Uniform exponential growth of non-positively curved groups

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ABSTRACT

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The ping-pong lemma was introduced by Klein in the late 1800s to show that certain subgroups of isometries of hyperbolic 3-space are free and remains one of very few tools that certify when a pair of group elements generate a free subgroup or semigroup. Quantitatively applying the ping-pong lemma to more general group actions on metric spaces requires a blend of understanding the large-scale global geometry of the underlying space with local combinatorial and dynamical behavior of the action. In the 1980s, Gromov publish a sequence of seminal works introducing several metric notions of non-positive curvature in group theory where he asked which finitely generated groups have uniform exponential growth. We give an overview of various developments of non-positive curvature in group theory and past results related to building free semigroups in the setting of non-positive curvature. We highlight joint work with Radhika Gupta and Kasia Jankiewicz and with Carolyn Abbott and

Davide Spriano that extends these tools and techniques to show several groups with that act on cube complexes and many hierarchically hyperbolic groups have uniform exponential growth.

ACKNOWLEDGEMENTS

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To my grandparents

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CHAPTER 1

INTRODUCTION

This document explores connections between non-positive curvature and exponential growth in the area of geometric group theory. Non-positive curvature of Riemannian manifolds and its relationship with the exponential growth of volumes of balls as a function of radius dates back to work of Schwarz and Milnor in the mid 1900s [Š55, Mil68]. Generalizing beyond manifold groups to the less continuous settings of arbitrary finitely generated groups, we can discretize volume computations by counting lattice points within a fixed distance of a base point. This idea is classical and dates back to the late 1700s with Gauss and his circle problem [Dic05, Page ix] for which a precise asymptotic solution is still unknown. By endowing a finitely generated group with the metric structure of a graph (see Section 2.2), we are led to study growth of groups.

1.1 History of exponential growth in groups

A finitely generated group has *(uniform) exponential growth* if the number of elements that can be spelled with words of bounded length grows (uniformly) exponentially fast with respect to any finite generating set (see Definition 2.7.4 for a detailed definition). Growth is a group invariant generalizing the idea of volume growth for Reimannian manifolds. Important examples of spaces with exponential growth are hyperbolic n-space and regular trees with valence at least 3. From this we see that Cayley graphs of nonabelian free groups and free semigroups have exponential growth because balls of radius n contain more than 2^n vertices.

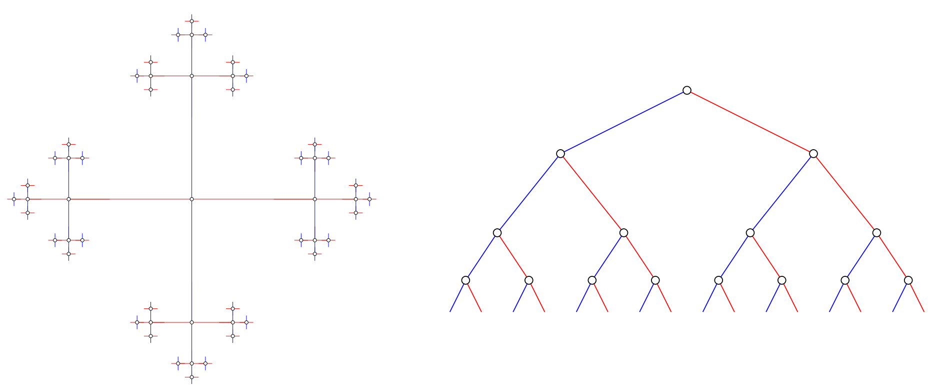


Figure 1.1: Exponentially growing Cayley graphs

Bounds on exponential growth rates are of significant interest in a broad range of areas. Indeed, the exponential growth rate bounds the entropy of group actions viewed as dynamical systems [Gro81]. Moreover, the dynamics of infinite order isometries can often be leveraged to verify uniform exponential growth. This is especially true for actions on coarsely hyperbolic spaces [Kou98, BF18]. There has been a flurry of recent activity on characterizing the dynamics of generic group elements for a given action. An essential step in doing this is computing the asymptotic growth rate of a group explicitly [Wie17, CW18, GTT18a, GTT18b].

The quantitative nature of uniform exponential growth also provides tools needed to effectively study other notions in geometric group theory. For example, the bounds in Theorem 1.2.2 depend only on the dimension of the underlying cube complex. This gives an obstruction for when certain groups can act on cube complexes of fixed dimension [Jan19]. The results presented here are motivated by providing a deeper understanding of the geometry of non-positively curved groups and towards addressing the following variation of a question of Gromov [Gro81, Remark 5.12].

Question 1.1.1. Does every finitely presented group with exponential growth have uniform exponential growth?

There are examples of finitely generated groups first provided by Wilson [Wil04b] that have exponential growth, but not uniform exponential growth. Subsequent examples were constructed [Wil04a, Bar03, Nek10], however, none of them are finitely presented. Many classes of finitely presented groups are known to have uniform exponential growth. For instance, non-elementary hyperbolic groups [Kou98], relatively hyperbolic groups [Xie07], groups that split nontrivially as amalgamated free products or HNN extensions [BdlH00], one-relator groups [GdlH91], non-nilpotent solvable groups [Alp02, Osi03], linear groups over a field of characteristic zero [EMO05], finitely generated subgroups of the mapping class group [Man10], linearly growing subgroups of the outer automorphism group of a free group [Ber19], and groups acting freely on CAT(0) square complexes [KS19]. Our goal is to extend these results to other groups suggesting a positive answer to Question 1.1.1.

1.2 Overview of main results

Our method for addressing Question 1.1.1 involves controlling the dynamics of group actions on non-positively curved spaces in order to exhibit N-short free semigroups (see Definition 2.6.5). Free semigroups are known to grow exponentially, so uniform bounds on the length of generators translates to uniform bounds on exponential growth (see Proposition 2.7.7). Many of our bounds (Theorem 1.2.1, Theorem 1.2.2, Corollary 1.2.7, Corollary 1.2.6) also apply to all finitely generated subgroups. As a consequence, we quantify the Tits alternative for certain classes of groups, giving a numerical certificate for when a group is either virtually abelian or contains a free subgroup.

The first setting where we show uniform exponential growth is for groups acting on CAT(0) cube complexes. Our result on this topic are obtained jointly with Radhika Gupta and Kasia Jankiewicz [GJN19] extending work of Kar and

Sageev [KS19] to allow more exotic actions on CAT(0) square complexes and also allow actions on cube complexes of arbitrary dimension. Specifically, we prove the following pair of theorems.

Theorem 1.2.1 ([GJN19, Theorem A])**.** Let G be a finitely generated group acting on a CAT(0) square complex X. Then either

(1) G has a global fixed point in X, or

(2) G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600} \approx 0.0012$, or

(3) G stabilizes a flat or line in X.

Here, $\lambda(G)$ is the exponential growth rate (see Definition 2.7.4). We recall that for the rank two free group $\lambda(\mathbb{F}_2) = \ln(2)$.

Theorem 1.2.2 ([GJN19, Theorem B])**.** Let X be a CAT(0) cube complex of dimension d with isolated flats that admits a geometric group action. Let G be a finitely generated group acting *freely* on X. Then either G has uniform exponential growth with $\lambda(G)$ depending only on d or G is virtually abelian.

To show Theorem 1.2.1, we construct loxodromic isometries of uniformly bounded word length for arbitrary collections of isometries of low-dimensional CAT(0) cube complexes. We achieve this with the following result where for two dimensional cube complexes the constant is $L = 12$.

Proposition 1.2.3 ([GJN19, Proposition 1.3])**.** Let a and b be a pair of isometries of a CAT(0) cube complex X of dimension two or three. Then either

(1) there exists a loxodromic element in $\langle a, b\rangle$ whose length in a, b is at most L, where L is a constant that only depends on $\dim(X)$, or,

(2) $\langle a, b\rangle$ fixes a point in X.

From these theorems we obtain the following corollaries on locally-uniform exponential growth, whose proofs we also describe in Chapter 3.

Corollary 1.2.4 ([GJN19, Corollary 1.1])**.** Let G be a finitely generated group that acts *properly* on a CAT(0) square complex. Then either G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$, or G is virtually abelian.

Corollary 1.2.5 ([GJN19, Corollary 1.4])**.** Let X be a CAT(0) cube complex of dimension d that is also *hyperbolic*. Let G be a finitely generated group admitting a *free* and *WPD* action on X. Then there exists a constant $\lambda_0 > 0$ depending only on d such that either G has uniform exponential growth bounded below by λ_0 or G is virtually infinite cyclic. In particular, groups acting *freely* and *acylindrically* on hyperbolic cube complexes have uniform exponential growth depending only on d.

The following two corollaries are particularly notable because they apply to groups that are known to not act properly on any CAT(0) cube complex. Nevertheless, they admit improper actions, which we leverage to recover uniform exponential growth.

Corollary 1.2.6 ([GJN19, Example 6.2])**.** Let G be any finitely generated subgroup of the Higman group H. Then either G is virtually abelian or G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$.

Corollary 1.2.7 ([GJN19, Theorem 6.4])**.** Let G be any finitely generated subgroup of a triangle-free Artin group A. Either G is virtually abelian or it has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$.

We move on to work with the example of the mapping class group to understand the definitions and relationship between acylindrical and hierarchical hyperbolicity. We state and prove several preliminary and structural results of hierarchically hyperbolic groups in Section 4.4. This is followed by the proof of the following theorem obtained with Carolyn Abbott and Davide Spriano.

Theorem 1.2.8 ([ANS19, Theorem 1.1])**.** Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group. Then either G has uniform exponential growth, or there is a space E such that the Cayley graph of G is quasi-isometric to $\mathbb{Z} \times E$.

This result gives the first proof of uniform exponential growth for torsion-free cocompactly cubulated groups that are also hierarchically hyperbolic and that have geometric dimension 3 or more. We note that there are no known examples of cocompactly cubulated group that are not hierarchicallly hyperbolic [HS19]. Building off of Theorem 1.2.8, we show that in several common settings we are able to exhibit free subgroups rather than free semigroups.

Proposition 1.2.9 ([ANS19, Proposition 1.8])**.** Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group such that G is not quasi-isometric to $\mathbb{Z} \times E$ for any metric space E. Suppose that either

(1) $\mathscr{C}S$ is a non-elementary hyperbolic space; or

(2) G is hierarchically acylindrical.

Then for any generating set S of G, there exists a free subgroup of G generated by two elements whose word length with respect to S is uniformly bounded.

In order to prove these, we rely on work of Breuillard and Fujiwara [Fuj15, BF18] and the following consequence of statements in [ANS19].

Corollary 1.2.10 (Not orthogonal implies free)**.** Let $a, b \in G$ be a pair of distinct axial elements of a hierarchically hyperbolic group with domains $A \in \mathrm{Big}\,(a)$ and $B \in \mathrm{Big}\,(b)$ such that $A \neq B$ and A and B are not orthogonal. Then there exists a constant $k = k(\mathfrak{S})$ such that $\langle a, b \rangle$ contains a k–short free subgroup.

The specific machinery involved in working with hierarchically hyperbolic groups is rather technical. We describe several conditions that do not require deep understanding of hierarchical hyperbolicity to verify which certify when a hierarchically hyperbolic group has uniform exponential growth. Most notably, we show the following.

Corollary 1.2.11 ([ANS19, Corollary 1.3])**.** Virtually torsion-free hierarchically hyperbolic groups which are acylindrically hyperbolic have uniform exponential growth.

1.3 Organization

Throughout this thesis, we provide an overview of non-positive curvature in group theory. We emphasize connections with low-dimensional topology as well as motivating examples that continue to guide the development of this theory. In Chapter 2, we review tools from coarse geometry that highlight the interplay between the coarse geometry of a metric spaces and algebraic properties of groups that act on them. We then recall several important classical results that are useful for showing uniform exponential growth. Chapter 3 is describes groups acting on CAT(0) cube complexes and joint work with Radhika Gupta and Kasia Jankiewicz on locally-uniform exponential growth [GJN19]. Chapter 4 describes hierarchically hyperbolic groups and uniform exponential growth results obtain jointly with Carolyn Abbott and Davide Spriano [ANS19].

CHAPTER 2

PRELIMINARIES IN GEOMETRIC GROUP THEORY

Geometric group theory concerns itself with studying groups acting on topological spaces (often by isometries on a metric space). Geometric properties of the action can be leveraged to give algebraic information about the group. Conversely, algebraic information about the group can be used to characterize the geometry or topology of the space.

2.1 Motivation from geometric topology

One fundamental result in this area is the following consequence of the Uniformization Theorem of Poincaré and the classification of surfaces.

Theorem 2.1.1 (Uniformization Theorem)**.** The fundamental group uniquely determines closed 2-manifolds up to diffeomorphism. In particular, when the manifold is orientable the fundamental group is either:

(1) The trivial group, in which case the underlying surface is a sphere, which admits a metric with constant sectional curvature $\kappa = +1$.

(2) $\mathbb{Z}^2$, in which case the underlying surface is a torus, which admits a metric with constant sectional curvature $\kappa = 0$.

(3) Has presentation $\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \cdots \cdot [a_g, b_g] \rangle$ for $g \geq 2$, in which case the underlying surface is a genus $g \geq 2$ surface, which admits a metric with constant sectional curvature $\kappa = -1$.

When the surface is non-orientable then its orientable double cover is one of the above surfaces, so it shares the same curvature. Note that we use the notation $[g, h] := ghg^{-1}h^{-1}$ for the commutator.

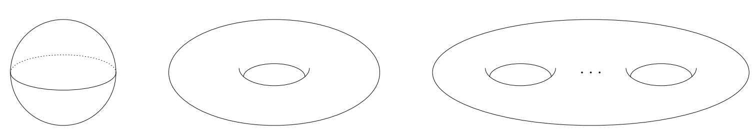


Figure 2.1: Classification of closed surfaces

This connection between fundamental groups and the geometry and topology of manifolds extends also to 3-dimensions in the Geometrization theorem proved by Perelman that also proved the Poincaré conjecture.

Theorem 2.1.2 (Geometrization Theorem)**.** The fundamental group uniquely determines closed aspherical and atoroidal 3-manifolds up to diffeomorphism. In particular, they admit one of Thurston's 8 model geometries.

These results help to motivate the following converse.

Question 2.1.3. To what extent can the geometry of a manifold or topological space be leveraged to determine a group that acts on it?

While every finitely presented group can be realized as the fundamental group of a compact 4-manifold, the geometry of 4-manifolds is not yet well-understood enough to provide much information about the group. Instead it is useful to study actions on more discrete spaces such as graphs or cell complexes. Working in this more discrete context, will be concerned with groups that are finitely generated.

2.2 Viewing groups as metric spaces

Every group acts on itself by left multiplication. The Cayley graphs of finitely generated groups are natural spaces that encode this action geometrically.

Definition 2.2.1 (Cayley graph)**.** Let G be a group with a fixed finite generating set S. The *Cayley graph of* G *with respect to* S has vertex set the elements of G and adjacency given by right (or left) multiplication by elements of S.

Cayley graphs allow us to regard such groups as metric spaces. Fix a presentation for $G = \langle S \mid \mathcal{R} \rangle$. By giving each edge length 1, G naturally acts isometrically on each of its Cayley graphs. The quotient object will be a graph with a single vertex and a loop for each generator in S. We can build a combinatorial model for the fundamental group by gluing in 2-cells whose boundary is labelled by each relator in $\mathcal{R}$. The resulting object is called the *presentation 2-complex* and has fundamental group equal to G.

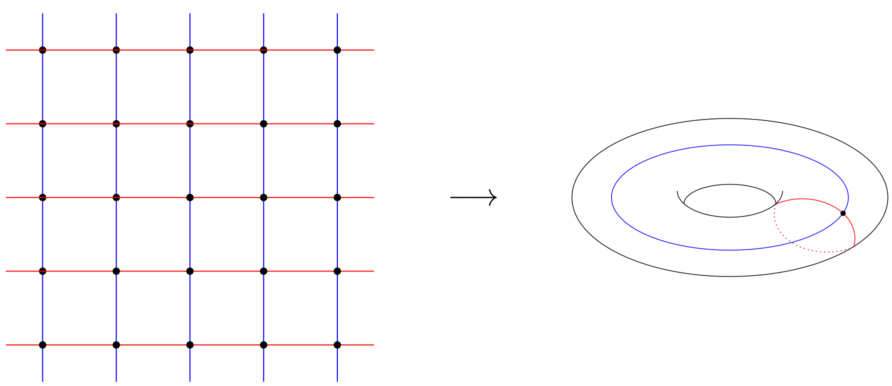


Figure 2.2: Presentation 2-complex of $\mathbb{Z}^2$

Distinct generating sets can give Cayley graphs that are not isometric.

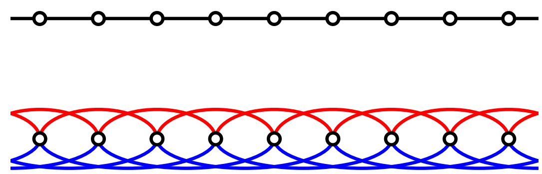


Figure 2.3: Non-isometric Cayley graphs of $\mathbb{Z}$ with different generating sets.

In search of properties characterizing a group rather than specific generating sets, it is customary to study metric spaces up to large-scale equivalence. We recall some of these notions of coarse geometry initiated by Schwarz [Š55] and Milnor [Mil68] and synthesized by Gromov [Gro81] and many others (see also [Yag00, Section 2]). In what follows, X and Y are metric spaces equipped with metrics d_X and d_Y respectively.

Definition 2.2.2 (Coarse map)**.** Let $f : X \to Y$ be a map of metric spaces. We say that f is a *coarse map* when it is both

(1) *metrically proper*: bounded sets have bounded preimages, and

(2) *coarsely uniform*: balls of fixed radius in X have images with uniformly small diameter in Y, that is, $\forall R > 0, \exists S > 0$ such that

$$d_X(p, q) \leq R \qquad \Rightarrow \qquad d_Y(f(p), f(q)) \leq S.$$

Note that f need not be continuous. We can also expand this notion to maps between power sets. A simple example to have in mind is a map between simplicial complexes that sends vertices to simplices.

Definition 2.2.3 (Hausdorff distance)**.** The *Hausdorff distance* between two subsets A, B of a metric space X is the smallest constant $R \geq 0$ such that each subset is contained in the closed R-neighborhood of the other.

Definition 2.2.4 (Coarse equivalence)**.** Let $f, g : X \to Y$ be coarse maps. We say f and g are *coarsely equivalent* when there exists a uniform constnat $C \geq 0$ such that for any point x the images $f(x)$ and $g(x)$ have Hausdorff distance at most C.

Definition 2.2.5 (Coarse inverse)**.** Let $f : X \to Y$ and $F : Y \to X$ be coarse maps. We say that F is a *coarse inverse* of f when $f \circ F$ is coarsely equivalent to the identity.

These definitions define a category in which standard rigid metric notions are relaxed slightly. Roughly speaking, we study geometry where distances are allowed to stretch and tear. We review useful notions for dealing explicitly with the metrics and concrete examples of coarse maps that show up often in geometric group theory.

Notation 2.2.6 (Coarse inequality)**.** We say that A is *coarsely less than* B, denoted $A \overset{(K,C)}{\preceq} B$, when there exists $K \geq 1$, $C \geq 0$, and $L \geq 0$ such that

$$A \leq KB + C$$

Two quantities are *coarsely equal*, denoted $A \overset{(K,C)}{\asymp} B$ when

$$B \overset{(K,C)}{\preceq} A \overset{(K,C)}{\preceq} B.$$

For functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that f *coarsely lower bounds* g, denoted $f \overset{(K,C,L)}{\preceq} g$, when for each $n \in \mathbb{N}$

$$f(n) \leq g(Kn + C) + L.$$

f and g are *coarsely equal*, $f \overset{(K,C,L)}{\asymp} g$, when $g \overset{(K,C,L)}{\preceq} f \overset{(K,C,L)}{\preceq} g$

Definition 2.2.7 (Lipschitz map)**.** Let $f : X \to Y$ be a map of metric spaces. The map f is (K-)*Lipschitz* for $K \geq 1$ when for any $p, q \in X$

$$d_Y(f(p), f(q)) \overset{(K,0)}{\preceq} d_X(p, q)$$

We say that f is (K)-*bi-Lipschitz* for $K \geq 1$ when

$$d_X(p, q) \overset{(K,0)}{\asymp} d_Y(f(p), f(q)).$$

Remark 2.2.8. Cayley graphs of the same group with different finite generating sets have the same 0-skeleton. The inclusion map on the 0-skeleton of one Cayley graph endowed with the graph metric into the other gives a bi-Lipschitz equivalence.

Definition 2.2.9 (Coarsely Lipschitz map)**.** A coarse map is (K, C)-*coarsely Lipschitz* where $K \geq 1$ and $C \geq 0$ when we can take $S = KR + C$ in Definition 2.2.2.

Definition 2.2.10 (Quasi-isometric embedding)**.** Let $f : X \to Y$ be a map of metric spaces. We say that f is a *quasi-isometric embedding* when f has a coarse inverse F such that both f and F are (K, C)-coarsely Lipschitz.

Definition 2.2.11 (Coarsely surjective)**.** A map $f : X \to Y$ is *coarsely surjective* if there exists a constant $C \geq 0$ such that the C-neighborhood of $f(X)$ covers Y.

Definition 2.2.12 (Quasi-isometry)**.** A map $f : X \to Y$ is a *quasi-isometry* when it is both a quasi-isometric embedding and coarsely surjective.

From the definition, we see that any bounded space is quasi-isometric to a point. Hence, coarse geometry tools are designed to study spaces that have infinite diameter.

2.3 Geodesics, subsets, flats, and boundaries

(Bi-infinite) lines and rays are a crucial tool in understanding the large-scale gometry of infinite diameter spaces. They allow us to keep track of length minimizing paths and "directions to infinity". We will see that these serve as basic pieces used to define a notion of boundary for certain spaces.

