta)\, d\theta$$

as an oscillatory integral; this is a distribution (despite the notation) on $\overset{\circ}{\mathcal{M}} \times \partial\mathcal{M}$, more precisely, on $(\overset{\circ}{\mathcal{M}} \cap U) \times (\partial\mathcal{M} \cap U)$. Let

$$\pi : \overset{\circ}{\mathcal{M}} \times \partial\mathcal{M} \to \partial\mathcal{M}$$

be the canonical projection. We need to see that the product $I(\phi, a)\pi^* u$ is defined whenever $u \in C_c^{-\infty}(\partial\mathcal{M} \cap U)$. For this, we use Hörmander's theorem on products of distributions (see [5] Chapter 2).

Suppose Γ_1 and Γ_2 are closed cones in $T^*(X)$ such that there is no $(x, \xi) \in \Gamma_1$ for which $(x, -\xi) \in \Gamma_2$. Then, for any distributions u and v with $WF(u) \subset \Gamma_1$ and $WF(v) \subset \Gamma_2$ there is a unique product uv that is sequentially continuous.

If ϕ is one of our phase functions and a is one of our amplitudes, then

$$WF(I(\phi, a)) \subset \{(t, x, y, \phi_t, \phi_x, \phi_y) : (t, x, y, \theta) \in C_\phi\}$$

Since the differential of the projection $\pi : \overset{\circ}{\mathcal{M}} \times \partial\mathcal{M} \to \partial\mathcal{M}$ is surjective, for any distribution u on $\partial\mathcal{M}$, the pull back

$$\pi^* : T^*\partial\mathcal{M} \to T^*(\partial\mathcal{M} \times \overset{\circ}{\mathcal{M}})$$

is injective, and furthermore, $WF(\pi^* u) = \pi^* WF(u)$. In coordinates, the pull back by π of the wavefront set of u consists of vectors of the form $\{(t, x, y, 0, 0, \eta)\}$, owing to the fact that it is independent of t, x.

Recalling the construction of the phase function using the eikonal equation (4.2), $\tau = \phi_t$ vanishes at the boundary, but $\dot{\tau}$ does not. Therefore, on the interior near the boundary, $\tau = \phi_t \neq 0$, and $WF(I(\phi, a))$ contains no vector of the form $(t, x, y, 0, 0, -\eta)$ with small $t > 0$.

Hence, there is a well defined product of the distributions $I(\phi, a)$ and $\pi^* u$ defined on $\overset{\circ}{\mathcal{M}} \times \partial\mathcal{M}$ if ϕ is one of our phase functions, a is one of our symbols and $u \in C_c^{-\infty}(\partial\mathcal{M})$. $\square$

Lemma 7.2. *The distribution* $E\binom{u_0}{u_1}$ *satisfies the initial conditions in the sense that*

$$\lim_{t_0 \searrow 0} E\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\Bigg|_{t=t_0} = u_0 \tag{7.2}$$

$$\lim_{t_0 \searrow 0} t^{-1/2} \partial_t E\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\Bigg|_{t=t_0} = u_1 \tag{7.3}$$

modulo smooth errors.

Proof. We only need to argue in the case where u_0 and u_1 are smooth. Simplifying the notation, let ϕ be one of our phase functions, a one of the amplitudes, and u one of u_0 or u_1. Recalling that ϕ has the form

$$\phi(t, x, y, \theta) = (x - y) \cdot \theta + t^{3/2} \psi(t, x, \theta),$$

we need find the limits of

$$\int e^{i\phi(t,x,y,\theta)} a(t,x,y,\theta)\varphi(x)u(y)\,dy\,d\theta$$

and

$$\frac{1}{t^{1/2}}\partial_t \int e^{i\phi(t,x,y,\theta)} a(t,x,y,\theta)\varphi(x)u(y)\,dy\,d\theta$$

as $t \searrow 0$. Both integrals are to be taken in the oscillatory sense and φ is a compactly supported smooth function. In either integral, using that

$$L = \frac{1}{1+|\theta|^2}(1 - i\theta\cdot\partial_y),$$

has the property that $Le^{i\phi} = e^{i\phi}$, that u is smooth and compactly supported, and integration by parts, allows us to assume that a has large negative order, so that the oscillatory integral is an actual integral. In the first case we use that a is continuous down to $t = 0$ (see Lemma 6.11) to conclude that the limit exists. In the second case, we note that

$$\frac{1}{t^{1/2}}\partial_t e^{i\phi} a = e^{i\phi}\big[\big(\frac{3i}{2}\psi + t\psi_t\big)a + \frac{1}{t^{1/2}}\partial_t a\big]$$

In this expression, ϕ is continuous in t at $t = 0$, $t\psi_t$ is also continuous, in fact vanishes at $t = 0$ because ψ is a smooth function of $t^{1/2}$, and the last term is also continuous (see Lemma 6.11).