Definition 2.3.1 (Quasi-geodesic)**.** A $((K, C)$-*quasi*$)$-*geodesic* from p to q is a $((K, C)$-quasi-$)$isometric embedding of the interval $[0, \ell]$ such that $0 \mapsto p$ and $\ell \mapsto q$. A *quasi-line* is a quasi-isometric embedding of $\mathbb{R}$.

Definition 2.3.2. A metric space is called a $((K,C)$-*quasi*$)$-*geodesic space* when every pair of points can be joined by a $((K,C)$-quasi-$)$geodesic.

It is not hard to see that coarse maps of quasi-geodesic metric spaces are always coarsely Lipschitz by looking at images of segments of length 1.

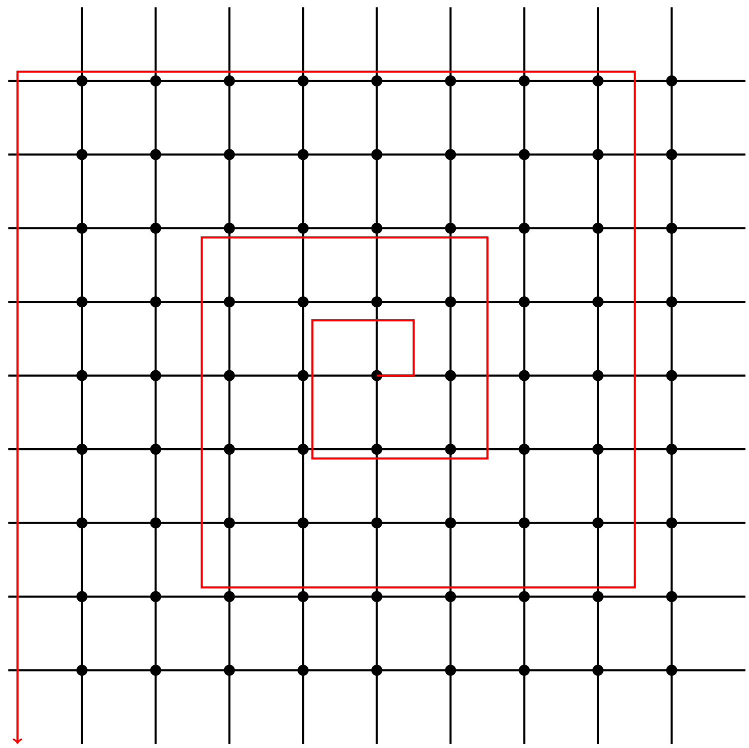


Figure 2.4: A distorted quasi-geodesic in the plane

In general, quasi-geodesics can be distorted. These quasi-geodesics do not tell us much about a space. We may therefore try to restrict out attention to quasi-geodesics with some version of convexity.

Definition 2.3.3 (Quasi-convex [RSC18, Definition 2.4])**.** Let X be a quasi-geodesic space. A subset $V \subseteq X$ is Q-*quasi-convex* if there is a function

$Q : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every (K, C)-quasi-geodesic segment with endpoints in Y is contained in the $Q(K, C)$-neighborhood of Y. The function Q is called the *convexity gauge* of Y.

When V is a quasi-geodesic and quasi-convex, V is called *Morse* and Q is called the *Morse gauge*.

We will be concerned with understanding generalizations of negative curvature to non-positive curvature. One way to do this is to keep track of subsets that witness zero curvature. The following definitions are not in the coarse category.

Definition 2.3.4 (Flat)**.** Let X be a geodesic space. For $k \geq 2$, a (k)-*flat* in X is an isometrically embedded copy of Euclidean space $\mathbb{E}^k$.

A *half flat* is defined similarly where a single coordinate in the Euclidean space is required to be non-negative.

Definition 2.3.5 (Rank-one)**.** A geodesic is *rank-one* when it does not bound a half-flat.

These notions have been widely studied in the setting of CAT(0) spaces (discussed in Chapter 3). In that setting, Bestvina and Fujiwara show a correlation of rank-one geodesics and Morse geodesics.

Theorem 2.3.6 ([BF09, Theorem 5.4])**.** Let X be a proper CAT(0) space. A geodesic on which an isometry of X acts by translation is Morse if and only if it is rank-one.

One way to distinguish the large-scale geometry of spaces is to compare their geometry at infinity. The following notion of boundary was introduced by Gromov.

Definition 2.3.7 (Visual boundary)**.** The *visual boundary* of a geodesic metric space consists of equivalence classes of geodesic rays emanating from a base point, where two rays are said to be equivalent if they have finite Hausdorff distance.

This boundary can be topologized using a neighborhood basis given by quantifying how long a pair of rays fellow-travel. In the setting where we will be working, coarsely hyperbolic spaces and CAT(0) spaces, the homeomorphism type of the visual boundary does not depend on the choice of base point.

While easy to work with, the visual boundary has some drawbacks. Croke and Kleiner showed that the homeomorphism type of the boundary is not a quasi-isometry invariant [CK00]. To remedy this, Charney and Sultan introduced a boundary consisting of the Morse geodesics [CS15].

Definition 2.3.8 (Morse boundary)**.** The *Morse boundary* of a geodesic metric space consists of equivalence classes of geodesic rays emanating from a base point that have the Morse property. As before, equivalence is given by finite Hausdorff distance.

This boundary was shown to be a quasi-isometry invariant of proper metric spaces by Cordes [Cor17]. To do this, however, it is not enough to give the Morse boundary the subspace topology coming from the visual boundary (see [CS15, Section 3.1] for details on topologizing the Morse boundary). We will make use of the fact that dynamical behavior of isometries on this boundary are easier to understand [Mur19, Liu19] than on the visual boundary.

2.4 Types of Group actions

We have already seen that finitely generated groups act on their Cayley graphs by isometries. A given group, however, may act on other topological spaces whose geometry is simpler or more well-understood. The following well-known result for δ-hyperbolic spaces (see Definition 2.6.6) gives one way to produce actions on other topological spaces.

Proposition 2.4.1 ([CDP90, GdlH91])**.** Quasi-isometries of proper δ-hyperbolic metric spaces extend to canonical self homeomorphisms of the visual boundary.

We will assume unless stated otherwise that all actions on topological spaces are by homeomorphisms and that all spaces are first countable. An action of a group G on a space X thus induces a homomorphism

$$G \to \mathrm{Homeo}(X).$$

We are most concerned with actions that "see" the whole group. Indeed every group acts trivially on every space, however, we learn nothing from that action.

An action is called *faithful* when the above homomorphism is an inclusion. In particular, every non-identity element of G acts non-trivially on X.

Definition 2.4.2 (Proper action)**.** An action $G \to \mathrm{Homeo}(X)$ is *proper* (also called properly discontinuous) when $G \times X \to X \times X$ given by $(g, x) \mapsto (g.x, x)$ is a proper map, that is, compact sets have compact preimages.

We can characterize properness by looking at compact sets.

Theorem 2.4.3 ([Bou66, Ch. III, Sec. 4.4, Proposition 7])**.** An action is proper if and only if every compact set has an open neighborhood for which all but finitely many group elements move the open set off of itself.

For cellular group actions, properness is equivalent to having finite cell stabilizers (see for example [Kap19, Theorem 9(11)]).

Discrete actions are closely related to proper actions. The distinction between the two is subtle because in the setting of locally compact spaces many different notions become the same. The result we will discuss however, will sometimes apply to spaces that are not locally compact.

Definition 2.4.4 (Discrete action)**.** A faithful action $G \subset \mathrm{Homeo}(X)$ is *discrete* if, in the subspace topology inherited from the compact-open topology on $\mathrm{Homeo}(X)$, one-point sets are open.

One remarkable result that highlights the duality between geometry and group theory is the following.

Theorem 2.4.5 (Bieberbach Theorem [Thu97, Corollary 4.1.13])**.** A group that acts discretely by isometries on $\mathbb{R}^n$ is virtually abelian.

Improper actions, however, occur naturally. Recall that the group of orientation preserving isometries of the hyperbolic plane is $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$ where the action is by fractional linear transformations.

Example 2.4.6 (Irrational rotation)**.** Consider the isometry represented by

$$r = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix},$$

a rotation about i in the upper half plane model by 1 radian. Because the hyperbolic plane is a proper metric space the infinite cyclic group $\langle r \rangle$ acts improperly on $\mathbb{H}^2$.

Example 2.4.7 (Figure-eight knot complement)**.** The fundamental group of the figure-eight knot complement can be generated by a pair of parabolic isometries of hyperbolic 3-space. This group acts properly on $\mathbb{H}^3$, but the quotient is not compact.

Definition 2.4.8 (Cocompact group action)**.** A group action is cocompact when the quotient space is compact.

Important examples of cocompact actions come from groups acting on Bass–Serre trees of a splitting or developable complexes of groups. In these actions, the 1-skeleton is frequently locally infinite. To handle actions of finitely generated groups on a locally infinite graphs, we sometimes use the following generalization of cocompact actions.

Definition 2.4.9 (Cobounded group action)**.** A group action on a metric space is *cobounded* when the action has a bounded fundamental set.

Definition 2.4.10 (Geometric group action)**.** A group action on a metric space is called *geometric* if the action is isometric, proper, and cobounded.

Hyperbolic knot complement groups do not act geometrically on $\mathbb{H}^3$. They do however act geometrically on the neutered space obtained by deleting an invariant collection of disjoint horoballs based at the parabolic points (see for example Farb and Schwartz [FS96]).

Lemma 2.4.11 (Milnor [Mil68], Schwarz[Š55])**.** The Cayley graph of a group is quasi-isometric to any proper geodesic space on which it acts geometrically.

Because finitely presented groups act geometrically on their Cayley graphs, we see that Cayley graphs with respect to different generating sets are quasi-isometric by Lemma 2.4.11. Moreover, finite index subgroups also act geometrically on the Cayley graph of their parent group. For this reason, results about the coarse geometry of a group only make sense up to finite index.

Definition 2.4.12 (Virtual)**.** Let G be a group and $\mathscr{P}$ a group property. We say that G is *virtually* $\mathscr{P}$ when there exists a finite index subgroup of G that has property $\mathscr{P}$.

Dually, properties of topological spaces are satisfied virtually if they hold for some finite cover.

Definition 2.4.13 (Hereditary)**.** Let G be a group and $\mathscr{P}$ a property. We say that $\mathscr{P}$ is *hereditary* if it passes to subgroups.

Using Lemma 2.4.11, any finitely generated group inherits the large-scale geometry of any space on which it acts geometrically by isometries. We also see other actions coming from the induced action on the boundary of the universal cover.

Example 2.4.14 (Actions by homeomorphism on a circle)**.** Fundamental groups of hyperbolic surfaces act by isometries on the hyperbolic place. By Proposition 2.4.1, each isometry induces an action on the visual boundary of the hyperbolic plane, which is easily seen to be a circle. Hence, hyperbolic surface groups act by homeomorphisms on the circle.

Homeomorphism groups of other low-dimensional manifolds are also of particular interest. The homeomorphism group of a line is closely related to orderability in groups [GdI99, Theorem 6.8]. Moreover, the homeomorphism group of surfaces leads naturally to the study of the mapping class group.

Example 2.4.15 (Mapping class group)**.** The *mapping class group*, denoted $\mathrm{MCG}(\Sigma)$, is the group of isotopy classes of orientation preserving homeomorphisms of a surface. This group admits many interesting isometric actions on several graphs that are useful for understanding the geometry of its Cayley graph. Moreover, mapping class groups have served as motivational examples for certain notions of non-positively curved groups (see Chapter 4 or for more details [FM12]).

2.5 Classifying isometries

When trying to understand isometric group actions, we can try to study the shape of an orbit. We start by looking at single elements and cyclic subgroups.

Definition 2.5.1 (Displacement)**.** Let X be a metric space and $S \in \mathrm{Isom}(X)$ a finite collection of isometries and $x \in X$ a point. The *joint displacement* of S at x is

$$L(S,x) = \max_{a \in S} d_X(x, a.x).$$

The *joint minimal displacement* is S is

$$L(S) = \inf_{x \in X} L(S,x).$$

When A is a single element $A = \{g\}$ we call $L(g)$ the *displacement* of g.

We will sometimes use the following alternative notion that assigns a translation length to infinite order elements with unbounded orbits.

Definition 2.5.2 (Stable translation length)**.** Let X be a metric space with basepoint $x \in X$ and $g \in \text{Isom}(X)$. The *stable translation length* of g is

$$\tau(g) = \lim_{n \to \infty} \frac{d_X(x, g^n(x))}{n}.$$

Notice that if X is a graph and g is a cellular isometry of X that translates along a geodesic line then both $L(g)$ and $\tau(g)$ are bounded from below by 1.

Example 2.5.3 (Parabolic isometry)**.** Consider the element represented by

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

in $\text{PSL}(2, \mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2)$. This isometry has displacement zero approximated by translates of the points $\{iN\}_{N \in \mathbb{N}}$ in the upper half plane model. However, P has no fixed point in $\mathbb{H}^2$.

Isometries of metric spaces can thus be classified as follows.

Definition 2.5.4 (Classification of isometries)**.** Isometries of a metric space are one of the following:

(1) *loxodromic*: displacement is positive and attained, or

(2) *elliptic*: displacement is zero and the element has a fixed point, or

(3) *parabolic*: displacement is not attained for any point.

An action is called *semisimple* when none of its elements act by parabolic isometries. We call the set of points that attain the minimal displacement of g

the *minimal set* of g and denote it by $\mathrm{Min}(g)$. Because loxodromic isometries have positive displacement, points in their min set are translated along paths called the *axes* of the element. In semisimple actions, we can better understand the group by associating to each element the collection of points for which the displacement is attained. The following lemma of Serre is an example of this.

Proposition 2.5.5 ([Ser03, I Proposition 26])**.** Let a, b be elliptic isometries of a simplicial tree with disjoint fixed sets. The element ab is an infinite order element that acts loxodromically on the tree.

2.6 Ping-pong and hyperbolicity

Minimal sets of elements can be used to understand the interactions between elements. In particular, they serve as a starting point to building free subgroups and free semigroups.

Lemma 2.6.1 (Ping-pong lemma [dlHoCP00, Chapter II.B])**.** Let G be a group acting on a set X and $a, b \in G \smallsetminus \{1_G\}$ where one of a and b has order at least 3. If there are disjoint nonempty subsets $A, B \subset X$ such that

$$a^{\pm 1}.B \subseteq A \qquad \text{and} \qquad b^{\pm 1}.A \subseteq B$$

then a and b freely generate a non-abelian free group of rank 2 in G.

Note that the ping-pong lemma only requires a set action. We will also be interested in using the following well-known variation that produces free semigroups. The statement appears for example in [KS19, Lemma 9] [BF18, Lemma 11.2].

Lemma 2.6.2 (Semigroup Ping-Pong)**.** Let X be a set and $u, w : X \hookrightarrow X$ If there are disjoint nonempty subsets $U, W \subset X$ such that

$$u(U \cup W) \subseteq U \qquad \text{and} \qquad w(U \cup W) \subseteq W$$

then a and b generate a free semigroup.

One immediate application of Lemma 2.6.2 is the following, which is immediate from either [KS19, Proposition 10] or [BF18, Proposition 12.1]. We include a proof for completeness.

Proposition 2.6.3. Let a and b be a pair of isometries of a simplicial tree, T. Then either

(1) $\langle a, b\rangle$ has a global fixed point, or

(2) a and b stabilize a line in T, or

(3) a pair of words u and w each of length at most 4 in a and b that generate a free semigroup.

Proof. Suppose a is loxodromic with axis A. If b stabilizes A then we are in case (2). Otherwise, bab^{-1} is another loxodromic with distinct axis B. The nearest point projection $\pi_A(B)$ will be a point, segment, or ray A because T is a tree.

In particular, any infinite component of $B\setminus\pi_B(A)$ will project to the endpoints of $\pi_A(B)$. Let $u = a$ and let A^+ be the infinite component of $A \setminus \pi_A(B)$ such that $u.A^+ \subset A^+$. This set exists because u is loxodromic. Observe that one of the endpoints, p, of $\pi_A(B)$ will be translated into A^+ by u. Similarly, let B^+ be an infinite component of $B \setminus \pi_B(A)$ that is contained in the preimage $\pi_A^{-1}(p)$. Pick $w = ba^{\pm1}b^{-1}$ such that $w.B^+ \subset B^+$. Taking U to be the component of $T\setminus\pi_A(B)$ containing A^+ and W to be the component of $T\setminus\pi_B(A)$ containing B^+ the collection u, w, U, W satisfies the conditions of Lemma 2.6.2. This is because T is a tree, so $\pi_A(W) = \pi_A(B^+)$ and $\pi_B(W) = \pi_B(A^+)$. Hence, u and w generate a free semigroup and case (3) is satisfied.

If neither a nor b act loxodromically on T then either their fixed sets overlap, putting us in case (1), or ab is loxodromic on T. Repeating the above argument considering whether both a and b stabilize the axis of ab, we are done. □

Proposition 2.6.3 easily extends to all finitely generated groups acting faithfully on a tree. This is because fixed sets of elements are convex, so we can apply Helly's property, and because axes of loxodromic elements are unique.

Corollary 2.6.4. Let G be group of tree automorphisms with finite generating set $S \subset \mathrm{Aut}(T)$. Then either

(1) G has a global fixed point in T, or

(2) G stabilizes a line, or

(3) there exists $u, w \in G$ with S-length at most 4 that generate a free semigroup.

The existence of free semigroups in Corollary 2.6.4(3) is an example of the following phenomena.

Definition 2.6.5 (N-short subgroups)**.** Let G be a group, with a finite generating set S. We say that a finitely generated sub(semi)group H in G is *N-short with respect to S*, if there exists a finite collection of words with S-length at most N that generates H. We say G contains a *uniformly N-short H*, if for every finite generating set S there exists a sub(semi)group isomorphic to H in G that is N-short with respect to S.

The ping-pong lemma was originally studied by Klein to produce free groups acting on hyperbolic n-space. This idea generalizes to the setting of spaces with a metric notion of hyperbolicity introduced by Gromov [Gro87] who recognized that triangles witness large-scale negative curvature.

Definition 2.6.6 (Coarse hyperbolicity)**.** A (K, C)–quasi-geodesic metric space is (δ)-*hyperbolic* when the δ-neighborhood of the union of any two sides of a (K, C)–quasi-geodesic triangle contains the third side.

Such spaces are also called *coarsely hyperbolic* or *Gromov hyperbolic*.

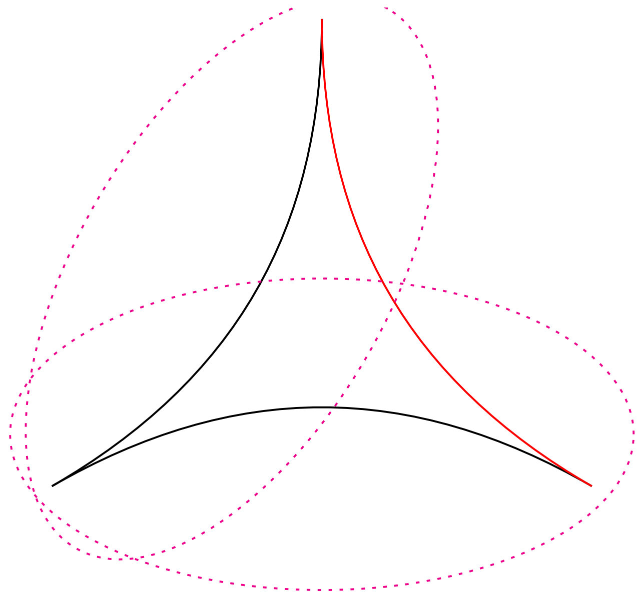


Figure 2.5: Gromov's thin triangle condition

Some examples include every bounded set, trees, which are 0-hyperbolic, and hyperbolic n-space, which is 1-hyperbolic. This notion is intended to study large-scale geometry. For this reason, it is useful to consider *non-elementary* hyperbolic spaces, which are coarsely hyperbolic spaces that are neither bounded nor quasi-isometric to the real line. The large-scale geometry of these spaces are also particularly nice. Quasi-geodesics in coarsely hyperbolic spaces are all Morse [Gro87, Proposition 7.2.A].

Coarse hyperbolicity is a quasi-isometry invariant [GdlH91, Chapitre 5, Proposition 15]. In light of the Milnor–Schwarz Lemma 2.4.11, this gives us a notion of negative curvature in groups that do not necessarily arise as the fundamental group of manifolds.

Definition 2.6.7 (Hyperbolic group)**.** A finitely generated group is *word hyperbolic* when it acts geometrically on a proper coarsely hyperbolic space.

By quasi-isometry invariance and Lemma 2.4.11, every Cayley graph of a word hyperbolic group with respect to a finite generating set is δ-hyperbolic for some constant $\delta \geq 0$. For this reason, a group is said to be *δ-hyperbolic* if it admits a Cayley graph that is a δ-hyperbolic space.

Examples of hyperbolic groups include free groups and fundamental groups of closed hyperbolic manifolds. Generalizing Corollary 2.6.4, Koubi demonstrated that, unless the group is virtually cyclic, one can always find uniformly short free subgroups of word hyperbolic groups.

Theorem 2.6.8 ([Kou98, Theorem 5.1])**.** Let G be a δ-hyperbolic group. There exists a constant $N = N(\delta)$ such that either

(1) G contains uniformly N-short free subgroups, or

(2) G is virtually cyclic.

Koubi's proof relies on producing a uniformly short infinite order element of G and taking powers of that element to apply the ping-pong lemma. The dichotomy in Koubi's result is a quantitative example of the following more general phenomenon.

Definition 2.6.9 (Tits alternative)**.** A groups satisfies the *Tits alternative* when each of its finitely generated subgroups either

(1) contains a nonabelian free subgroup, or

(2) is virtually solvable.

The Tits alternative was first shown to hold for $\mathrm{GL}(n, \mathbb{K})$ where $\mathbb{K}$ is a field [Tit72]. In the fundamental group of a hyperbolic n-manifold, all virtually solvable subgroups are virtually abelian. This is common among many other important groups that can be considered "non-positively curved". Thus, we say that a group satisfies the *strong Tits alternative* when virtually solvable is replaced with virtually abelian in Definition 2.6.9.

2.7 Growth of groups

In the previous section, we have seen ways to build free subgroups and free semigroups. Both of these objects can be seen as trees where the number of vertices grows exponentially fast in the depth. This is in stark contrast to the Euclidean plane, where the area of balls grows quadratically in the radius.

Definition 2.7.1 (Growth function)**.** Let G be a group with finite subset S. The *growth function of S in G* is

$$\beta_S(n) = \left|(S^{\pm 1} \cup \{1_G\})^n\right|.$$

This function $\beta_S(n)$ counts the number of elements that can be expressed as words in the alphabet S with length at most n.

The set S is often taken to be a finite generating set for G, in which case $\beta_S(n)$ is called the *growth of G with respect to S*. The growth functions of a group with respect to different generating sets will be coarsely equal (see Notation 2.2.6). For this reason, we study the asymptotic rate of growth of G rather than computing its growth function explicitly.

It is easy to see that abelian groups have polynomial growth. Bass and Guivarc'h independently observed that this is also true for virtually nilpotent groups [Bas72, Gui73]. The converse of their result is the contribution of Gromov's celebrated polynomial growth theorem.

Theorem 2.7.2 (Polynomial growth [Gro81])**.** A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Example 2.7.3 (Baumslag-Solitar groups)**.** Baumslag-Solitar groups are given by the presentation: $BS(n,m) = \langle a, t \mid ta^n = a^m t \rangle$. It is known that every Baumslag-Solitar group $BS(1,m)$ is solvable. Looking at the Cayley graphs of these groups, we see that they have exponential growth because there exists a surjective graph morphism onto a tree that is not a line.

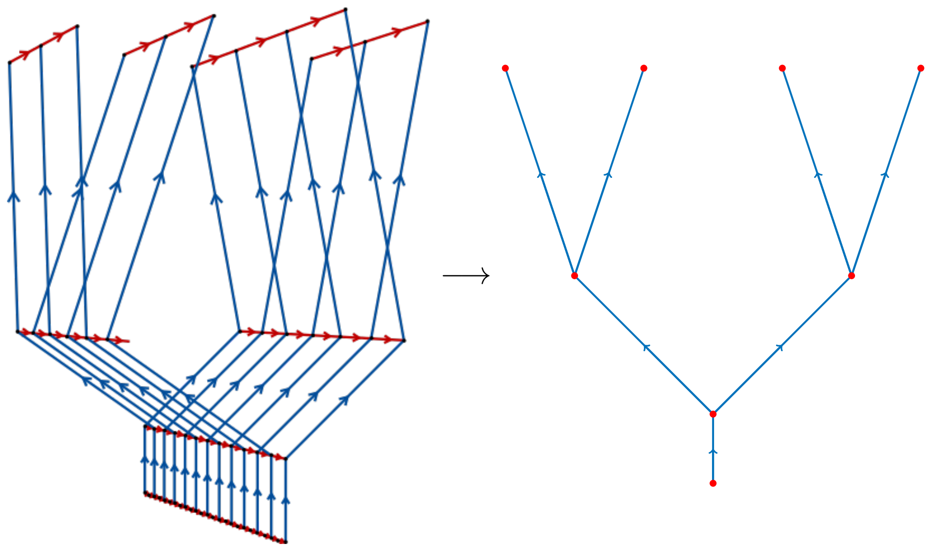


Figure 2.6: The Baumslag–Solitar group $BS(1, m)$ has exponential growth

The *exponential growth rate* of a finite subset, $S \subset G$, of a group is

$$\lambda(G, S) := \limsup_{n \to \infty} \frac{\log(\beta_S(n))}{n}.$$

Definition 2.7.4 ((Uniform) exponential growth)**.** A finitely generated group is said to have *exponential growth* if there is a finite generating set S such that

$$\lambda(G, S) > 0.$$

Such a group has *uniform exponential growth* if the infimum over all generating sets is bounded away from 0, that is,

$$\lambda(G) := \inf_{\substack{\langle S \rangle = G \\ |S| < \infty}} \lambda(G, S) > 0.$$

Further, we say that a group has *uniform uniform exponential growth* or *locally-uniform exponential growth* if

$$\inf \{\lambda(H) \mid H \leq G \text{ finitely generated with exponential growth}\} > 0$$

We have seen examples of groups with exponential growth and polynomial growth. There are, however, examples the lie in between. The first such group was produced by Grigorchuk [Gri84].

Example 2.7.5 (The first Grigorchuk group)**.** There exists a finitely generated, but not finitely presented group $R < \mathrm{Aut}(T)$ acting faithfully on a binary tree whose growth function, $\beta(n)$ with respect to any generating set lies between

$$e^{\sqrt{n}} \preceq \beta(n) \preceq e^{n^{0.767}}.$$

Here the lower bound was shown by Grigorchuk [Gri84] and the upper bound given here is an improvement shown by Bartholdi [Bar98].

In the setting of negative curvature we have already seen one example where intermediate growth does not occur.

Theorem 2.7.6 ([Kou98, Theorem 1.1])**.** Let G be a δ-hyperbolic group. There exists a constant $M = M(\delta)$ such that either

(1) G has uniform exponential growth with $\lambda(G) \geq M$, or

(2) G is virtually cyclic.

This result uses the same basic strategy that was used by Grigorchuk and de la Harpe [GdlH97, Section (A)] to show uniform exponential growth of torsion-free hyperbolic groups. The following result originates in work of Gromov

[Gro87, Theorem 5.3(E)] that was proved by Delzant [Del91, Théorème I]. To our knowledge, every known result on uniform exponential growth can be shown using some version of this strategy.

Proposition 2.7.7 (Uniform exponential growth from short free semigroups)**.** Let G be a finitely generated group. If G contains uniformly N-short free semigroups (or subgroups) in every finite generating set then G has uniform exponential growth with $\lambda(G) \geq \frac{1}{N}\ln(2)$.

In 1981, Gromov asked whether every finitely generated exponentially growing group has uniform exponential growth [Gro81, Remark 5.12]. This was refuted by Wilson, who produced a finitely generated, but not finitely presented group having exponential growth where $\lambda(G) = 0$ [Wil04b]. Other counterexamples have been produced since by Bartholdi [Bar03], Wilson [Wil04a], and Nekrashevych [Nek10], all of which are not finitely presented. The following question is still open.

Question 2.7.8. Does every finitely presented group with exponential growth also have uniform exponential growth?

When studying the orbits of points, it can be helpful to pass to a finite index subgroup where the action is more controlled. The following result of Shalen and Wagreich gives bounds on the growth of a group given the growth of a finite index subgroup.

Lemma 2.7.9 ([SW92, Lemma 3.4])**.** Let G be a group with finite generating set S, and let H be a finite index subgroup with $[G : H] = d$. Then there exists a generating set for H all of whose elements of have S–length at most $2d - 1$.

An immediate consequence of Lemma 2.7.9 is that if $[G : H] = d$ then

$$\lambda(G) \geq \frac{1}{2d-1}\lambda(H).$$

From this we see that uniform exponential growth passes up to finite index supergroups.

CHAPTER 3

UNIFORM EXPONENTIAL GROWTH OF GROUPS ACTING ON CAT(0) CUBE COMPLEXES

While the fundamental group of closed hyperbolic manifolds are word hyperbolic, the fundamental group of compact hyperbolic manifolds with cusps in dimensions 3 and above are not. The cusps obstruct hyperbolicity because their fundamental groups are virtually abelian of rank at least 2, by Bierberbach's Theorem 2.4.5. Indeed, the fundamental groups act geometrically on horospheres, which are Euclidean planes. Nevertheless, such groups exhibit many features of negative curvature coming from their action by isometries on hyperbolic n-space.

3.1 Products and CAT(0) spaces

Much like how points in a product can be specified in terms of coordinates in each of the factor spaces, isometries of products can be understood by how they permute the factor spaces and their projected actions on each factor. In this sense, many results that hold for some appropriate notion of "non-positive curvature" should continue to hold for products of such spaces.

The following was proved by Caprace and Sageev for actions on CAT(0) cube complexes. However, the statement holds for more general actions on products.

Proposition 3.1.1 (Automorphisms on products [CS11, Proposition 2.6])**.** Consider a space with product decomposition

$$X = X_1 \times \cdots \times X_p$$

where each X_i does not further decompose as a proper direct product and every automorphism of X preserves this decomposition up to permutation of the factors. Every group $G \leq \mathrm{Aut}(X)$ has a finite index subgroup that stabilizes each factor, that is,

$$\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_p) \overset{\text{finite index}}{\leq} \mathrm{Aut}(X).$$

In particular, actions on products of irreducible CAT(0) cube complexes have this property.

The following fact is a key part of the proof of [KS19, Proposition 15] and highlights the behavior of automorphisms of products.

Lemma 3.1.2 ([GJN19, Lemma 5.2])**.** Let S is a finite collection of loxodromic isometries of $\mathbb{R} \times T$ where T is a simplicial tree. Either $\langle S \rangle$ contains uniformly 4-short free semigroups or $\langle A \rangle$ stabilizes a flat or line in $\mathbb{R} \times T$.

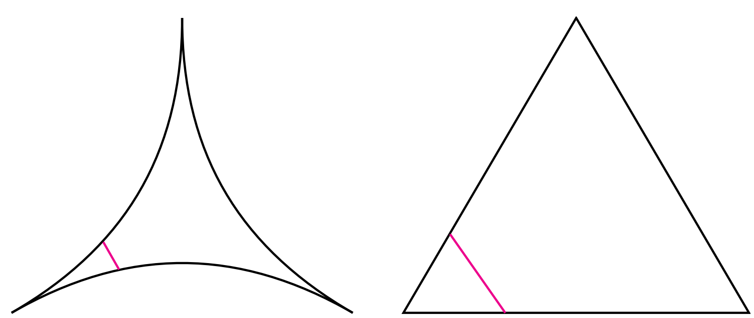
Gromov gave the following notion of non-positive curvature in which triangles are no fatter than Euclidean triangles.

Definition 3.1.3. A geodesic metric space X is *CAT(0)* when for every geodesic triangle with sides α, β, and γ in X there is a length preserving map to a comparison triangle in Euclidean space, $\mathbb{E}$

$$\alpha \cup \beta \cup \gamma \to \bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma}$$

such that for any $s \in [0, |\alpha|]$ and $t \in [0, |\beta|]$

$$d_X(\alpha(s), \beta(t)) \leq d_{\mathbb{E}}(\bar{\alpha}(s), \bar{\beta}(t))$$



More generally, a space is called *CAT(κ)* when the same inequality holds for comparison triangles taken in the unique complete simply connected Riemannian surface of constant curvature κ. One important feature of CAT(0) spaces is that geodesics joining any pair of points are unique.

A group is *CAT(0)* when it acts geometrically on a CAT(0) space. This class of groups has the advantage that it includes the fundamental groups of all compact hyperbolic manifolds and is closed under taking direct products.

3.2 CAT(0) cube complexes

Fundamental groups of compact hyperbolic 3-manifolds (possibly with cusps) are examples of CAT(0) groups. Another large class of examples of CAT(0) groups come from cube complexes. We review select notions about CAT(0) cube complexes here that will be used later. For a detailed overview we refer the reader to Sageev's notes [Sag14].

Definition 3.2.1 (Cube complex)**.** A cell complex is called a *cube complex* if all the n-cells are *n-cubes*, that is a copy of $[-1,1]^n$, and the attaching maps are isometric identifications along faces. Cube complexes where the top dimensional cells have dimension 2 are called *square complexes*.

Using a result stated by Gromov, any such complex built from identifying polyhedra along their faces admits a CAT(0) metric when they are simply

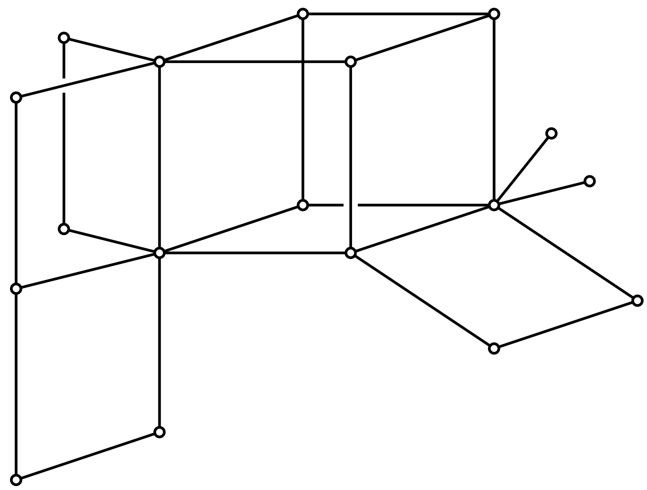


Figure 3.1: An example of a CAT(0) cube complex

connected and the link of each point is CAT(1) (see [CD93, Theorem 3.1] for a proof). For cube complexes, this local condition is equivalent to requiring that the link of every vertex be a *flag simplicial complex*, that is, a simplicial complex where $k+1$ pairwise adjacent vertices always span a k-simplex.

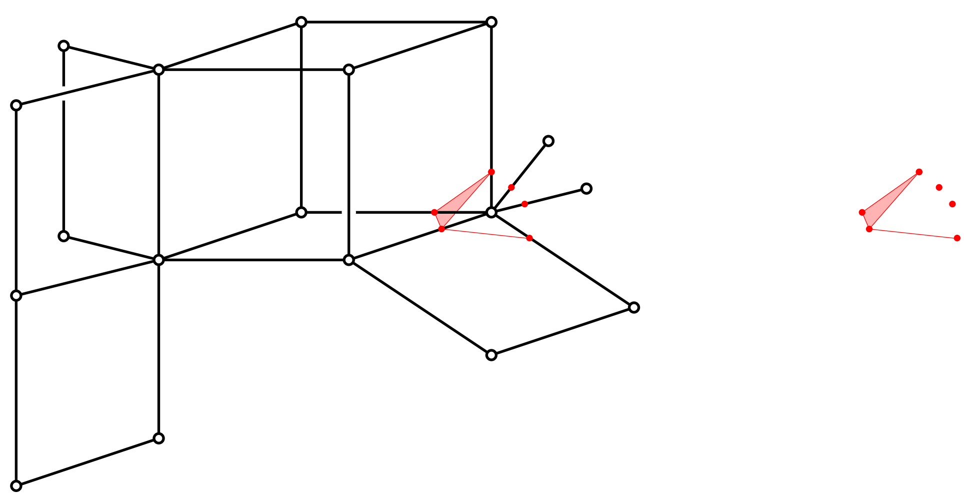


Figure 3.2: Gromov's link condition for CAT(0) cube complexes

Throughout this chapter X will be a CAT(0) cube complex, and $\mathrm{Aut}(X)$ will denote the group of isometries that preserve the cubical structure of X. The *cubical subdivision* of cube complex, X, is the complex $X_{\boxplus}$, obtained by subdividing each n-cube into 2^n subcubes.

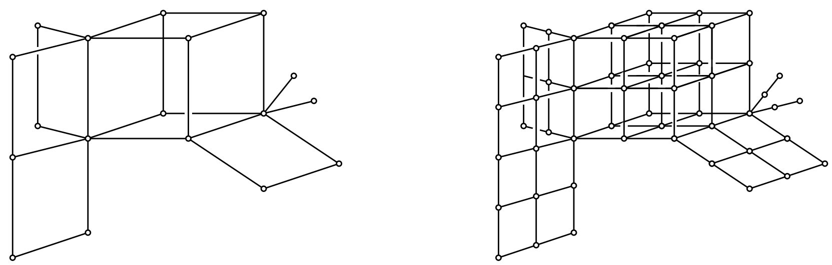


Figure 3.3: An example of cubical subdivision

Cubical isometries of a cube complex X naturally act on the cubical subdivision $X_{\boxplus}$. Haglund shows that even for infinite dimensional cube complexes, the action of $\mathrm{Aut}(X)$ on $X_{\boxplus}$ is combinatorially semisimple, that is, every element either fixes a vertex of $X_{\boxplus}$ or acts by a loxodromic isometry that leaves invariant an axis (with respect to the L^1-metric) contained in the 1-skeleton of X [Hag07]. For this reason, we will assume that all cube complexes have been subdivided once.

Definition 3.2.2 (Cubulated)**.** A group is *cubulated* if it acts properly on a CAT(0) cube complex. If the action is also cocompact we say the group is *cocompactly cubulated*.

Many groups act faithfully on CAT(0) cube complexes, however, cocompactly cubulated groups already provide a rich source of examples of groups.

Definition 3.2.3 (BMW groups)**.** Burger-Mozes-Wise (BMW) groups (a term coined by Caprace [Cap19, Section 4.1]) act freely, transitively, and by isometries on the vertex set of a product of two trees that preserves the product decomposition. This lets us identify the Cayley graphs of these groups with the 1-skeleta of CAT(0) square complexes.

This family of groups provide concrete examples of non-residually-finite infinite groups [Wis96] as well as the first example of an infinite simple group [BM97].

Definition 3.2.4. A *right-angled Artin group* (RAAG) is a finitely generated group where the only defining relations are that some of the generators are declared to commute.

These groups are presented using a finite simplicial graph, Γ. The right-angled Artin group associated to Γ is given by the presentation

$$A(\Gamma) = \left\langle v \in \Gamma^{(0)} \mid [v, w] = 1 \iff (v, w) \in \Gamma^{(1)} \right\rangle$$

Since the defining relations have length 4, we can build a cube complex whose fundamental group is $A(\Gamma)$, called the Salvetti complex. To do this, start with a single vertex and for each generator of $A(\Gamma)$ attach a loop to form the 1-skeleton. For each maximal collection of pairwise commuting generators, attach an n-torus such that its fundamental group is generated by the loops corresponding to the commuting generators. The universal cover of the Salvetti complex is a CAT(0) cube complex and also a classifying spaces for the right-angled Artin group [Sal87].

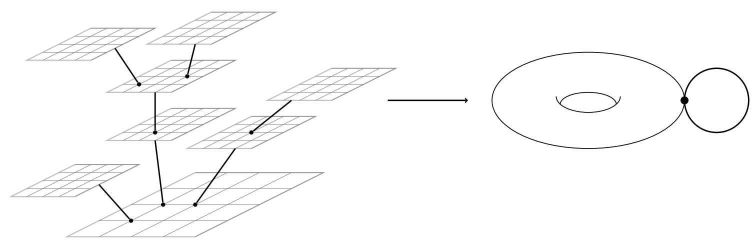


Figure 3.4: Universal cover of the Salvetti Complex of $\mathbb{Z}^2 * \mathbb{Z}$

These groups featured heavily in the work of Agol and Wise (with contributions from many others), resolving Thurston's virtual conjectures [Ago13, Wis].

Definition 3.2.5 (Convex)**.** A subspace A of a geodesic metric space X is *convex* when A contains every geodesic segment with endpoints in A.

One of the benefits of working with CAT(0) cube complexes is that convex subcomplexes are more organized and easier to study than in other spaces. One natural family of convex subspaces are *hyperplanes*. These are subspaces $\hbar \subset X$ that separate the complex into two distinct half spaces, $\hbar^+, \hbar^-$, by cutting every cube they intersect exactly in half. That is, the intersection of $\hbar$ with an n-cube is either empty or the subset given by restricting a single coordinate to 0.

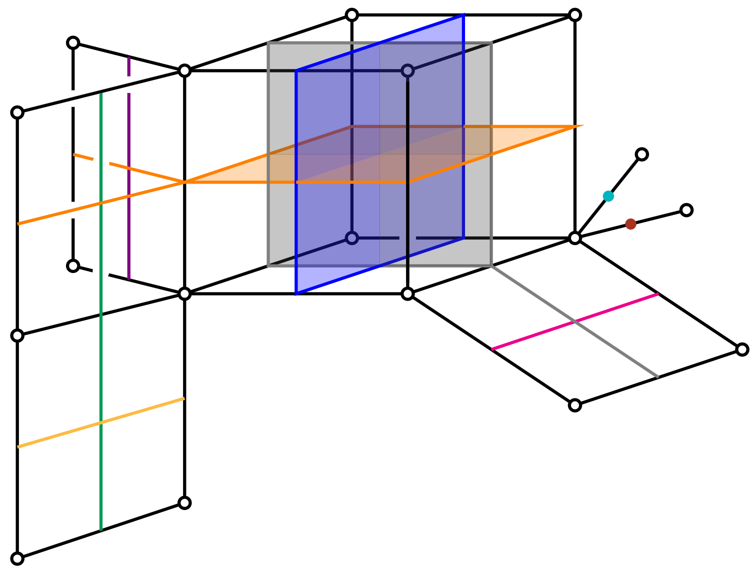
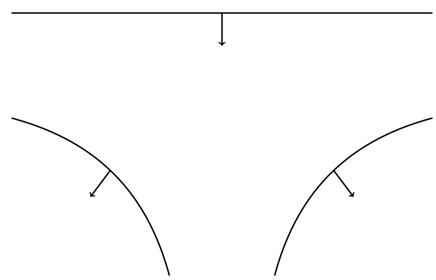


Figure 3.5: Hyperplanes in a cube complex

A cubical isometry a is said to *invert a hyperplane* if there is a hyperplane $\mathfrak{h}$ such that $a\mathfrak{h}^+ = \mathfrak{h}^-$. In order to show combinatorial semisimplicity of cubical isometries, Haglund showed that no element of $\text{Aut}(X)$ acts by hyperplane inversion on $X_{\boxplus}$. The presence of hyperplanes makes CAT(0) cube complexes particularly well-suited to detecting loxodromic isometries and free semigroups. Indeed, if $a \in \text{Aut}(X)$ and there exists a halfspace $\mathfrak{h}^+$ such that $a\mathfrak{h}^+ \subsetneq \mathfrak{h}^+$ then a acts loxodromically on X because it has no fixed point.