Assembling E we see that the limits (7.2, 7.3) exist. Using now (6.8) we get

$$\lim_{t_0\searrow 0} E\Big|_{t=t_0}\begin{pmatrix}u_0\\ u_1\end{pmatrix} = u_0 - R_0\begin{pmatrix}u_0\\ u_1\end{pmatrix}$$

where R_0 is a smoothing operator and using (6.11), we get

$$\lim_{t_0 \searrow 0} \frac{1}{t^{1/2}} \partial_t E \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \Bigg|_{t=t_0} = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\theta} u_1(y)\, dy\, d\theta.$$

Hence,

$$\lim_{t_0 \searrow 0} \frac{1}{t^{1/2}} \partial_t E \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \Bigg|_{t=t_0} = u_1 - R_1 \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

again with some smoothing operator R_1. □

Finally, we show that this distribution solves the equation. Here, we exploit the fact that $a^{i,\pm} \sim \sum_j a^{i,\pm}_{-j}$.

Lemma 7.3. *The distribution* $E\binom{u_0}{u_1}$ *solves*

$$\Box\, E \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = 0 \bmod C^\infty \textit{ for } t > 0$$

Proof. The symbols in this proof are all of type $1, 0$ and defined in $(U \cap \mathring{\mathcal{M}}) \times \mathbb{R}^n$. Therefore, we will omit the type and domain and only indicate the order of the symbol throughout the proof.

Observe that with $b^{i,\pm}_J = a^{i,\pm} - \sum_{j=0}^J a^{i,\pm}_{-j}$, we have

$$I(\phi^\pm, a^{i,\pm}) = I(\phi^\pm, b^{i,\pm}_J) + I(\phi^\pm, \sum_{j=0}^J a^{i,\pm}_{-j}).$$

By construction, using the variable $s = t^{1/2}$, $b^{i,\pm}_J \in S^{-J-1-i}$. Moreover, $\Box$ is a second order operator. Therefore, $\Box\, I(\phi^\pm, b^{i,\pm}_J) = I(\phi^\pm, p b^{i,\pm}_J)$ where p is a symbol of order 2 (the full left symbol of $\Box$). Thus, $p b^{i,\pm}_J \in S^{-J-i+1}$.

Furthermore, the $a^{i,\pm}_{-j}$ satisfy the transport equations, so that for each $J \geq 0$, $\Box(I(\phi^\pm, \sum_{j=0}^J a^{i,\pm}_{-j}))$ is given by $I(\phi^\pm, \tilde{b}^{i,\pm}_J)$ for some $\tilde{b}^{i,\pm}_J \in S^{-J-i}$. Hence,

$$\Box\, I(\phi^{\pm}, a^{i,\pm}) = I(\phi^{\pm}, p b_J^{i,\pm} + \tilde{b}_J^{i,\pm})$$

where the sum $b_J^{i,\pm} + \tilde{b}_J^{i,\pm} \in S^{-J-i+1}$. This is true for any J. Hence,

$$\Box\, I(\phi^{\pm}, a^{i,\pm}) = I(\phi^{\pm}, \tilde{a}^{i,\pm})$$

for some $\tilde{a}^{i,\pm}$ in $S^{-\infty}$. Therefore, each $I(\phi^{\pm}, a^{i,\pm})u$ is smooth, and $\Box\, E\binom{u_0}{u_1} \in C^{\infty}$. □

All together, we have proven the following theorem.

Theorem 7.4. *Let $\mathcal{T}$ be a smooth vector field whose restriction to $\partial\mathcal{M}$ spans $\mathscr{K}$ and let (t, x) be coordinates in a neighborhood U of a point $p_0 \in \partial\mathcal{M}$ with t a defining function of $\partial\mathcal{M}$ such that $\partial_t|_{\partial\mathcal{M}} = \mathcal{T}$. Let u_0 and u_1 be elements of $C_c^{-\infty}(\partial\mathcal{M})$. With $\phi^{\pm}$ and $a^{i,\pm}$ as constructed, the operator given by the oscillatory integral in* (7.1) *solves the problem* (3.3) *modulo C^{∞}, with the initial conditions being satisfied in the sense of distributions. Moreover, the description of the initial conditions is independent of coordinates, and depends only on the vector field $\mathcal{T}$.*

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