Definition 3.2.6 (Ping-pong triple)**.** A triple of halfspaces, $(\mathfrak{h}^+, \mathfrak{k}^+, \mathfrak{r}^+)$, is called a *ping-pong triple* when $\mathfrak{k}^+ \subset \mathfrak{h}^+$ and $\mathfrak{r}^+ \subset \mathfrak{h}^+$ and $\mathfrak{k}^+ \cap \mathfrak{r}^+ = \varnothing$



Ping-pong triples were introduced by Kar and Sageev to build free semigroups in groups acting on CAT(0) square complexes.

Lemma 3.2.7 ([KS19, Lemma 11])**.** Let $a, b \in \mathrm{Aut}(X)$ and $\boldsymbol{h} \in \mathscr{H}(X)$. If $(\boldsymbol{h}^+, a\boldsymbol{h}^+, b\boldsymbol{h}^+)$ forms a ping-pong triple then a and b generate a free semigroup.

Let $\mathscr{H}(X)$ denote the collection of all hyperplanes of X. For a subcomplex Y of X, let $\mathscr{H}(Y)$ be the collection of hyperplanes of X that intersect Y. A path joining two vertices of X is called a *combinatorial geodesic* if it is a path of minimum length in the 1-skeleton of X joining the two points. Note that every edge in a combinatorial geodesic uniquely corresponds to a hyperplane separating the vertices.

Definition 3.2.8 (Cubical convex hull)**.** Let Y be a subset of X. The *cubical convex hull* of Y, denoted $\mathrm{CHull}(Y)$, is the smallest convex subcomplex of X containing Y.

CAT(0) cube complexes are often regarded as high dimensional generalizations of trees because convex subcomplexes satisfy the *Helly property*, that is, any collection of pairwise intersecting convex subcomplexes have nonempty intersection.

Hyperplanes give the cube complex a wallspace structure in the sense of Haglund and Paulin [HP98]. This wallspace structure can be used to view the cube complex as being *dual* to the collection of hyperplanes all of X via Sageev's construction [Sag95] (see also [Sag14, Lecture 2] for more details). Since hyperplanes encode combinatorial geodesics in a cube complex, the hyperplanes that cross a subcomplex determine its cubical convex hull. We record this observation.

Observation 3.2.9. Let Y be a subcomplex of X. Then the subcomplex $\mathrm{CHull}(Y)$ is isomorphic to the cube complex dual to $\mathscr{H}(Y)$.

We say that two subcomplexes $Y, Z \subset X$ are *parallel* when there exists $p \geq 0$ such that $Y \times [0, p]$ embeds isometrically in X such that $Y \times \{0\} = Y$ and $Y \times \{p\} = Z$. In light of Observation 3.2.9, hyperplanes can be used to detect when two cubically convex subcomplexes are parallel.

Lemma 3.2.10 ([Hua17, Lemma 2.8], [HJP16, Lemma 2.7])**.** Two cubically convex subcomplexes are parallel when they are dual to the same hyperplanes. Moreover, they are equal if and only if there are no hyperplanes separating them.

3.3 Isolated flats and relative hyperbolicity

Wise showed that the fundamental group of a cusped hyperbolic 3-manifold is cubulated [Wis] building off work of the surface subgroup theorem of Kahn and Markovic[KM12]. The cube complexes on which these groups act enjoy the isolated flats property. The reader is referred to [Hru05] for details on CAT(0) spaces with isolated flats. We recall the definition here.

Definition 3.3.1 (Isolated Flats)**.** A CAT(0) space has *isolated flats* if there is a non-empty $\mathrm{Aut}(X)$-invariant collection of flats $\mathcal{F}$ such that

(1) **(Tubular neighborhood)** There is a constant $D < \infty$ such that each flat in X lies in the D-tubular neighborhood of some $F \in \mathcal{F}$.

(2) **(Isolated)** For every $\rho < \infty$ there is a constant $k(p) < \infty$ such that for any two distinct flats $F_1, F_2 \in \mathcal{F}$, $\mathrm{diam}(N_\rho(F_1) \cap N_\rho(F_2)) < k(p)$.

We say a CAT(0) cube complex X has isolated flats if X is a CAT(0) space with isolated flats and it also has a cube complex structure. These cube complexes are particularly well-adapted to studying isolated flats because hyperplanes inherit the isolated flats property. This allows for arguments that induct on dimension.

Lemma 3.3.2 ([GJN19, Lemma 2.7])**.** Let X be a CAT(0) cube complex with isolated flats. Let $\boldsymbol{\hbar}$ be a hyperplane of X. Then either $\boldsymbol{\hbar}$ does not have any flats or it is also a CAT(0) cube complex with isolated flats.

Proof. Let F be a flat contained in $\boldsymbol{\hbar}$. Every flat in $\boldsymbol{\hbar}$ is a flat in X and hence there exists a maximal flat F' in $\mathcal{F}$ containing F. Since intersections of two convex sets are convex in a CAT(0) space, $\boldsymbol{\hbar} \cap F'$ is convex. Thus, $\boldsymbol{\hbar} \cap \mathcal{F}$ is the non-empty collection of maximal flats in $\boldsymbol{\hbar}$ that satisfy Definition 3.3.1. □

Hruska and Kleiner showed the following result demonstrating the connection between isolated flats and relative hyperbolicity. We note that length spaces that admit a geometric group action are always proper [BH99, Chapter I.8 Exercise 8.4(1)]

Theorem 3.3.3 ([HK05, Theorem 1.2.1])**.** Let X be a CAT(0) space and Λ a group acting geometrically. The following are equivalent.

(1) X has isolated flats;

(2) X is hyperbolic relative to a family of flats;

(3) Λ is hyperbolic relative to a family of virtually $\mathbb{Z}^n$ subgroups with $n \geq 2$.

We review some background on relatively hyperbolic spaces and groups that will be useful later.

Definition 3.3.4 (Combinatorial horoball)**.** Let Γ be a connected graph. The *combinatorial horoball* on Γ, denoted $\mathcal{C}(\Gamma)$, is a graph obtained from $\Gamma \times \mathbb{Z}_{\geq 0}$ by attaching the following two types of edges:

- (v, k) is joined to $(v, k+1)$ for each $v \in \Gamma^{(0)}$ and $k \geq 0$, and

- (v,k) is joined to (w,k) for $k \geq 1$ when $d_\Gamma(v,w) \leq 2^k$.

A subset P of a metric space is *C-coarsely connected* when there exists a $(1,C)$ quasi-geodesic joining every pair of points in the subset. Such subsets can be approximated by a graph, Γ_P, by taking vertices to be an ε-separated net (for $\varepsilon \leq C$) and joining vertices v and w when $d(v,w) \leq C$.

Definition 3.3.5 (Relative hyperbolicity)**.** Let X be a quasi-geodesic metric space and $\mathcal{P}$ a collection of C-coarsely connected subsets such that for each $P \in \mathcal{P}$ the inclusion map to X is not coarsely surjective. The space X is said to be *hyperbolic relative to* $\mathcal{P}$ when the *cusp space* $\mathrm{Cusp}(X,\mathcal{P})$, obtained by identifying an approximating graph Γ_P with $\Gamma \times \{0\}$ in the combinatorial horoball $\mathcal{C}(\Gamma_P)$ for each $P \in \mathcal{P}$, is coarsely hyperbolic. We will call the collection, $\mathcal{P}$, the *peripheral sets* of the relatively hyperbolic space.

A group G is said to be *hyperbolic relative to a finite collection of finitely generated subroups* $\bar{\mathcal{P}}$ when any Cayley graph of G with respect to a finite generating set is relatively hyperbolic, with peripheral set $\mathcal{P}$ being the left cosets of every $P \in \bar{\mathcal{P}}$.

The above definition of relative hyperbolicity was shown to be equivalent to several others by Sisto [Sis12]. This definition is advantageous because it gives rise to the following boundary.

Definition 3.3.6 (Bowditch boundary)**.** The *Bowditch boundary* of a relatively hyperbolic group is the visual boundary of the cusped space $\mathrm{Cusp}(X, \mathscr{P})$.

For each coset, this boundary has a single isolated point whose stabilizer is a conjugate of one of the peripheral subgroups.

3.4 Constructing loxodromic cubical isometries

In this section, our goal is to effectively produce loxodromic isometries of a CAT(0) cube complex from elliptic isometries. Specifically, we generalize Serre's Proposition 2.5.5 for trees to CAT(0) cube complexes of dimension 2 or 3.

Lemma 3.4.1 ([GJN19, Lemma 4.1])**.** Let X be a d-dimensional CAT(0) cube complex and let a be an elliptic isometry of X. Then $\mathrm{CHull}(\mathrm{Fix}(a))$ is pointwise fixed by $a^{d!}$. In fact, $a^k \in \mathrm{Stab}(\mathrm{CHull}(\mathrm{Fix}(a)))$ for $k = \mathrm{LCM}\{1, 2, \ldots, d\}$.

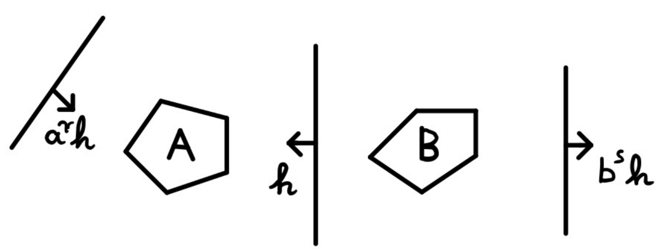
Proof. Let $A := \mathrm{Fix}(a)$. By Observation 3.2.9, $\mathrm{CHull}(A)$ is dual to the collection of hyperplanes $\mathscr{H}(A)$. We will show that $a^{d!}$ fixes each hyperplane in $\mathscr{H}(A)$, which implies that $a^{d!}$ pointwise fixes $\mathrm{CHull}(A)$.

Let A_0 be the union of all open cubes of X that intersect A non-trivially. Then a preserves every cube $c \in A_0$. Moreover, if a hyperplane $\boldsymbol{h}$ crosses and $c \in A_0$ then $\boldsymbol{h} \in \mathscr{H}(A)$. Since c contains fixed points of a, the action of a on c is determined by a permutation of at most d hyperplanes. However, $a^{d!}$

is a trivial permutation of hyperplanes crossing a cube. Therefore, $a^{d!}$ fixes each hyperplane in $\mathscr{H}(A)$. In fact, a^m, where $m = \text{LCM}\{1, 2, \ldots, d\}$, is also a trivial permutation of hyperplanes crossing a cube. $\square$

By using Lemma 3.4.1 and considering how the cubical hull of elliptic fixed sets overlap, we can sometimes build loxodromic cubical isometries.

Lemma 3.4.2 ([GJN19, Lemma 4.2])**.** Let a, b be two elliptic isometries of a CAT(0) cube complex X of dimension d. When their fixed sets are separated by a hyperplane then there exists a loxodromic isometry $g \in \langle a, b \rangle$, such that g has length at most $2d$ in a and b. When there does not exist a hyperplane separating the fixed sets of a and b, the subgroup $\left\langle a^k, b^k \right\rangle$ fixes the intersection $\text{CHull}(\text{Fix}(a)) \cap \text{CHull}(\text{Fix}(b))$ pointwise where $k = \text{LCM}\{1, 2, \ldots, d\}$, .



Proof. Supposed that there is a hyperplane $\boldsymbol{h}$ that separates $A := \text{Fix}(a)$ and $B := \text{Fix}(b)$. Since X is d-dimensional, there are two hyperplanes in $\{\boldsymbol{h}, a\boldsymbol{h}, a^2\boldsymbol{h}, \ldots, a^d\boldsymbol{h}\}$ that are either equal or disjoint. If $\boldsymbol{h} = a\boldsymbol{h}$, then the point in $\boldsymbol{h}$ closest to A is fixed by a, which is not possible since $\boldsymbol{h}$ separates

A and B. For every $2 \leq n \leq d$, suppose $\boldsymbol{\hbar} = a^n\boldsymbol{\hbar}$, $\boldsymbol{\hbar} \neq a^m\boldsymbol{\hbar}$ for $m < k$ and $\boldsymbol{\hbar} \cap a^m\boldsymbol{\hbar} \neq \emptyset$. Then the collection $H_n := \{\boldsymbol{\hbar}, a\boldsymbol{\hbar}, \ldots, a^{n-1}\boldsymbol{\hbar}\}$ is invariant under a. Also, each pair of hyperplanes in H_n intersects nontrivially. Therefore, by the Helly property, they all have a common point of intersection which is invariant under a. This is again not possible since $\boldsymbol{\hbar}$ separates A and B. Therefore, there exists $r, s \in \{1, 2, \ldots, d\}$ such that $a^r\boldsymbol{\hbar}$ and $b^s\boldsymbol{\hbar}$ are disjoint from $\boldsymbol{\hbar}$ and contained in different halfspaces determined by $\boldsymbol{\hbar}$. Let $\boldsymbol{\hbar}^+$ be the halfspace containing A. Then we have $b^s\boldsymbol{\hbar}^+ \subset a^r\boldsymbol{\hbar}^+$. Thus $a^{-r}b^s$ is a hyperbolic isometry of length at most $2d$ in a and b.

If there is no hyperplane separating A and B, then CHull(A) and CHull(B) intersect in a non-empty set. Therefore, by Lemma 3.4.1, a^k and b^k fix CHull(A) $\cap$ CHull(B), where $k = \text{LCM}\{1, 2, \ldots, d\}$ or $k = d!$. □

The previous lemma involves the subgroup $\langle a^k, b^k \rangle$ instead of $\langle a, b \rangle$. Subgroups generated by powers of cubical isometries often behave significantly differently than the original group. When a or b are torsion elements, raising to powers may make them act trivially on the cube complex. Passing to powers can also change the geometry of groups generated by infinite order elements.

Example 3.4.3 ($\mathbb{R}\times$(Tree))**.** Consider the space $X = \mathbb{R} \times T$ where T is a 3-valent tree. Let G be generated by a pair of elements, $a, b \in \text{Isom}(X)$ be given by translation by 1 in $\mathbb{R}$ and an order three permutation of branches about adjacent vertices in T. It is easy to see that G acts geometrically on X, so it has exponential growth. In fact, X can be given the structure of a cube complex by considering the orbit of the point $(0, v)$ and joining orbit points p to q when $a.p = q$. Since $a^3 = b^3$, the subgroup $\langle a^3, b^3 \rangle \cong \mathbb{Z} < G$ is abelian and does not act cocompactly on X.

We can even find examples where taking squares gives different behaviors. The following example was communicated to us by Giles Gardam.

Example 3.4.4 (A free-by-cyclic group)**.** Let $G = \langle x, y \mid [x^2, y^2] = 1 \rangle$. The group G clearly acts freely on a CAT(0) cube complex given by tiling the Cayley complex with squares. The subgroup $\langle x^2, y^2 \rangle$ is free abelian of rank 2. However, G is far from being virtually abelian because it is isomorphic to a free-by-cyclic group.

A brief outline is as follows: we change the presentation by replacing the generator y by $z = xy$ and get

$$G = \left\langle x, z | x^2 z x^{-1} z x^{-2} z^{-1} x z^{-1} = 1 \right\rangle.$$

Consider the epimorphism $\varphi \colon G \to \mathbb{Z}$ which sends x to 1 and z to 0. Then by Brown's criterion [Bro87], $\ker(\varphi)$ is finitely generated and by Magnus's Freiheitssatz [Mag30] it is a free group. Thus G is isomorphic to $\ker(\varphi) \rtimes \mathbb{Z}$.

We now show it is possible to upgrade Lemma 3.4.2 to avoid passing to powers for low-dimensional cube complexes by building a homomorphism to a finite Burnside group.

Definition 3.4.5 (Burnside groups)**.** The *free Burnside group* on m generators and n relations, denoted $\mathrm{Burn}(m,n)$, is the quotient of the rank m free group by the normal subgroup generated by the n^{th} powers of all the elements.

Any group in which the n^{th} power of any element is trivial for fixed exponent n is called a *Burnside group* and is a quotient of the free Burnside group.

Many Burnside groups are known to be infinite. However, for small values of m and n, some are known to be finite. For instance, $\mathrm{Burn}(2,2) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Burn}(2,6)$ is finite by work of Hall [Hal58]. In order to use our proof of Proposition 1.2.3 for cube complexes of dimension $d > 3$, we would need $\mathrm{Burn}(2,k)$, where $k = \mathrm{LCM}\{1,2,\ldots,d\}$, to be finite. However, already for $d = 4$, it is not known whether $\mathrm{Burn}(2,12)$ is finite or not. For odd $k > 666$, the free Burnside groups $\mathrm{Burn}(2,k)$ were shown to be infinite by Adian [ALW79]. We recall the statement of Proposition 1.2.3 from Chapter 1.

Proposition 1.2.3 ([GJN19, Proposition 1.3])**.** Let a and b be a pair of isometries of a CAT(0) cube complex X of dimension two or three. Then either

(1) there exists a loxodromic element in $\langle a,b\rangle$ whose length in a,b is at most L, where L is a constant that only depends on $\dim(X)$, or,

(2) $\langle a,b\rangle$ fixes a point in X.

Proof. Let $F(\alpha,\beta)$ be the free group on two generators, generated by α and β.

First assume X is 2-dimensional. Let K be the kernel of the map from $F(\alpha, \beta)$ to $B(2, 2)$. Then K is finitely generated by $\alpha^2, \beta^2, \alpha\beta^2\alpha^{-1}, \beta\alpha^2\beta^{-1}$. Let $w_1 = \alpha, w_2 = \beta, w_3 = \alpha\beta\alpha^{-1}, w_4 = \beta\alpha\beta^{-1}$. Then K is generated by $w_1^2, w_2^2, w_3^2, w_4^2$ and each w_i has length at most 3 in $F(\alpha, \beta)$. Let $\phi\colon F(\alpha, \beta) \to \langle a, b\rangle$ be the map that sends α to a and β to b. Then $\overline{w_i} := \phi(w_i)$ for $1 \leq i \leq 4$ also has length at most 3 in $\langle a, b\rangle$. The following is a schematic:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \hookrightarrow & F(\alpha,\beta) & \twoheadrightarrow & \mathrm{Burn}(2,2) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi & & \downarrow & & \\ 1 & \longrightarrow & \phi(K) & \hookrightarrow & \langle a, b\rangle & \twoheadrightarrow & (\text{finite}) & \longrightarrow & 1 \end{array}$$

If $\overline{w_i}$ is a loxodromic isometry of X for some $1 \leq i \leq 4$, then we are done. We may thus suppose each $\overline{w_i}$ is elliptic in X. For $i \neq j$, by Lemma 3.4.2, either there exists a loxodromic isometry in $\langle \overline{w_i}, \overline{w_j}\rangle$ of length at most 4 in $\overline{w_i}, \overline{w_j}$, equivalently length at most $4 \cdot 3 = 12$ in $\langle a, b\rangle$, or, $\langle \overline{w_i}^2, \overline{w_j}^2\rangle$ fixes $\mathrm{CHull}(\mathrm{Fix}(\overline{w_i})) \cap \mathrm{CHull}(\mathrm{Fix}(\overline{w_j}))$. Suppose the latter happens for each pair $\overline{w_i}, \overline{w_j}$. By Helly's property for cubically convex sets, there exists a point $x \in \bigcap \mathrm{CHull}(\mathrm{Fix}(\overline{w_i}))$, which is fixed by each $\overline{w_i}^2$ and hence by $\phi(K)$. Since K is a finite index subgroup of $F(\alpha, \beta)$, $\phi(K)$ is a finite index subgroup of $\langle a, b\rangle$. Thus $\langle a, b\rangle$ has a global fixed point in X. Here $L = 12$.

If X is 3-dimensional, then we consider the group $\mathrm{Burn}(2, 6)$ instead of $\mathrm{Burn}(2, 2)$ because $6 = 3!$. The group $\mathrm{Burn}(2, 6)$ is also finite by Hall [Hal58]. Let $M > 0$ be the maximum length in α, β of the elements in the smallest finite generating set of K. Then as above, we can either find a loxodromic isometry of length at most $6M = L$ in $\langle a, b\rangle$ or $\langle a, b\rangle$ fixes a point in X. □

3.5 Actions on CAT(0) square complexes that are not free

The first result on uniform exponential growth of groups acting on CAT(0) square complexes was proved by Kar and Sageev. They made use of the following result for groups generated by pairs of cubical isometries.

Proposition 3.5.1. [KS19, Proposition 15] Let a and b be two distinct loxodromic isometries of a CAT(0) square complex X. Then either

(1) $\langle a, b\rangle$ contains a 10-short free semigroup, or

(2) there exists a Euclidean subcomplex of X invariant under $\langle a, b\rangle$.

Kar and Sageev restrict their attention to free actions. Freeness implies that every element of G acts as a loxodromic isometry of X. Wise shows, however, that cubical groups need not even be virtually torsion-free [Wis07, Section 9].

In non-free actions, a given generating set may consist partially or entirely of elements acting elliptically on X. We will use Proposition 1.2.3 to guarantee existence of a short loxodromic element and then generalize the proof of [KS19, Proposition 15] to prove the following.

Theorem 1.2.1 ([GJN19, Theorem A]). Let G be a finitely generated group acting on a CAT(0) square complex X. Then either

(1) G has a global fixed point in X, or

(2) G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600} \approx 0.0012$, or

(3) G stabilizes a flat or line in X.

We briefly recall some terminology describing the relation between hyperplanes and loxodromic cubical isometries that we will use in the proof.

Given a hyperplane $\boldsymbol{\hbar}$ of X and g a loxodromic isometry of X, we say g *skewers* $\boldsymbol{\hbar}$ if for some choice of halfspace $\boldsymbol{\hbar}^+$ associated to $\boldsymbol{\hbar}$ we have $g^d\boldsymbol{\hbar}^+ \subset \boldsymbol{\hbar}^+$ where $d = \dim(X)$. We say g is *parallel* to $\boldsymbol{\hbar}$ if any axis for g is contained in the R-neighborhood of $\boldsymbol{\hbar}$ for some $R \geq 0$.

Definition 3.5.2 (Skewer set). Given a loxodromic isometry g of a cube complex, the *skewer set* of g, denoted $\text{sk}(g)$, is the collection of all hyperplanes skewered by g. A *disjoint skewer set* of g is a collection of disjoint hyperplanes in $\text{sk}(g)$ that is invariant under g^k where $k \leq \dim(X)$.

These sets play a key role in Kar and Sageev's proof of Proposition 3.5.1. Besides skewer sets, it is beneficial to identify subcomplexes stabilized by subgroups. One useful candidate is the parallel subcomplex of a loxodromic isometry. This subcomplex, $P(g)$, is dual to

$$\mathcal{H}(P(g)) = \text{sk}(g) \cup \{\boldsymbol{\hbar} \mid \boldsymbol{\hbar} \text{ crosses every hyperplane in } \text{sk}(g)\}.$$

The subcomplex naturally decomposes as a product, where one factor is isometric to the cubical hull of any axis of g. An important example of this for CAT(0) square complexes is when $P(g) = \mathbb{R} \times T$ where T is a simplicial (possibly locally infinite) tree, in which case we can apply Lemma 3.1.2.

Proposition 3.5.3 ([GJN19, Proposition 5.3])**.** Let $S = \{s_1, \ldots, s_n, e_1, \ldots, e_m\}$, with $n, m \geq 1$, be a finite collection of isometries of a CAT(0) square complex, X, such that each s_i is a loxodromic isometry and each e_j is an elliptic isometry of X. Then either

(1) $\langle S \rangle$ contains a 50-short free semigroup, or

(2) $\langle S \rangle$ stabilizes a flat or line in X.

Proof. Let $S = S_0 \sqcup S_{fix}$ where S_0 is the set of the loxodromic generators, and S_{fix} is the set of elliptic generators. Let $S_1 = S_0 \cup \{ese^{-1} \mid s \in S_0, e \in S_{fix}\}$ and $S_2 = S_1 \cup \{ese^{-1} \mid s \in S_1, e \in S_{fix}\}$. Note that elements of S_1 and S_2 have word length at most 5 in the original generating set.

Assume no pair of word of length at most 50 in S generates a free semigroup. By [KS19, Main Theorem], there are Euclidean subcomplexes E_0, E_1, E_2 in X that are stabilized by S_0, S_1, S_2, respectively. We take E_0, E_1 to be the minimal such subcomplexes with respect to inclusion.

There are two possibilities: either $E_0 = E_1$, or $E_0 \subsetneq E_1$. First, suppose $E_0 = E_1$. If $S_{fix} \subset \mathrm{Stab}(E_1)$, we are done. Suppose there exists $e \in S_{fix}$ such that $eE_1 \neq E_1$. If the subcomplexes eE_1 and E_1 are not parallel, then there exists a hyperplane $\boldsymbol{h} \in \mathcal{H}(E_1)$ such that $e\boldsymbol{h} \notin \mathcal{H}(E_1)$. Since $\boldsymbol{h}$ intersects E_1, which is a minimal Euclidean complex stabilized by S_0, there exists $g \in \langle S_0 \rangle$ such that $\boldsymbol{h} \in \mathrm{sk}(g)$. It follows that $e\boldsymbol{h} \in e\,\mathrm{sk}(g) = \mathrm{sk}(ege^{-1})$. Since $ege^{-1} \in \langle S_1 \rangle$ and S_1 stabilizes E_1, $e\boldsymbol{h}$ must intersect E_1. Thus eE_1 and E_1 must be parallel. By 2-dimensionality of X, we get $\dim E_1 = \dim eE_1 = 1$. Since E_1 is a subcomplex, it must be a combinatorial line. Its parallel subcomplex is isometric to $E_1 \times T$ where T is a tree. In this case, we are done by Lemma 3.1.2.

Now consider the case where $E_0 \subsetneq E_1$. We must have $\dim E_0 = 1$ and $\dim E_1 = 2$. Since $\mathcal{H}(E_1) \subseteq \mathcal{H}(E_2)$, and E_1 is a flat of maximal dimension in X, we get that E_2 is also 2-dimensional and in fact $E_1 = E_2$. If $S_{fix} \subset \mathrm{Stab}(E_2)$, then we are done. Suppose there exists $e \in S_{fix}$ such that $eE_2 \neq E_2$. Now eE_2 and E_2 are not parallel because they are maximal dimension flats in X. Therefore, there exists $\boldsymbol{h} \in \mathcal{H}(E_2)$ such that $e\boldsymbol{h} \notin \mathcal{H}(E_2)$. This yields a contradiction as in the previous paragraph and we conclude that in fact $S_{fix} \subset \mathrm{Stab}(E_2)$. □

We are now ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Let $S = \{s_1, \ldots, s_n, e_1, \ldots, e_m\}$ be a finite generating set for G where each s_i is a loxodromic isometry and each e_j is an elliptic isometry of X. Assume G has no global fixed point. If S contains no loxodromic isometries then by Proposition 1.2.3 we may replace S with a new generating set containing a loxodromic isometry whose word length is at most 12 in the e_i's. In doing this, we may assume $n \geq 1$. If $m = 0$ then the result follows from [KS19, Main Theorem].

Suppose then that $m \geq 0$. By Proposition 3.5.3, either there exist a pair of words of length at most 50 in S that freely generate a free semigroup or S stabilizes a flat or line in X. Thus, G has uniform exponential growth with $w(G) \geq \frac{1}{600}\ln(2)$ because the loxodromic element needed to apply Proposition 1.2.3 could have length 12 in the original generating set. □

When a group G acts by homeomorphism on a metric space X and stabilizes a subspace K then there is a natural action of G on K. This restricted action, $G \to \mathrm{Homeo}(K)$, however, need not inherit properties of the original action on X. Indeed the kernel of such an action will be the elements that act trivially on K and may be quite large.

Example 3.5.4 (Faithfulness does not pass to actions on stabilized subsets)**.** Let $G = \mathbb{Z}^2 \oplus R$ where R is the first group of intermediate growth introduced by Grigorchuk. The group R acts faithfully on a tree, T, with global fixed point v, so G acts faithfully on the universal cover of the wedge product a torus with T along the vertex v. The torus lifts to a 2-flat stabilized by G, but G does not act faithfully on this flat because R acts trivially. Moreover, G is neither virtually abelian nor contains a free semigroup because it has intermediate growth.

Certain group properties, however, do extend to the restricted action on a stabilized subset.

Lemma 3.5.5. Let G be a finitely generated finite-by-$\mathscr{P}$ group, that is, there exists a finite subgroup $K < G$ such that G/K has property $\mathscr{P}$, where $\mathscr{P}$ is a hereditary property. Then G is virtually $\mathscr{P}$.

Proof. From the definition of a finite-by-$\mathscr{P}$ group we have the short exact sequence

$$1 \longrightarrow K \hookrightarrow G \overset{\pi}{\twoheadrightarrow} G/K \longrightarrow 1.$$

Since K is a normal subgroup, G acts on K by conjugation. This gives a map $\varphi : G \to \mathrm{Aut}(K)$. The kernel $H := \ker(\varphi)$ consists of all elements $g \in G$ such that $gkg^{-1} = k$ for every $k \in K$, that is $H = \mathrm{Cent}_G(H)$. Moreover, H is finite index in G because automorphism groups of finite sets are finite. Hence, we can expand the diagram to the following where $N = K \cap H$.

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Aut}(K) & & & & \\
 & & & & \uparrow \varphi & & & & \\
1 & \longrightarrow & K & \hookrightarrow & G & \overset{\pi}{\twoheadrightarrow} & G/K & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & N & \hookrightarrow & H & \twoheadrightarrow & \pi(H) & \longrightarrow & 1.
\end{array}$$

Note that H splits as a direct product $H = N \times \bar{H}$ because elements of H all commute with those in $N \leq K$. Moreover, $\bar{H} \cong \pi(H)$ by the first isomorphism theorem. If G/K has property $\mathscr{P}$ then so does $\pi(H)$ since $\mathscr{P}$ is hereditary. Now,

N is finite because K was, so H is virtually $\mathscr{P}$. Therefore, G is virtually $\mathscr{P}$ because the index of a subgroup is multiplicative. □

This allows us to say more for groups that act properly on CAT(0) square complexes.

Corollary 1.2.4 ([GJN19, Corollary 1.1])**.** Let G be a finitely generated group that acts *properly* on a CAT(0) square complex. Then either G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$, or G is virtually abelian.

Proof. Suppose G stabilizes a flat F. Consider the image $\bar{G}$ of G in $\mathrm{Isom}(F)$. Since the action of G on X is proper, the action on F is also proper, and so $\bar{G}$ is a discrete subgroup of $\mathrm{Isom}(F)$ (this is an exercise in topology). Also, the properness of the action implies that $K = \ker(G \to \bar{G})$ is finite. By Theorem 2.4.5, $\bar{G}$ is virtually abelian. Therefore, G is virtually abelian by Lemma 3.5.5. □

As seen in Example 3.5.4, the properness hypothesis cannot be dropped from Corollary 1.2.4. Nevertheless, it is curious to study when groups acting improperly on CAT(0) square complexes may still satisfy the conclusion of Corollary 1.2.4.

3.5.1 Improper actions on CAT(0) square complexes

In this section, we will show that by understanding vertex stabilizers it is possible to use Theorem 1.2.1 to prove a locally-uniform exponential growth result. We use this to give the first known proof that the Higman group and triangle-free Artin groups have locally-uniform exponential growth. In each case, we make use of the following.

Corollary 3.5.6 ([GJN19, Corollary 1.6])**.** Suppose G acts faithfully and by isometries on a CAT(0) square complex X, such that finitely generated subgroups of the vertex stabilizers are either virtually abelian or have uniform exponential growth bounded below by λ_0. Then for any finitely generated subgroup $H \leq G$ either

(1) H has uniform exponential growth with $\lambda(H) \geq \min\left\{\frac{1}{600}\ln(2), \lambda_0\right\}$, or

(2) H stabilizes a flat or line in X, or

(3) H is virtually abelian.

Proof. If H acts without global fixed point on X, then by Theorem 1.2.1, either it contains a uniformly short free semigroup, or it stabilizes a flat or line. If H stabilizes a point in X, then it is a finitely generated subgroup of one of the vertex groups, so either is virtually abelian or has uniform exponential growth bounded by λ_0. □

Remark 3.5.7. This proof also works when "virtually abelian" is replaced with "virtually nilpotent".

The *Higman group* [Hig51], H, is given by the following presentation.

$$H := \left\langle a_i \mid a_i(a_{i+1})a_i^{-1} = a_{i+1}^2 \right\rangle_{i \in \mathbb{Z}/4\mathbb{Z}}$$

This presentation gives a decomposition of H as a square of groups with the following local groups. Each vertex group is a copy of $BS(1,2)$. Each edge group is a copy of $\mathbb{Z}$. Each 2-cell group is trivial. This decomposition gives a cocompact action of H on a CAT(0) square complex, X, whose vertex stabilizers are the vertex groups mentioned above. Martin used this structure to show that certain generalizations of the Higman group act acylindrically hyperbolically [Mar15, Theorem B] on CAT(0) square complexes. The Higman group itself is acylindrically hyperbolic, coming from its structure as a free product with amalgamation and [MO15].

To understand exponential growth in the Higman group, we first show locally-uniform exponential growth of Baumslag-Solitar groups. Uniform exponential growth of solvable Baumslag–Solitar groups follows from work of Bucher and de la Harpe [BdlH00]. However, they do not address subgroups.

Lemma 3.5.8 (UEG in Baumslag–Solitar groups [GJN19, Lemma 6.3])**.** Any finitely generated subgroup of a solvable Baumslag-Solitar group, $BS(1,m)$, is either cyclic or has uniform exponential growth at least $\frac{1}{4}\ln(2)$.

Proof. Assume $m \neq 1$ or else the group is virtually abelian and we are done. Let S be any finite collection of elements of the Baumslag-Solitar group $\langle a, t \mid tat^{-1} = a^m \rangle$. Let T be the Bass–Serre tree for G with $\mathbb{Z}$ vertex and edge groups. This tree can be obtained from the Cayley complex by collapsing in the a-direction.

Suppose S contains only elliptic isometries of T. If the fixed sets of all these elliptic isometries pairwise intersect then by the Helly property, $\langle S \rangle$ has a global fixed point in T. Thus, $\langle S \rangle$ is cyclic because vertex stabilizers are infinite cyclic. If two of the isometries, $a, b \in S$ have disjoint fixed sets then by [Ser03, I. Proposition 26] ab is a loxodromic isometry of T.

Hence, up to increasing the word length by 1, we may assume some element $g \in S$ is a loxodromic isometry of T. Consider the action of each element of S on ℓ, the axis of g. If every element of S stabilizes ℓ we will show that $\langle S \rangle$ is cyclic. Every element of $BS(1, m)$ can be written in the form $h = a^p t^q$ because $ta = a^m t$. We claim elements of the form a^p cannot stabilize ℓ. Indeed, if a^p stabilizes ℓ then it would fix the line pointwise because such elements fix a vertex in T. Vertex stabilizers are conjugates of a, so segments of length n can only be fixed pointwise by elements that are powers of a^{mn}. Taking n larger than p gives a contradiction. If $q \neq 0$ then h is also a loxodromic isometry of T. The only loxodromic isometries that will stabilize the axis of g are roots and powers of g. The axes of all other loxodromic isometries will diverge from

ℓ in the tree, T. It follows that $\mathrm{Stab}(\ell)$ is cyclic.

If some element $c \in S$ does not stabilize the axis of g then one of $\langle g^{\pm}, cg^{\pm}c\rangle$ is a free semigroup by [KS19, Proposition 10]. The bound on growth follows from these words having length at most 4. □

Combining Lemma 3.5.8 with Theorem 1.2.1, we obtain the following.

Corollary 1.2.6 ([GJN19, Example 6.2])**.** *Let G be any finitely generated subgroup of the Higman group H. Then either G is virtually abelian or G has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$.*

Proof. From its presentation as a square of groups, we see that the Higman group, H, acts cocompactly on a CAT(0) square complex, X. By Corollary 3.5.6 and Lemma 3.5.8, any finitely generated subgroup $G \leq H$ either has $\lambda(G) \geq \frac{1}{600}\ln(2)$, or is virtually abelian, or stabilizes a flat or line in X. In the third case, let E be the minimal line or flat stabilized by G. By Theorem 1.2.1, we have a homomorphism $\pi : G \to \mathrm{Isom}(E)$ where the image $\mathrm{im}(G) = \bar{G}$ is virtually abelian. The kernel is contained in $\bigcap_{p\in E}\mathrm{Stab}(p)$. The stabilizers of squares in X are trivial, and edge stabilizes are distinct cyclic subgroups of H. Therefore, $\ker(\pi)$ is trivial, so G is virtually abelian. □

Artin groups include both right-angled Artin groups and braid groups. They admit presentations corresponding to finite labeled graphs where the labels are $m_{ij} \geq 2$. Vertices correspond to generators and an edge labeled by

m joining vertices a and b corresponds to the relation:

$$\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$$

Every Artin group has a quotient Coxeter group obtained by declaring that every generator squares to the identity. *Triangle-free* Artin groups are those whose defining graphs have girth ≥ 4. *Finite-type* Artin groups are those whose quotient Coxeter group is finite. An Artin group is *FC-type* when every clique in the defining graph is associated to a finite-type Artin subgroup. An Artin group is 2-*dimensional* if every finite-type Artin subgroup has rank at most 2. It is easy to see that an Artin group is triangle-free if and only if it is 2-dimensional FC-type.

While it is not known whether Artin groups are CAT(0), many subclasses of Artin groups exhibit properties of nonpositive curvature and have attracted much attention in recent years. Indeed, FC-type Artin groups are acylindrically hyperbolic (see Definition 4.1.3 for a definition) by Chatterji and Martin [CM19, Theorem 1.2].

Many Artin groups cannot act properly on CAT(0) cube complexes [HJP16, Hae15]. Nevertheless, Martin and Przytycki recently exploited an improper action of FC-type Artin groups on CAT(0) cube complexes in order to prove that these groups satisfy the strong Tits alternative [MP19].

Charney and Davis showed an Artin group A is FC-type if and only if its Deligne complex $\mathcal{D}$ is a CAT(0) cube complex (see for detailed [MP19]).

Moreover, they showed that $\mathcal{D}$ is a $K(\pi, 1)$ space for A so it has the same cohomological dimension as the Artin group [CD95, Theorem 4.3.5]. In the action of A on $\mathcal{D}$, vertex stabilizers are conjugates of standard parabolic subgroups, which correspond to subgraphs of the defining graph of A.

Corollary 1.2.7 ([GJN19, Theorem 6.4])**.** Let G be any finitely generated subgroup of a triangle-free Artin group A. Either G is virtually abelian or it has uniform exponential growth with $\lambda(G) \geq \frac{\ln(2)}{600}$.

The base case of this theorem is the following.

Lemma 3.5.9 (Rank-2 Artin groups [GJN19, Lemma 6.5])**.** Any finitely generated subgroup of a rank-2 Artin group is either virtually abelian or has uniform exponential growth bounded by $\frac{\ln(2)}{4}$.

Proof. Brady and McCammond showed that rank-2 Artin groups act geometrically on $\mathbb{R} \times T$ where T is a simplicial tree [BM00] (see also [HJP16, Lemma 4.3]). Since the group is torsion-free [Del72], all the elements act by loxodromic isometries. The bounds on growth, hence, come from Lemma 3.1.2.

□

Proof of Corollary 1.2.7. The Deligne complex $\mathcal{D}$ of a triangle-free Artin group is a CAT(0) square complex. Every edge in $\mathcal{D}$ lies in a square and every square in $\mathcal{D}$ has one vertex with trivial stabilizer, two vertices with cyclic stabilizers that are conjugates of subgroups generated by two distinct standard generators, and one vertex that is a conjugate of a rank-2 standard parabolic subgroup (see [CD95, MP19]).

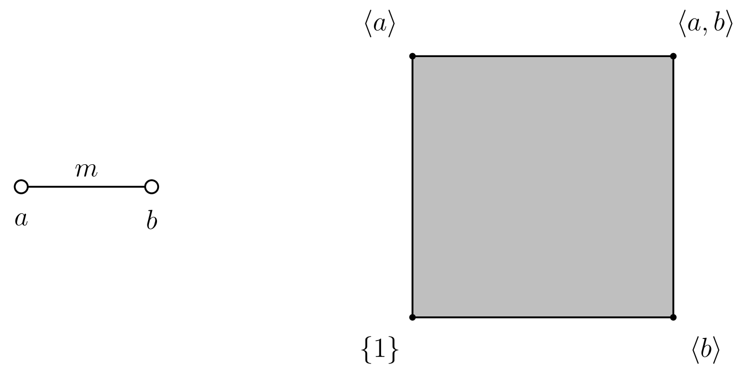


Figure 3.6: Fundamental domain of the Deligne complex

Suppose G does not have uniform exponential growth with $\lambda(G) \geq \frac{1}{600}\ln(2)$. By Theorem 1.2.1 G stabilizes a flat or line E and so the image $\bar{G}$ of G in $\mathrm{Isom}(E)$ is virtually abelian. The kernel $\ker(G \to \bar{G})$ has a subgroup K of index at most 2 that pointwise stabilizes the convex hull of E. We will show that K is trivial by showing that the larger group $T = \bigcap_{v \in \mathrm{CHull}(E)^{(0)}} \mathrm{Stab}(v)$ is trivial.

If $\mathrm{CHull}(E)$ contains any vertex with trivial stabilizer, then T is clearly also trivial. Suppose that E is a combinatorial line that does not contain any vertices with trivial stabilizers. Then E must contain a vertex w such that $\mathrm{Stab}(w)$ is rank-2. Let v and u be the two vertices adjacent to w in E. Every vertex adjacent to w has cyclic stabilizer corresponding to conjugates of generators, so T is trivial. Indeed, $\mathrm{Stab}(u)$ and $\mathrm{Stab}(v)$ will either correspond to distinct generators or distinct conjugates of the same generator. In either case $\mathrm{Stab}(u) \cap \mathrm{Stab}(v)$ is trivial (see also [Mor19]). Thus G is a finitely generated finite-by-virtually abelian group, so it is virtually abelian by Lemma 3.5.5. □

Throughout this section we exploited that we were working with a CAT(0) square complex. Hyperplane orbits in CAT(0) square complexes are significantly simpler than in higher dimensional cube complexes. Most notably, in a CAT(0) square complex if $\boldsymbol{h}, \boldsymbol{k}$ are distinct hyperplanes that are parallel to the same loxodromic isometry then $\boldsymbol{h} \cap \boldsymbol{k} = \varnothing$. In the following section, we work with CAT(0) cube complexes of dimension 3 or more where this nice behavior fails.

3.6 Free actions on CAT(0) cube complexes with isolated flats

The goal of this section is to show a generalization of Kar and Sageev's work on CAT(0) square complexes to free actions on higher dimensional cube complexes with isolated flats. As seen in Example 3.4.3, passing to powers can drastically change how a group acts on a CAT(0) space. In the setting of isolated flats, however, passing to powers does not change whether a group of cubical automorphisms stabilizes a flat.

Lemma 3.6.1 ([GJN19, Lemma 3.1])**.** Let a, b be a pair of loxodromic isometries of a CAT(0) cube complex X with isolated flats such that $\langle a^N, b^M \rangle \leq \mathrm{Aut}(X)$ stabilizes a flat for some $N, M \in \mathbb{Z}$ nonzero. Then $\langle a, b \rangle \leq \mathrm{Aut}(X)$ stabilizes a flat in X.

Proof. Let $\langle a^N, b^M \rangle$ stabilize a flat F_0. By Definition 3.3.1**(tubular neighborhood)**, there exists a maximal flat $F \in \mathcal{F}$ such that F_0 is contained in a D-neighborhood of F. Let ℓ_a and ℓ_b be axes of a and b respectively. Then ℓ_a and ℓ_b are contained in a bounded neighborhood of F. This is because ℓ_a is also an axis of a^N and any two axes of a^N are parallel in X. The same is true for ℓ_b and b^M. The axis ℓ_a (resp. ℓ_b) is also in a bounded neighborhood of aF (resp. bF). Since the collection $\mathcal{F}$ is $\mathrm{Aut}(X)$-invariant, $aF, bF \in \mathcal{F}$. Now by Definition 3.3.1**(isolated)** and maximality of F, we get that $F = aF$ and $F = bF$. Thus a and b stabilize F. □

Lines in CAT(0) spaces with isolated flats can also be controlled when the space admits a geometric group action, because either they lie close to a flat or they connect a pair of distinct points in the Morse boundary.

Lemma 3.6.2 ([GJN19, Lemma 3.3])**.** Let a and b be a pair of loxodromic isometries of a CAT(0) cube complex X with isolated flats that admits a geometric group action such that $\langle a^N, b^M \rangle \leq \mathrm{Aut}(X)$ stabilizes a line for some $N, M \in \mathbb{Z}$. Then $\langle a, b \rangle \leq \mathrm{Aut}(X)$ either stabilizes a flat or a quasi-line in X.

Proof. By [BCG$^+$18, Lemma 2.11], the line stabilized by $\langle a^N, b^M\rangle$ is either rank one or its endpoints are identified with a parabolic point p in the Bowditch boundary. In the latter case, the end points of axes of a and b also get identified with p. This is because the axes of powers fellow-travel every axis of the original element. By [HK05, Theorem 1.2.1], the stabilizers of peripheral subgroups are virtually abelian. Hence, the Flat Torus Theorem [BH99, Theorem II.7.1] implies that the group $\langle a, b\rangle$ stabilizes a flat in X.

If the line is rank one then it is a Morse geodesic by Theorem 2.3.6. By the same reasoning as above, the elements a and b are Morse elements that share endpoints in the Morse boundary. The cubical hull of these two boundary points will be a subcomplex $L \subseteq X$ quasi-isometric to a line. □

A similar situation occurs for conjugates.

Lemma 3.6.3 ([GJN19, Lemma 3.2])**.** Let a, b be a pair of loxodromic isometries of a CAT(0) cube complex with isolated flats, X, such that $\langle a, bab^{-1}\rangle \leq \mathrm{Aut}(X)$ stabilizes a flat or line. Then $\langle a, b\rangle$ stabilizes a flat or quasi-line in X.

Proof. Let E be the line of flat stabilized by $\langle a, bab^{-1}\rangle$ and suppose E is contained in a tubular neighborhood of some maximal flat $F \in \mathcal{F}$. Let ℓ_a and $\ell_{bab^{-1}} = b\ell_a$ be axes of a and bab^{-1} respectively. Then ℓ_a and $b\ell_a$ are contained in a bounded neighborhood of F. The axis ℓ_a (resp. $b\ell_a$) is also in a bounded neighborhood of aF (resp. bF). Since the collection $\mathcal{F}$ is $\mathrm{Isom}(X)$-invariant,

$aF, bF \in \mathcal{F}$. Now by the isolated condition of Definition 3.3.1, we get that $F = aF$ and $F = bF$. Therefore a and b stabilize F.

We may thus assume that E does not lie in a tubular neighborhood of a maximal flat. It follows that $\dim(E) = 1$ and E is an axis for a. As in the proof of Lemma 3.6.1,E must be a rank one geodesic by [BCG$^+$18, Lemma 2.11].

The element b must also be rank one and have the same endpoints as a in the visual boundary ∂X. Indeed, any group element c that is not rank one has its min set is contained in a tubular neighborhood of a maximal flat F, so $\partial \operatorname{Min}(c) \subseteq \partial F$ in the visual boundary. The points in ∂X fixed by c in the visual boundary is $\partial \operatorname{Min}(c)$ by a result of Ruane [Rua01, Theorem 3.3] Hruska and Kleiner showed that every component of the visual boundary of CAT(0) spaces with isolated flats is an isolated point or a sphere where the sphere components are the visual boundary of each maximal flat. In particular, rank one elements never have endpoints in a sphere component. Hence, loxodromic elements that stabilize E must be rank one.

By work of Hamestädt, rank one isometries of CAT(0) spaces act with north-south dynamics on the visual boundary [Ham08, Lemma 4.4], so the attracting and repelling fixed points of b must agree with the endpoints of E. Therefore, the cubical hull of ∂E will be a subcomplex of X quasi-isometric to a line that is stabilized by $\langle a, b\rangle$. □

Now that we understand how pairs of loxodromic isometries stabilize flats and lines, we are able to extend this to any finite collection.

Lemma 3.6.4 ([GJN19, Lemma 3.4])**.** Let $s_1, \ldots, s_n$ be a collection of loxodromic isometries of X such that $\langle s_i, s_j \rangle$ stabilizes a flat or a line for every $1 \leq i \neq j \leq n$. Then $\langle s_1, \ldots, s_n \rangle$ stabilizes a flat or quasi-line in X.

Proof. Let $E_{i,j}$ denote the flat or line stabilized by each pair s_i and s_j. Suppose one of the spaces $E_{i,j}$ is contained in a tubular neighborhood of a maximal flat $F \in \mathcal{F}$ for some $i \neq j$. Up to reordering the generators we may assume that it is $E_{1,2}$. Let ℓ be an axis of s_1. Then ℓ is contained in a bounded neighborhood of $E_{1,j}$ for all $1 \leq j \leq n$. This means that for D large enough the D-tubular neighborhoods of $E_{1,j}$ and F have unbounded intersection. By the isolated condition of Definition 3.3.1, every $E_{1,j}$ is contained in a D-tubular neighborhood of F. Similarly, every axis for s_j is also contained in a bounded neighbohood of F. It follows that every $E_{i,j}$ is contained in a tubular neighborhood of F. Hence, as in Lemma 3.6.1, $s_j F = F$ for every $1 \leq j \leq n$. Thus, $\langle s_1, \ldots, s_n \rangle$ stabilizes the flat F.

It remains to consider when every $E_{i,j}$ is a rank one geodesic. There are a pair of distinct points $p, q \in \partial X$ such that $\partial E_{ij} = \{p, q\}$ because rank one elements act with north-south dynamics and fix exactly two points in the visual boundary [Ham08, Lemma 4.4]. The cubical hull of $\{p, q\}$ will be a quasi-line stabilized by $\langle s_1, \ldots, s_n \rangle$. □

In order to exhibit small cancellation groups with arbitrarily high cubical dimension, Jankiewicz obtains the following generalization of Proposition 3.5.1 to higher dimensional cube complexes.

Lemma 3.6.5 ([Jan19, Lemma 4.2])**.** Let a, b be two distinct loxodromic isometries of a d-dimensional CAT(0) cube complex X. Then one of the following hold:

(1) **(short free semigroup)** there exists a constant $L = L(d) < \infty$ such that $\langle a, b\rangle$ contains an L-short free semigroup, or,

(2) **(stabilize hyperplane)** one of $\left\langle b^N, a^{-d!}b^N a^{d!}\right\rangle$ or $\left\langle a^N, b^{-d!}a^N b^{d!}\right\rangle$ stabilizes a hyperplane of X, or

(3) **(stabilize Euclidean)** the subgroup $\left\langle a^N, b^N\right\rangle$ stabilizes a Euclidean subcomplex of X.

where $N = d!K_3!$ and K_3 is the Ramsey number $\mathrm{Ram}(d+1, 3)$.

The exponents in Lemma 3.6.5 make it insufficient to identify when a group has uniform exponential growth. By working with flat and line stabilizers and using Lemmas 3.6.1, 3.6.2, and 3.6.3, we are able to upgrade Lemma 3.6.5 to the following.

Lemma 3.6.6 ([GJN19, Lemma 3.5])**.** Let a and b be a pair of loxodromic isometries of a CAT(0) cube complex with isolated flats that admits a geometric group action, X. There exists a constant $M = M(d) < \infty$ such that either:

(1) $\langle a, b\rangle$ contains an M-short free semigroup, or

(2) the subgroup $\langle a, b\rangle$ stabilizes a flat or a quasi-line in X.

Proof. The proof is by induction on the dimension of X. CAT(0) cube complexes with isolated flats must have dimension 2 or more because the collection of flats is assumed to be non-empty. For the base case, if $\dim(X) = 2$ then the conclusions are satisfied by [KS19, Main Theorem]. with $M(2) = 10$.

For induction, apply Lemma 3.6.5. Let $M(d) = \max\{M(d-1), L(d)\}$ where $L(d)$ is the constant from Lemma 3.6.5. If either conditions **(short free semigroup)** or **(stabilize Euclidean)** are satisfied then the conclusions are satisfied. If these conditions are not satisfied then, up to switching a and b, the group $H = \left\langle a^N, b^{-d!}a^N b^{d!}\right\rangle$ stabilizes a hyperplane. Since hyperplanes are convex in the cube complex, the elements a^N and $b^{-d!}a^N b^{d!}$ have axes in $\boldsymbol{\hbar}$ [BH99, Chapter II.6 Proposition 6.2(4)]. Hence, they are both loxodromic in the restricted action of H on $\boldsymbol{\hbar}$. Therefore, $\langle a, b\rangle$ either contains an $M(d-1)$-short free semigroup or stabilizes a flat contained in a hyperplane by the induction hypothesis. This completes the proof.

□

We are now ready to show uniform exponential growth.

Theorem 1.2.2 ([GJN19, Theorem B])**.** Let X be a CAT(0) cube complex of dimension d with isolated flats that admits a geometric group action. Let G

be a finitely generated group acting *freely* on X. Then either G has uniform exponential growth with $\lambda(G)$ depending only on d or G is virtually abelian.

Proof. Let $S = \{s_1, \ldots, s_n\}$ be a finite generating set for G. Since the action is free, each s_i is a loxodromic isometry of X. For every $1 \leq i \neq j \leq n$, consider the pair s_i and s_j. By Lemma 3.6.6 applied to s_i and s_j, if there exists a constant $M = M(d) < \infty$ such that $\langle s_i, s_j \rangle$ contains an M–short free semigroup, then we are done. So suppose $\langle s_i, s_j \rangle$ stabilizes a flat or quasi line for all pairs i, j.

From Lemma 3.6.2 the quasi line L_{ij} is the cubical hull of common boundary points of s_i and s_j in the Morse boundary of X. Since X supports a geometric group action, it is a proper space and hence L_{ij} is also a proper subcomplex. Murray shows that a group acts geometrically on a proper CAT(0) space with exactly 2 points in its Morse boundary if and only if it is virtually cyclic [Mur19, Proposition 4.8]. Since G acts freely on X, the subgroup $\langle s_i, s_j \rangle$ also acts freely on L. Since s_i, s_j act by translation on L_{ij}, $\langle s_i, s_j \rangle$ acts properly and coboundedly, hence geometrically, on L_{ij}. Thus $\langle s_i, s_j \rangle$ is virtually infinite cyclic, so it stabilizes a line in X.

Now by Lemma 3.6.4, $G = \langle s_1, \ldots, s_n \rangle$ also stabilizes a flat or a line. Since G acts freely on X, it is a discrete subgroup of isometries of the flat or line. Thus by Bieberbach's theorem, the group G is virtually abelian. □

One source of examples of CAT(0) cube complexes with isolated flats are complexes that do not contain any flats at all. If instead of isolated flats, we impose the condition of hyperbolicity on our CAT(0) cube complex X, then we can use the same proof strategy as for Theorem 1.2.2 to show uniform exponential growth where the requirement of a geometric group action is replaced with the siginificantly milder assumption of WPD (see Definition 4.1.5 for the definition of WPD). We recall a lemma of Dahmani, Guirardel, and Osin.

Lemma 3.6.7. [DGO16, Lemma 6.5] Let G be a group acting on a δ-hyperbolic space X and let $h \in G$ be a loxodromic WPD element with quasi-geodesic axis ℓ in X. Then h is contained in a unique maximal virtually abelian subgroup of G of the form $L(h) = \{g \in G | d_{Haus}(g(\ell), \ell) < \infty\}$.

In general actions on hyperbolic spaces, stabilizers of endpoints of loxodromic isometries need not be virtually cyclic. By requiring that loxodromic isometries be WPD, we obtain the following.

Corollary 1.2.5 ([GJN19, Corollary 1.4])**.** Let X be a CAT(0) cube complex of dimension d that is also *hyperbolic*. Let G be a finitely generated group admitting a *free* and *WPD* action on X. Then there exists a constant $\lambda_0 > 0$ depending only on d such that either G has uniform exponential growth bounded below by λ_0 or G is virtually infinite cyclic. In particular, groups acting *freely* and *acylindrically* on hyperbolic cube complexes have uniform exponential growth depending only on d.

Proof. Let us first prove analogs of Lemma 3.6.1 and Lemma 3.6.3. Let $a, b \in G$ be two distinct loxodromic isometries of X. Suppose $\langle a^M, b^N \rangle$ stabilizes a line ℓ in X. Then a^M and b^N fix the endpoints of ℓ. Since a loxodromic isometry and its powers share the same fixed points in the visual boundary, ∂X, a and b fix the same pair of points in ∂X. This implies that $b(\ell_a)$ has finite Hausdorff distance from ℓ_a. Thus by Lemma 3.6.7, $b \in L(a)$ and $\langle a, b \rangle$ is virtually cyclic. A similar argument works for the analogue of Lemma 3.6.3 in the current setting.

Let $\{s_1, \ldots, s_n\}$ be a finite generating set of G such that each s_i acts as a hyperbolic isometry of X. We obtain the desired result by appling the proof of Lemma 3.6.6 and Theorem 1.2.2 where the stabilized subcomplex is always a rank one geodesic because a hyperbolic CAT(0) cube complex do not contain any flats. $\square$

CHAPTER 4

HIERARCHICAL HYPERBOLICITY

This chapter describes joint work with Carolyn Abbott and Davide Spriano that further develops the structure of hierarchically hyperbolic spaces and hierarchically hyperbolic groups towards understanding uniform exponential growth of these groups. Under mild additional assumptions, we obtain a quantitative Tits alternative, generalizing work of Mangahas [Man10].

Before getting to hierarchical hyperbolicity, let us return to the example of the mapping class group of a hyperbolic surface, which we denote by $\mathrm{MCG}(\Sigma)$, where Σ is an orientable surface with genus g and k marked points defined in Example 2.4.15. We refer the reader to the book by Farb and Margalit for basics on the mapping class group [FM12].

4.1 Acylindrical actions

Surface homeomorphisms play a big role in low-dimensional geometry and topology. For example, the following theorem of Thurston provides a bridge to understanding hyperbolic 3-manifolds in terms of the geometry and topology of surfaces and certain elements of the mapping class group.

Theorem 4.1.1 (Hyperbolization [Thu98, Theorem 0.1])**.** Let φ be a homeomorphism of a connected surface Σ with negative Euler characteristic. The mapping torus M_φ, obtained as

$$M_\varphi = \Sigma \times [0,1]/(p,0) \sim (\varphi(p),1),$$

admits a complete hyperbolic structure if and only if no power of φ fixes the isotopy class of an essential curve.

We have already seen several notions of non-positively curved groups: word hyperbolic (Definition 2.6.6), CAT(0) (Definition 3.1.3), and relatively hyperbolic (Definition 3.3.5). However, none of these notions are satisfied by the mapping class group of a surface of genus $g \geq 3$. Pairs of Dehn twists generate a $\mathbb{Z}^2$ subgroup obstructing word hyperbolicity. Kapovich and Leeb showed that when $3g-3+k \geq 4$ the mapping class group is not CAT(0) [Lee96]. Anderson, Aramayona, and Shackleton showed that when $3g-3+k \geq 1$, the mapping class group fails to be relatively hyperbolic with respect to any finite collection of subgroups. The quantity $\xi(\Sigma) = 3g-3+k$ is called the *complexity of a surface*. Surfaces for which $\xi(\Sigma) \geq 1$ admit a complete hyperbolic metric.

The mapping class group does, however, admit actions on negatively curved spaces. Harvey introduced the following graph associated to hyperbolic surfaces, and showed connectivity for $\xi(\Sigma) \geq 2$ [Har81].

Definition 4.1.2 (Curve graph)**.** Let Σ be a surface of genus g with k marked points such that $\xi(\Sigma) \geq 2$. The *curve graph*, denoted $\mathscr{C}(\Sigma)$, has vertex set the isotopy classes of essential non-peripheral simple closed curves on the surface with adjacency given by pairs admitting representatives that can be realized disjointly.

Henceforth, unless specified otherwise we will assume that surfaces have $\xi(\Sigma) \geq 2$. The mapping class group acts by isometries on the corresponding curve graph. It is easily seen that curve graphs are locally infinite. Nevertheless, Luo pointed out that $\mathscr{C}(\Sigma)$ is infinite diameter [MM99, Proposition 3.6], and Masur and Minsky showed that it is a *non-elementary hyperbolic space* [MM99, Theorem 1.1], that is, a coarsely hyperbolic space with at least 3 points in its visual boundary. The following notion of action was introduced by Bowditch to unify the action on the mapping class group on the curve complex with the action of hyperbolic groups on their Cayley graphs [Bow08].

Definition 4.1.3 (Acylindrically hyperbolic)**.** The action of a group, G, on a metric space X is *acylindrical* if there exist positive integer valued functions

$R, N : \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}$ such that for all $\varepsilon > 0$ and all $x, y \in X$ with $d(x, y) \geq R(\varepsilon)$,

$$\#\{g \in G \mid d(x, gx) \leq \varepsilon \text{ and } d(y, gy) \leq \varepsilon\} \leq N(\varepsilon).$$

The quantities $R(\varepsilon)$ and $N(\varepsilon)$ are sometimes referred to as the *acylindricity constants* of the action.

A group is called *acylindrically hyperbolic* if it admits an acylindrical action on a non-elementary hyperbolic space.

The term acylindrically hyperbolic was coined by Osin [Osi16, Definition 1.3]. The flexibility of coarsely hyperbolic spaces allows infinite order elements with fixed points and finite order isometries that need not fix any point. The following result is due to Bowditch.

Lemma 4.1.4 ([Bow08, Lemma 2.2])**.** Let G act acylindrically by isometries on a coarsely hyperbolic space X. There is a global constant τ_0 such that every element of G either has bounded orbit or has stable translation length at least τ_0.

In light of Lemma 4.1.4, we adjust our classification of isometries and say that an element g acts *elliptically* if any orbit of the cyclic group $\langle g \rangle$ is bounded and loxodromic otherwise. One of the characterizing features of acylindrical actions on non-elementary hyperbolic spaces is the behavior of the loxodromic isometries. The following definition was introduced by Bestvina and Fujiwara.

Definition 4.1.5 (Weakly properly discontinuous [BF02, Section 3])**.** The action of a group G on a coarsely hyperbolic space X is *weakly properly discontinuous* (or *WPD*) if is satisfies the following:

(1) G is not virtually cyclic,

(2) at least one element of G acts loxodromically on X,

(3) for every loxodromic element $g \in G$, every point $x \in X$, and every constant $\varepsilon > 0$ there exists a power $P \in \mathbb{N}$ such that

$$\#\left\{h \in G \mid d(x, h(x)) \leq \varepsilon \text{ and } d(g^P(x), hg^P(x)) \leq \varepsilon\right\} < \infty$$

For a general isometric action of a group G on a coarsely hyperbolic space X, a loxodromic isometry $g \in G$ that satisfies the above inequality is called a *WPD element*.

The best known way to verify that a group fails to be acylindrically hyperbolic is to show that it has infinite center [Osi16, Corollary 7.2]. On the other hand, it is challenging to verify when an arbitrary action on a coarsely hyperbolic space is acylindrical. While WPD is a weaker condition than acylindricity, Osin established the following.

Theorem 4.1.6 ([Osi16, Theorem 1.2])**.** A group, G, is acylindrically hyperbolic if and only if it is not virtually cyclic and acts on a coarsely hyperbolic space such that one element of G is a WPD element.

To our knowledge, WPD is the weakest condition for an action on a coarsely hyperbolic space for which the stabilizer of both endpoints of loxodromic elements in the visual boundary is virtually cyclic, which was a key component of verifying uniform exponential growth in Corollary 1.2.5. For general coarsely hyperbolic spaces, the following question remains open.

Question 4.1.7. Does every finitely presented acylindrically hyperbolic group have uniform exponential growth?

If we weaken the "finitely presented" assumption to "finitely generated," a counterexample was produced by Minasyan and Osin who exhibited a group Q such that for every $n, R \in \mathbb{N}$ there exist generating sets for Q such that for every element g in the R-ball of the Cayley graph, g has order at most n. [MO18, Lemm 5.2]. In particular, even though acylindrically hyperbolic groups contain many free subgroups [DGO16, Theorem 6.14(c)], this group Q does not even contain uniformly N-short free semigroups.

4.2 Building free semigroups on hyperbolic spaces

It is nevertheless curious under what conditions we are able to build free subroups or free semigroups in an acylindrical action. The following quantitative result was obtained by Fujiwara.

Proposition 4.2.1 ([Fuj15, Proposition 2.3(2)])**.** Let G act acylindrically on a δ–hyperbolic space containing elements $a, b \in G$ such that a acts loxodromically and $ba^nb^{-1} \neq a^{\pm n}$ for any $n \neq 0$. There is a constant power p depending on δ, $R(20\delta)$, $N(20\delta)$, and $N(100\delta)$ such that $\left\langle a^k, ba^kb^{-1} \right\rangle = \mathbb{F}_2 < G$ for all $k \geq p$.

The following is an easy consequence of Fujiwara's proposition that we observed in conversation with Carolyn Abbott, but is certainly known to experts.

Corollary 4.2.2. Let G act acylindrically on a non-elementary δ-hyperbolic space, X, and $S \subseteq G$ a finite collection containing at least one loxodromic element. Then either

(1) $\langle S \rangle$ contains an N-short free subgroup where N depends on δ and the acylindricity constants, or

(2) $\langle S \rangle$ is virtually cyclic.

Proof. Let $a \in S$ act loxodromically on X. Let $a^{+\infty}, a^{-\infty} \in \partial X$ be the endpoints of any axis of a and consider the action of each element of S on them. If all of S stabilizes the pair $a^{\pm\infty}$ then $\langle S \rangle$ is virtually cyclic by Lemma 3.6.7. Otherwise, there is an element $c \in S$ that moves at least one of the endpoints. Let $b = cac^{-1}$. The elements a and b have unequal endpoints in ∂X, so none of their powers commute by [DGO16, Corollary 6.6]. Hence, $\left\langle a^k, ba^kb^{-1} \right\rangle \cong \mathbb{F}_2$ where k is the constant from Proposition 4.2.1, so we can take $N = k + 6$. $\square$

It is tempting to think that Corollary 4.2.2 can be leveraged to show that acylindrically hyperbolic groups have locally-uniform exponential growth. This, however, is false. Let W be any group with exponential growth, but not uniform exponential growth. For example, let W be Wilson's group [Wil04b] mentioned in Chapter 2. The free product $W * W$ is acylindrically hyperbolic by [MO15], and does not have locally-uniform exponential growth.

We saw in Theorem 2.7.6 that, for hyperbolic groups, it is possible to build free subgroups using the ping-pong lemma without explicitly relying on acylindricity. If we content ourselves with finding free semigroups then we can use the following result of Breuillard and Fujiwara.

Proposition 4.2.3 ([BF18, Proposition 11.1])**.** For $\delta \geq 0$ let X be a δ–hyperbolic space, and $g, h \in \text{Isom}(X)$. If $\tau(g), \tau(h) > 10000\delta$ and g and h do not have the same endpoints in ∂X then some pair in $\{g^{\pm}, h^{\pm}\}$ generates a free semigroup.

As in Corollary 4.2.2, this proposition requires a loxodromic isometry. Generalizing ideas in this section to show uniform exponential growth for acylindrically hyperbolic groups is quite challenging. Indeed, Minasyan and Osin's group, mentioned at the end of Section 4.1, has generating sets all of which act elliptically. Less exotically, however, the mapping class group is generated by Dehn twists [Deh87] all of which act elliptically on the curve complex.

4.3 Hierarchies and subsurface projection

While generating sets for $\mathrm{MCG}(\Sigma)$ need not contain any element acting loxodromically on the curve graph $\mathscr{C}(\Sigma)$, there is a sense in which every infinite order element acts loxodromically on some $\mathscr{C}(U)$ where U is a π_1-injective subsurface that is not homotopic into the boundary of Σ.

Pioneering work of Masur and Minsky showed that, by considering the collection of all isotopy classes of essential subsurfaces $U \subseteq \Sigma$ and their associated curve graphs, we obtain normal forms for quasi-geodesics in the mapping class group called *hierarchy paths* [MM00] (see also [Min10] for more on hierarchies). These hierarchy paths give a way to understand the large-scale geometry of the mapping class group and related spaces. They play a pivotal role in the proofs of the ending lamination theorem by Brock, Canary, and Minsky [BCM12], quasi-isometric rigidity of the mapping class group by Behrstock, Kleiner, Minsky, and Mosher [BKMM12], and quasi-isometric rigidity of Teichmüller Space by Eskin, Masur, and Rafi [EMR18]. One way to see this connection is the following projection.

$$\pi : \mathrm{MCG}(\Sigma) \to \prod_{U \subseteq \Sigma} \mathscr{C}(U) \tag{4.1}$$

This map is determined by the quasi-isometry between $\mathrm{MCG}(\Sigma)$ and the marking complex (see [MM00, Section 7.1] for the proof and more on the marking complex). Each marking has diameter at most 2 in each curve graph, so π

is a coarse map. The (quasi-)distance formula of Masur and Minsky [MM00, Theorem 6.12] shows that this projection resembles a quasi-isometric embedding where the product is equipped with the ℓ^1 metric. We will say a bit more about this in Section 4.4.2.

A key tool in developing Masur and Minsky's hierarchy machinery is the idea of *subsurface projections*, which give maps between curve complexes of different subsurfaces $U, V \subseteq S$.

$$\rho^U_V : \mathscr{C}(U) \to \mathscr{C}(V)$$

Building on work of Behrstock [Beh06], these maps are used to prove the Consistency theorem in [BKMM12], which coarsely determines the tuples in $\prod_{U \subseteq \Sigma} \mathscr{C}(U)$ that are in the image of the projection π. One way to interpret this result is that the space of consistent tuples in the product give a complicated coarse coordinate system for the mapping class group. This allows us to work entirely in the context of curve graphs.

We saw in Chapter 3 that the geometry of CAT(0) cube complexes are determined by their hyperplanes. Focusing on building an analogy between the mapping class group and the Salvetti complex of right-angled Artin groups, Behrstock, Hagen, and Sisto developed the notions of *hierarchically hyperbolic spaces* and *hierarchically hyperbolic groups* [BHS14, BHS19]. Their framework provides a unified way to understand aspects of the geometry of groups that are not necessarily visible from the point of view of acylindrical actions.

Definition 4.3.1 (Hierarchically hyperbolic space)**.** The quasigeodesic space $(\mathcal{X}, d_{\mathcal{X}})$ is a *hierarchically hyperbolic space* (HHS) if there exists $\delta \geq 0$, an index set $\mathfrak{S}$ called *domains*, and a set $\{\mathscr{C}W : W \in \mathfrak{S}\}$ of δ–hyperbolic spaces $(\mathscr{C}W, d_W)$, such that the following conditions are satisfied:

1. **(Projections.)** There is a set $\{\pi_W \colon \mathcal{X} \to 2^{\mathscr{C}W} \mid W \in \mathfrak{S}\}$ of *projections* sending points in $\mathcal{X}$ to sets of diameter bounded by some $\xi \geq 0$ in the various $\mathscr{C}W \in \mathfrak{S}$. Moreover, there exists K so that each π_W is (K, K)–coarsely Lipschitz and $\pi_W(\mathcal{X})$ is K–quasi-convex in $\mathscr{C}W$.

2. **(Nesting.)** $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$, and either $\mathfrak{S} = \emptyset$ or $\mathfrak{S}$ contains a unique $\sqsubseteq$–maximal element; when $V \sqsubseteq W$, we say V is *nested* in W. (We emphasize that $W \sqsubseteq W$ for all $W \in \mathfrak{S}$.) For each $W \in \mathfrak{S}$, we denote by $\mathfrak{S}_W$ the set of $V \in \mathfrak{S}$ such that $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with V properly nested in W there is a specified subset $\rho^V_W \subset \mathscr{C}W$ with $\operatorname{diam}_{\mathscr{C}W}(\rho^V_W) \leq \xi$. There is also a *projection* $\rho^W_V \colon \mathscr{C}W \to 2^{\mathscr{C}V}$.

3. **(Orthogonality and containers.)** $\mathfrak{S}$ has a symmetric and anti-reflexive relation called *orthogonality*: we write $V \perp W$ when V, W are orthogonal. Also, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$.

   We require that for each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \emptyset$, there exists $W \in \mathfrak{S}_T - \{T\}$, so that whenever $V \perp U$ and

$V \sqsubseteq T$, we have $V \sqsubseteq W$; we say W is a *container of* $U \perp T$. Finally, if $V \perp W$, then V, W are not $\sqsubseteq$–comparable.

4. **(Transversality and consistency.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$. There exists $\kappa_0 \geq 0$ such that if $V \pitchfork W$, then there are sets $\rho_W^V \subseteq \mathscr{C}W$ and $\rho_V^W \subseteq \mathscr{C}V$ each of diameter at most ξ and satisfying the following *transversality inequality*:

$$\min\left\{d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W)\right\} \leq \kappa_0$$

for all $x \in \mathcal{X}$.

For $V, W \in \mathfrak{S}$ satisfying $V \sqsubseteq W$ and for all $x \in \mathcal{X}$, we have:

$$\min\left\{d_W(\pi_W(x), \rho_W^V), \operatorname{diam}_{\mathscr{C}V}(\pi_V(x) \cup \rho_V^W(\pi_W(x)))\right\} \leq \kappa_0.$$

The preceding inequality is the *consistency inequality* for points in $\mathcal{X}$.

Finally, if $U \sqsubseteq V$, then $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either that V is properly nested in W or that $V \pitchfork W$ and $W \not\perp U$.

5. **(Finite complexity.)** There exists $n \geq 0$, the *complexity* of $\mathcal{X}$ (with respect to $\mathfrak{S}$), so that any set of pairwise–$\sqsubseteq$–comparable elements has cardinality at most n.

6. **(Large links.)** There exist $\lambda \geq 1$ and $E \geq \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathfrak{S}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda d_{_W}(\pi_W(x), \pi_W(x'))+$

λ. Then there exists $\{T_i\}_{i=1,\dots,\lfloor N\rfloor} \subseteq \mathfrak{S}_W - \{W\}$ such that for all $T \in \mathfrak{S}_W - \{W\}$, either $T \in \mathfrak{S}_{T_i}$ for some i, or $d_T(\pi_T(x), \pi_T(x')) < E$. Also, $d_W(\pi_W(x), \rho_W^{T_i}) \leq N$ for each i.

7. **(Bounded geodesic image.)** There exists $E > 0$ such that for all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of $\mathscr{C}W$, either $\mathrm{diam}_{\mathscr{C}V}(\rho_V^W(\gamma)) \leq E$ or $\gamma \cap N_E(\rho_W^V) \neq \emptyset$.

8. **(Partial Realization.)** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathfrak{S}$, and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq \mathscr{C}V_j$. Then there exists $x \in \mathcal{X}$ so that:

   - $d_{V_j}(x, p_j) \leq \alpha$ for all j,
   - for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubseteq V$, we have $d_V(x, \rho_V^{V_j}) \leq \alpha$, and
   - if $W \pitchfork V_j$ for some j, then $d_W(x, \rho_W^{V_j}) \leq \alpha$.

9. **(Uniqueness.)** For each $\kappa \geq 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d_{\mathcal{X}}(x, y) \geq \theta_u$, then there exists $V \in \mathfrak{S}$ such that $d_V(x, y) \geq \kappa$.

For ease of readability, given $U \in \mathfrak{S}$, we typically suppress the projection map π_U when writing distances in $\mathscr{C}U$, that is, given $x, y \in \mathcal{X}$ and $p \in \mathscr{C}U$ we write $d_U(x, y)$ for $d_U(\pi_U(x), \pi_U(y))$ and $d_U(x, p)$ for $d_U(\pi_U(x), p)$.

For any hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, the index set $\mathfrak{S}$ contains a domain which is largest under the nesting relation; we will always denote this domain by Σ and its associated hyperbolic space by $\mathscr{C}\Sigma$.

Heuristically, a hierarchically hyperbolic structure on a space $\mathcal{X}$ is a means of organizing the space by the coarse geometry of the product regions in $\mathcal{X}$ and their interactions. Nesting gives a notion of sub-product regions and subspaces. Transversality gives a notion of separate or isolated subspaces. Orthogonality gives a notion of independent subspaces that together span a product region in $\mathcal{X}$. The remaining conditions have been curated to imply the existence of a quasi-distance formula, which relates distances in $\mathcal{X}$ to distances in the hyperbolic spaces $\mathscr{C}U$ just as in the quasi-distance formula of Masur and Minsky [MM00]. The notation $\{\!\{x\}\!\}_s$ is a cut-off function that gives the value of x when $x > s$ and 0 otherwise.

Theorem 4.3.2 (Distance formula; [BHS19, Theorem 4.5])**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then there exists s_0 such that for all $s \geq s_0$, there exist C, K so that for all $x, y \in \mathcal{X}$,

$$d(x,y) \underset{K,C}{\asymp} \sum_{U \in \mathfrak{S}} \{\!\{d_U(x,y)\}\!\}_s .$$

We want to understand which groups can be studied using hierarchically hyperbolic spaces. Intuitively, a hierarchically hyperbolic group is a group whose Cayley graph is an HHS such that the action of the group on its Cayley graph is compatible with the HHS structure. The compatibility of the action is a key requirement: it is tedious but straightforward to verify that the definition of HHS is quasi-isometry invariant, whereas it is unknown if being an HHG is preserved under quasi-isometry. This compatibility will be packaged in terms of automorphisms of hierarchically hyperbolic spaces discussed in the next section.

4.4 Basics on hierarchical hyperbolicity

In this section we explain the definition of hierarchically hyperbolic groups of Berhstock, Hagen, and Sisto [BHS19] and the classification of automorphisms of Durham, Hagen, and Sisto [DHS17]. We will move on to see how using the hyperbolic spaces as coordinate spaces lets us put a metric (rather than a quasi-metric) on the space of consistent tuples in Section 4.4.2.

Some basic examples of hierarchically hyperbolic groups include hyperbolic groups, mapping class groups, many (conjecturally all) cocompactly cubulated groups, fundamental groups of most 3–manifolds, and various combinations of the above groups, including direct products, certain quotients, and graph products [BHS19, BR18].

4.4.1 Definition of hierarchically hyperbolic groups

To understand automorphisms that are compatible with the hierarchical structure we begin by considering morphisms between two hierarchically hyperbolic spaces.

Definition 4.4.1 (Hieromorphism; [BHS19, Definition 1.20])**.** A *hieromorphism* between the hierarchically hyperbolic spaces $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}', \mathfrak{S}')$ consists of a map $f \colon \mathcal{X} \to \mathcal{X}'$, an injection $f^\diamond \colon \mathfrak{S} \to \mathfrak{S}'$ and a collection of quasi-isometric embeddings $f^*(U) \colon \mathscr{C}U \to \mathscr{C}f^\diamond(U)$ such that the two following diagrams uniformly coarsely commute (whenever defined).

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_{f^\diamond(U)}} \\ \mathscr{C}U & \xrightarrow{f^*_U} & \mathscr{C}f^\diamond(U) \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{C}U & \xrightarrow{f^*_U} & \mathscr{C}f^\diamond(U) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^{f^\diamond U}_{f^\diamond V}} \\ \mathscr{C}V & \xrightarrow{f^*_V} & \mathscr{C}f^\diamond(V) \end{array}$$

As the functions f, f^*_U, and $f^\diamond$ all have distinct domains, it is often clear from the context which is the relevant map; in that case we periodically abuse notation slightly by dropping the superscripts and simply calling all of the maps f.

Note that the definition does not have any requirement on the map $f : \mathcal{X} \to \mathcal{X}'$. This is because the distance formula (Theorem 4.3.2) implies that f is determined up to uniformly bounded error by the map $f^\diamond$ and the collection $\{f^*_U \mid U \in \mathfrak{S}\}$. The fact that a hieromorphism is coarsely-determined by its action on the hierarchical structure is key in the definition of a hierarchically hyperbolic group.

Definition 4.4.2 (Hierarchical automorphism [BHS19, Definition 1.21])**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. An *automorphism* of $(\mathcal{X}, \mathfrak{S})$ is a hieromorphism $f\colon \mathcal{X} \to \mathcal{X}$, such that the map $f^{\diamond}\colon \mathfrak{S} \to \mathfrak{S}$ is a bijection, and the maps $f^*_U\colon \mathscr{C}U \to \mathscr{C}f^{\diamond}(U)$ are isometries. Two automorphisms f, f' are *equivalent* if $f^{\diamond} = (f')^{\diamond}$ and $\phi_U = \phi'_U$ for all U. Given f, we define a quasi-inverse $\bar{f}$ by setting $\bar{f}^{\diamond} = (f^{\diamond})^{-1}$ and $\bar{\phi}_{f^{\diamond}(U)} = \phi_U^{-1}$ (then $\bar{f}$ is determined by the distance formula). The set of such equivalence classes forms a group, denoted $\mathrm{Aut}(\mathfrak{S})$.

Just as for word hyperbolic and CAT(0) groups, a group is said to be hierarchically hyperbolic when it acts by hierarchical automorphisms with a certain co-finiteness condition.

Definition 4.4.3 (Hierarchically hyperbolic group [BHS19, Definition 1.21])**.** A group G is said to be *hierarchically hyperbolic* if there is a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ and an action $G \to \mathrm{Aut}(\mathfrak{S})$ such that the quasi-action of G on $\mathcal{X}$ is proper and cobounded and $\mathfrak{S}$ contains finitely many G–orbits.

Studying quasi-actions can be cumbersome. Fortunately, we can always assume that a hierarchically hyperbolic group is acting by isometries.

Remark 4.4.4. The hierarchically hyperbolic space on which a hierarchically hyperbolic group acts can be taken to be the Cayley graph of G with respect to any finite generating set. In this case, G acts on $\mathcal{X}$ by isometries. We adopt this convention for the remainder of the paper and use the notation $(G, \mathfrak{S})$ to denote this structure.

Thinking of the spaces $\mathscr{C}W$ as coordinate spaces, we wish to characterize the coarse geometry of a hierarchically hyperbolic group by only looking at the domains W for which the space $\mathscr{C}W$ is infinite diameter. This would pose a problem if there was a sequence of domains W_i such that $\operatorname{diam} \mathscr{C}W_i \to \infty$.

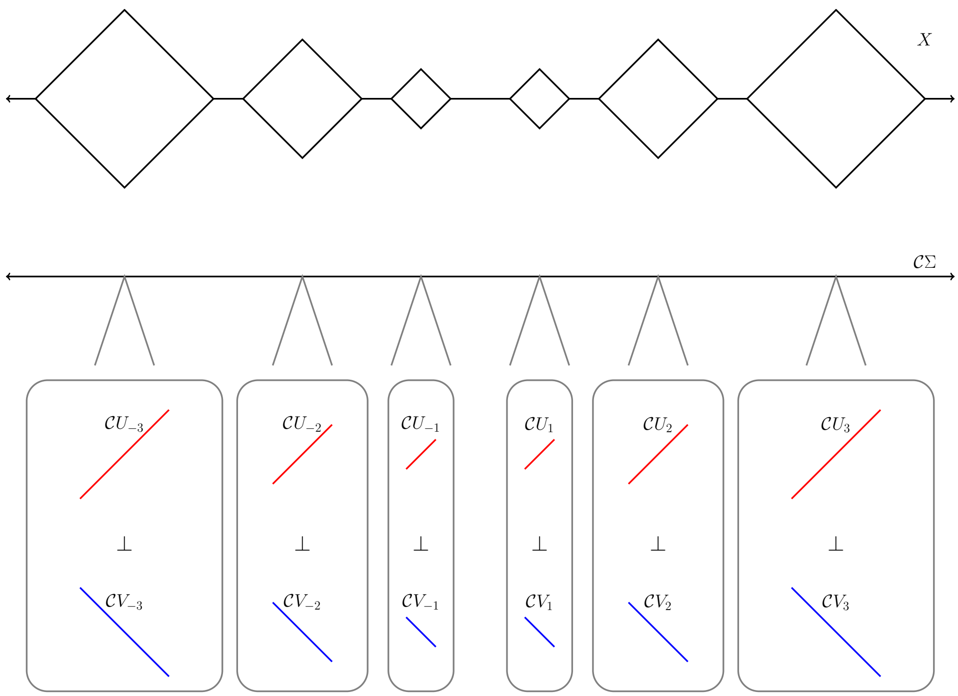


Figure 4.1: The real line where each integer n has been replaced by an $n \times n$ square admits a hierarchically hyperbolic structure that does not satisfy the bounded domain dichotomy

Fortunately, this does not happen in a hierarchically hyperbolic group.

Remark 4.4.5 (Bounded domain dichotomy)**.** By the definition of a hierarchically hyperbolic group, there is a finite set of domains $U_1, \ldots, U_k$ such that for every $W \in \mathfrak{S}$, there is some $i = 1, \ldots, k$ such that $\mathscr{C}W$ is isometric to $\mathscr{C}U_i$. It follows that for every $W \in \mathfrak{S}$, the diameter of $\mathscr{C}W$ is either infinite or *uniformly* bounded.

In what follows we will consider a hierarchically hyperbolic group $(G, \mathfrak{S})$ with respect to different finite generating sets. Let S and T be two finite generating sets for a group G, and suppose that a hierarchical structure $(G, \mathfrak{S})$ is given, where distances in G are measured with d_S. Then the identity provides an equivariant quasi-isometry between (G, d_S) and (G, d_T). Note that this provides a hierarchically hyperbolic group structure on (G, d_T), where all the constants of the hierarchy axioms are the same, except the ones that involve distances in G. In particular, the only two such constants are K of Axiom 1, and the constant θ_u of Axiom 9.

Remark 4.4.6. We say a constant $k = k(\mathfrak{S})$ depends only on $(G, \mathfrak{S})$ when k is a function of the constants in the definition of the hierarchically hyperbolic structure on G excluding K and θ_u, a function which is independent of the generating set. Further, we will frequently refer to $D = \max\{\delta, \xi, \kappa_0, n, E\}$ as the *hierarchy constant*, which is independent of generating set.

Before moving on to understanding what constraints the group structure puts on a hierarchically hyperbolic space, we review why Equation (4.1) "resembles" a quasi-isometric embedding. To begin with, this map π is a coarse map, so we consider tuples in the product of power sets to make it an honest function.

4.4.2 Quasi-isometries from hierarchical coordinates

Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and consider the map

$$\pi : \mathcal{X} \to \prod_{U \in \mathfrak{S}} 2^{\mathscr{C}U} \tag{4.2}$$

given by sending each $x \in \mathcal{X}$ to the tuple $\{\pi_W(x)\}_{W \in \mathfrak{S}}$. The goal of this section is to prove the following criteria for being a quasi-product.

Proposition 4.4.7 (Quasi-product from orthogonality [ANS19, Proposition 2.27])**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $\bar{\mathfrak{S}}$ consist of all $W \in \mathfrak{S}$ such that $\mathscr{C}W$ has infinite diameter. Suppose that $\bar{\mathfrak{S}}$ can be partitioned as $\mathfrak{T}_1 \sqcup \cdots \sqcup \mathfrak{T}_n$ where $\mathfrak{T}_i \neq \emptyset$ for all i and every element of $\mathfrak{T}_i$ is orthogonal to every element of $\mathfrak{T}_j$ for $i \neq j$. Then there are infinite diameter metric spaces Y_i such that $\mathcal{X}$ is quasi-isometric to $Y_1 \times \cdots \times Y_n$.

This result can be deduced from discussions in [BHS19, Sections 3 & 5]; we restate it here, along with justification, for the sake of clarity and completeness. Along the way, we describe the space of consistent tuples that justifies how the function π (from Equation (4.2)) behaves like a quasi-isometric embedding.

First we mention a useful convention that allows us to assume that the hyperbolic spaces $\mathscr{C}U$ should not have any extraneous directions.

Definition 4.4.8 (Normalized hierarchically hyperbolic space [DHS17, Definition 1.15])**.** A hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ is *normalized* if there exists C such that for each $U \in \mathfrak{S}$ one has $\mathscr{C}U = N_C(\pi_U(\mathcal{X}))$, that is, π_U is C–coarsely surjective.

Convention. By [DHS17, Proposition 1.16], we can and will assume that all hierarchically hyperbolic spaces are normalized.

While the restriction $\pi_U : \mathcal{X} \to \mathscr{C}U$ can be assumed to be coarsely surjective, not every tuple in $\prod_{W \in \mathfrak{S}} 2^{\mathscr{C}W}$ needs be close to a point in the image.

Definition 4.4.9 (Consistent tuple space (Ω_κ) [BHS19, Definition 1.17])**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Fix $\kappa \geq 0$, and let $\vec{b} \in \prod_{W \in \mathfrak{S}} 2^{\mathscr{C}W}$ be a tuple such that for each $W \in \mathfrak{S}$, the coordinate b_W is a subset of $\mathscr{C}W$ of diameter at most κ. The tuple $\vec{b}$ is κ*–consistent* when

(1) $d_W(b_W, \pi_W(\mathcal{X})) \leq \kappa$ for all $W \in \mathfrak{S}$;

(2) $\min\left\{d_W(b_W, \rho^V_W), d_V(b_V, \rho^W_V)\right\} \leq \kappa$, whenever $V \pitchfork W$;

(3) $\min\left\{d_W(b_W, \rho^V_W, \operatorname{diam}_{\mathscr{C}V}(b_V \cup \rho^W_V(b_W))\right\} \leq \kappa$, whenever $V \sqsubseteq W$.

We denote by Ω_κ the subset of $\prod_{W \in \mathfrak{S}} 2^{\mathscr{C}W}$ consisting of κ–consistent tuples.

Remark 4.4.10. Note that for κ large enough, the first condition holds for any normalized hierarchically hyperbolic space.

Because we are assuming that $(\mathcal{X}, \mathfrak{S})$ is normalized, these exists a constant C such that all projections π_W are C–coarsely surjective. Thus, by setting $\kappa_1 = \max\{C, \kappa_0, \xi\}$, Axioms 1 and 4 of Definition 4.3.1 give that for each $\kappa \geq \kappa_1$, the map π has image in Ω_κ. The following theorem should be thought of as roughly saying that the projection π has a quasi-inverse.

Theorem 4.4.11 ([BHS19, Theorem 3.1])**.** For each $\kappa \geq 1$ there exist $\theta_e, \theta_u \geq 0$ such that the following holds. Let $\vec{b} \in \Omega_\kappa$ be a κ–consistent tuple, and for each W let b_W denote the $\mathcal{C}W$–coordinate of $\vec{b}$. Then the set $\Psi(\vec{b}) \subseteq \mathcal{X}$ defined as all $x \in \mathcal{X}$ so that $d_W(b_W, \pi_W(x)) \leq \theta_e$ for all $\mathcal{C}W \in \mathfrak{S}$ is non empty and has diameter at most θ_u.

The reason why "Ψ is a quasi-inverse of π" is not a precise statement is that we did not equip Ω_κ with a metric. The quasi-distance formula (Theorem 4.3.2) gives a constant s_0 such that for each $s \geq s_0$ we can equip Ω_κ with a map $f_s \colon \Omega_\kappa \times \Omega_\kappa \to \mathbb{R}$ defined as

$$f_s(\vec{a}, \vec{b}) = \sum_{W \in \mathfrak{S}} \{\!\{ d_{\mathcal{C}W}(a_W, b_W) \}\!\}_s \, ,$$

such that for every $x, y \in \mathcal{X}$, the quantities $f_s(\pi(x), \pi(y))$ and $d_{\mathcal{X}}(x, y)$ are comparable. However, note that the map f_s is not a distance: it does not satisfy the triangle inequality and there exists $\vec{a} \neq \vec{b}$ such that $f_s(\vec{a}, \vec{b}) = 0$. To remedy this, we equip Ω_κ with the subspace metric coming from Ψ, which we abuse notation and denote by $d_{\mathcal{X}}$.

The next ingredient in the proof of Proposition 4.4.7 is to show that one needs only focus on domains whose associated hyperbolic spaces have sufficiently large diameter. We first concern ourselves with subdividing $\mathfrak{S}$ into blocks. Let $\mathfrak{S}' \subseteq \mathfrak{S}$ be any subset. It is straightforward to see that the concept of consistent tuples (Definition 4.4.9) restricts to $\prod_{W\in\mathfrak{S}'} 2^{\mathscr{C}W}$. Let $\Omega_\kappa^{\mathfrak{S}'}$ be the set of κ–consistent tuples of $\prod_{W\in\mathfrak{S}'} 2^{\mathscr{C}W}$.

Notation 4.4.12 (Restriction to large diameter spaces)**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and suppose that a basepoint $x \in \mathcal{X}$ is fixed. For $C < \kappa_0$ consider the set $\mathfrak{S}_C$ consisting of all $W \in \mathfrak{S}$ such that $\operatorname{diam}(\mathscr{C}W) > C$. Given $\vec{a} \in \Omega_\kappa^{\mathfrak{S}_C}$ we define

$\Psi_{\mathfrak{S}_C}(\vec{a}) := \Psi(\vec{b})$, where $\vec{b}$ coincides with $\vec{a}$ on $\mathfrak{S}_C$ and

$b_U := \pi_U(x)$ for $U \in \mathfrak{S} - \mathfrak{S}_C$.

Remark 4.4.13. The choice of basepoint is not very important: the distance formula shows that the Hausdorff distance between the images of $\Psi_{\mathfrak{S}_C}$ under different choices of basepoints is bounded in terms of C. For this reason, we will suppress the dependence.

Lemma 4.4.14 (Ignoring small diameter spaces gives quasi-isometry [ANS19, Lemma 2.31])**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. For each

$0 \le C < \kappa$ the spaces Ω_κ and $\Omega_\kappa^{\mathfrak{S}_C}$ equipped with the subspace metric are quasi-isometric.

Proof. Setting $s > C$, the coordinates associated to the elements of $\mathfrak{S} - \mathfrak{S}_C$ do not contribute to the distance formula. Thus the conclusion follows. □

Lemma 4.4.14 is particularly useful when an HHS satisfies the *bounded domain dichotomy*, that is, when there exists C such that for each $U \in \mathfrak{S}$ either $\operatorname{diam}(\mathscr{C}U) \le C$ or $\operatorname{diam}(\mathscr{C}U) = \infty$. Notably, Remark 4.4.5 states that all HHGs satisfy the bounded domain dichotomy. The following corollary is immediate.

Corollary 4.4.15 ([ANS19, Corollary 2.32])**.** Let $\mathcal{X}$ be an HHS satisfying the bounded domain dichotomy, and let $\bar{\mathfrak{S}}$ consist of all $W \in \mathfrak{S}$ such that $\mathscr{C}W$ has infinite diameter. Then there is a constant $\kappa > 0$ such that $\Psi_{\bar{\mathfrak{S}}} \colon \Omega_\kappa^{\bar{\mathfrak{S}}} \to \mathcal{X}$ is coarsely surjective, and so $\Omega_\kappa^{\bar{\mathfrak{S}}}$ with the subspace metric is quasi-isometric to $\mathcal{X}$.

We can now prove Proposition 4.4.7.

Proof of Proposition 4.4.7. By assumption, $\bar{\mathfrak{S}}$ can be partitioned as $\mathfrak{T}_1 \sqcup \cdots \sqcup \mathfrak{T}_n$ where every element of $\mathfrak{T}_i$ is orthogonal to every element of $\mathfrak{T}_j$ for $i \neq j$. By definition of consistent tuples, the set $\Omega_\kappa^{\bar{\mathfrak{S}}}$ can be written as $\Omega_\kappa^{\mathfrak{T}_1} \times \cdots \times \Omega_\kappa^{\mathfrak{T}_n}$. Fix a basepoint $x \in \mathcal{X}$ and for each $\Omega_\kappa^{\mathfrak{T}_i}$ consider the map $\Psi_{\mathfrak{T}_i} \colon \Omega_\kappa^{\mathfrak{T}_i} \to \mathcal{X}$ defined by $\Psi_{\mathfrak{T}_i}(\vec{a}) = \Psi_{\bar{\mathfrak{S}}}(\vec{b})$, where $\vec{b}$ coincides with $\vec{a}$ on $\mathfrak{T}_i$ and is defined to be

$\pi_U(x)$ otherwise. Let Y_i denote the resulting metric space. The quasi-distance formula yields that $Y_1 \times \cdots \times Y_n$ is quasi-isometric to $\Psi_{\tilde{\mathfrak{S}}}\left(\Omega_\kappa^{\tilde{\mathfrak{S}}}\right)$. By Corollary 4.4.15, the latter coarsely coincides with $\mathcal{X}$. □

4.4.3 Classification of elements

In this section, we recall the classification elements of a hierarchical group from [DHS17] and related results.

Definition 4.4.16 (Big set)**.** The *big set* of an element is the collection of all domains onto whose associated hyperbolic spaces the orbit map is unbounded, that is, for an element $g \in \mathrm{Aut}(\mathfrak{S})$ and base point $x \in \mathcal{X}$ the *big set* is

$$\mathrm{Big}\,(g) = \{U \in \mathfrak{S} \mid \mathrm{diam}_{\mathscr{C}U}\left(\langle g \rangle .x\right) \text{ is unbounded}\}\,.$$

Note that this collection is independent of base point.

Remark 4.4.17. The elements of $\mathrm{Big}\,(g)$ must all be pairwise orthogonal. It follows immediately that $|\,\mathrm{Big}\,(g)\,|$ is uniformly bounded by the constant from Axiom 5 of Definition 4.3.1. For the rest of the paper, we denote this number by N.

Definition 4.4.18 (Axial and elliptic elements)**.** An automorphism of a hierarchically hyperbolic space is *elliptic* if it acts with bounded orbits on $\mathcal{X}$. It is *axial* if its orbit map induces a quasi-isometric embedding of a line in $\mathcal{X}$.

We note that because of the hierarchically hyperbolic group structure infinite cyclic subgroups cannot be distorted. This version of semisimplicity makes the classification of hierarchically hyperbolic group elements significantly easier to deal with than the more general setting of all hierarchical automorphisms.

Proposition 4.4.19 ([DHS17, Lemma 6.3, Proposition 6.4, & Theorem 7.1])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. Then there exists $M = M(\mathfrak{S})$ between 0 and $N!$ so that for all $g \in \operatorname{Aut}(\mathfrak{S})$ the following hold.

1. g is either elliptic or axial;
2. g is elliptic if and only if $\operatorname{Big}(g) = \emptyset$;
3. for every $U \in \operatorname{Big}(g)$, we have $(g^{\diamond})^M(U) = U$.

Remark 4.4.20. An element $g \in G$ is finite order if and only if $\operatorname{Big}(g) = \emptyset$ [AB18, Lemma 1.7]. Therefore, if G is a torsion-free HHG, then every element of G has a non-empty big set.

Given an infinite order element $g \in G$ and a domain $U \in \operatorname{Big}(g)$ such that $g.U = U$, we let $\tau_U(g)$ denote the stable translation length of g in this action.

Lemma 4.4.21 ([AB18, Lemma 1.8])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. There exists a constant $\tau_0 > 0$ such that for every infinite order element $g \in G$ and every $U \in \operatorname{Big}(g)$, we have $\tau_U(g^M) \geq M \cdot \tau_0$ where M is the constant from Proposition 4.4.19.

It will be important for us to pass to certain finite index subgroups while maintaining the hierarchical structure of the group. We do this with the following lemma.

Lemma 4.4.22 (Passing to finite index [ANS19, Lemma 2.24])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group, and let H be a finite index subgroup of G. Then $(H, \mathfrak{S})$ is a hierarchically hyperbolic group with the same hierarchical structure as G. Moreover, the property of being normalized is preserved under passing to finite-index subgroups.

Proof. Since G is hierarchically hyperbolic, we have an embedding $G \hookrightarrow \mathrm{Aut}(\mathfrak{S})$, and hence an embedding $H \hookrightarrow \mathrm{Aut}(\mathfrak{S})$. Since H is finite index in G, we have that H still acts on $\mathfrak{S}$ with finitely many orbits. Moreover, since H coarsely coincides with G, the uniform quasi-action of H on G is metrically proper and cobounded. This proves that H is an HHG.

Suppose that $(G, \mathfrak{S})$ is normalized. For each $U \in \mathfrak{S}$, the map $\pi_U \colon H \to \mathscr{C}U$ is defined as the restriction of $\pi_U \colon G \to \mathscr{C}U$. Since the latter is coarsely surjective by hypothesis, and since H coarsely coincides with G, we obtain that $\pi_U \colon H \to \mathscr{C}U$ is coarsely surjective, yielding that H is normalized. $\square$

The tools in this section allow us to study axial elements of a hierarchically hyperbolic groups by associating a collection of domains whose associated spaces have infinite diameter.

4.5 Structure of hierarchically hyperbolic groups

In this section, we give several structural results which will be useful in the proof of Theorem 1.2.8. Specifically, we focus our attention on collections of G-invariant domains and domains whose associated space is a quasi-line.

Lemma 4.5.1 (Invariant orthogonal domains have finite diameter containers [ANS19, Lemma 3.1])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. Suppose $\mathfrak{U}$ is a G–invariant collection of pairwise orthogonal domains such that $\mathscr{C}U$ has infinite diameter for each $U \in \mathfrak{U}$. If there exists a domain $V \notin \mathfrak{U}$ with $\operatorname{diam}(\mathscr{C}V) = \infty$, then for any $U \in \mathfrak{U}$, we have $U \not\sqsubseteq V$.

Proof. Suppose by way of contradiction that there exists a domain $U \in \mathfrak{U}$ such that $U \sqsubseteq V$. For each $W \in \mathfrak{U}$, fix any point $p_W \in \mathscr{C}W$, and let $p \in G$ be given by partial realization (Axiom 8 of Definition 4.3.1). Pick any $g \in G$ and consider the points $\pi_V(g)$ and $\pi_V(p)$. By the choice of p,

$$d_V(p, \rho_V^U) \leq \alpha.$$

Now apply the isometry $\phi_{pg^{-1}} \colon \mathscr{C}V \to \mathscr{C}(pg^{-1}V)$ induced by pg^{-1}. It follows that

$$d_{pg^{-1}V}(\phi_{pg^{-1}}(\pi_V(p)), \phi_{pg^{-1}}(\rho_V^U)) \leq \alpha.$$

Since $\phi_{pg^{-1}}(\rho_V^U)$ uniformly coarsely coincides with $\rho_{pg^{-1}V}^{pg^{-1}U}$, we have that $\phi_{pg^{-1}}(\pi_V(p))$ uniformly coarsely coincides with $\rho_{pg^{-1}V}^{pg^{-1}U}$.

As the action of G on $\mathfrak{S}$ fixes $\mathfrak{U}$ setwise, it follows that $pg^{-1}U \in \mathfrak{U}$. Moreover, $pg^{-1}U \sqsubseteq pg^{-1}V$. Thus, by using partial realization as above, we have that $\pi_{pg^{-1}V}(p)$ uniformly coarsely coincides with $\rho^{pg^{-1}U}_{pg^{-1}V}$, and so $\phi_{pg^{-1}}(\pi_V(p))$ uniformly coarsely coincides with $\pi_{pg^{-1}V}(p)$, as well. Moreover, $\pi_{pg^{-1}V}(p) = \pi_{pg^{-1}V}(pg^{-1}g)$, hence applying the inverse isometry $\phi_{gp^{-1}}$ shows that the distance between $\pi_V(p)$ and $\pi_V(g)$ is uniformly bounded. Since g was arbitrary and π_V is coarsely surjective, it follows that $\mathscr{C}V$ has finite diameter, which contradicts our assumption on V. □

We next show that any G–invariant domain whose associated hyperbolic space is a quasi-line that contains a loxodromic axis must be nest minimal. To do this we need the following transversality criterion.

Lemma 4.5.2 (Transversality criterion [ANS19, Lemma 2.16])**.** Let $U, W, V \in \mathfrak{S}$ be such that U and W properly nest into V. If $d_V(\rho^U_V, \rho^W_V) > 2D$, then $U \pitchfork W$.

Proof. If $U \sqsubseteq W$ or $W \sqsubseteq U$, then $d_V(\rho^U_V, \rho^W_V) \leq D$ by the transversality and consistency axiom, which contradicts our assumption. If $U \perp W$, then there is a partial realization point $x \in \mathcal{X}$ such that $d_V(x, \rho^U_V) \leq D$ and $d_V(x, \rho^W_V) \leq E$. It follows that $d_V(\rho^U_V, \rho^W_V) \leq 2D$, which contradicts our assumption. Therefore $U \pitchfork W$. □

Proposition 4.5.3 (Quasi-line domains are nest minimal [ANS19, Proposition 3.2]). Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group, and suppose there exists $U \in \mathfrak{S}$ such that $G.U = U$ and $\mathscr{C}U$ is Q–quasi-isometric to $\mathbb{R}$. If G contains an element acting by translation on $\mathscr{C}U$, then for all $V \sqsubsetneq U$, $\operatorname{diam}(CV) < \infty$.

Proof. We remark that since we are solely concerned with understanding the spaces $\mathscr{C}W$ for $W \in \mathfrak{S}$, we can fix an arbitrary generating set to work with for the proof of this proposition.

Let D be the hierarchy constant introduced in Remark 4.4.6, and let $\partial\mathscr{C}U = \{\alpha_+, \alpha_-\}$. The nesting axiom (Axiom 2) gives that $\operatorname{diam}_{\mathscr{C}U}(\rho^V_U) \leq D$. Because $\mathscr{C}U$ is a quasi-line, there is a constant $R_1 > 2D$ such that the neighborhood $N_{R_1}(\rho^V_U)$ disconnects $\mathscr{C}U$. Let A_+ and A_- be the two connected components of $\mathscr{C}U \smallsetminus N_{R_1}(\rho^V_U)$ containing α_+ and α_-, respectively, and let $A_\pm = A_+ \bigcup A_-$ denote their union. Since $\mathscr{C}U$ is a path connected Q–quasi-line by assumption, we have $\operatorname{diam}_{\mathscr{C}U}(\mathscr{C}U \smallsetminus (A_\pm)) \leq 2\,(Q^2R_1 + Q^2 + Q)$. Take $R_2 = \max\{D,\ 2(Q^2R_1 + Q^2 + Q)\}$. The bounded geodesic image axiom (Axiom 7) states that every geodesic segment in A_+ (respectively A_-) projects to $\mathscr{C}V$ with diameter at most D, and thus $\operatorname{diam}_{\mathscr{C}V}(\rho^U_V(A_+)) \leq 2D$ (respectively $\operatorname{diam}_{\mathscr{C}V}(\rho^U_V(A_-)) \leq 2D$).

The proof follows by contradiction using the following two claims, each relying on the assumption that there is a domain properly nested into U whose curve graph has infinite diameter.

Claim 1. If $V' \subsetneq U$ and $\mathscr{C}V'$ is unbounded, then for all $L > 0$ there is an unbounded domain $V \subsetneq U$ such that

$$d_{\mathscr{C}V}\left(\pi_V(1), \rho_V^U(A_\pm)\right) > L. \tag{4.3}$$

Claim 2. If $V \subsetneq U$ and $\mathscr{C}V$ is unbounded, then for all $L > 0$ there is an element $h \in G$ such that

$$d_{\mathscr{C}V}\left(\pi_V(h), \rho_V^U(A_\pm)\right) > L \qquad \text{and} \qquad d_{\mathscr{C}U}\left(\rho_U^{hV}, \rho_U^V\right) > L. \tag{4.4}$$

We complete the proof assuming the claims, which will be addressed later. Take $L > R_2 + D$. Lemma 4.5.2 and the second statement of (2) give that $V \pitchfork hV$. Since $G.U = U$, every element $g \in G$ acts on $\mathscr{C}U$ by isometries. Using Claim 2 we get $\operatorname{diam}_{\mathscr{C}U}(\mathscr{C}U \smallsetminus A_\pm) = \operatorname{diam}_{\mathscr{C}U}(\mathscr{C}U \smallsetminus hA_\pm) \leq R_2$ and consequently that $\rho_U^{hV} \subset A_\pm$ and $\rho_U^V \subset hA_\pm$. The coarse commutativity of hieromorphisms (Definition 4.4.1) applied to Claim 1 yields

$$d_{\mathscr{C}hV}\left(\pi_{hV}(h), \rho_{hV}^U(hA_\pm)\right) > L - D > R_2.$$

It thus follows that

$$d_{\mathscr{C}V}\left(\pi_V(h), \rho_V^{hV}\right) \geq d_{\mathscr{C}V}\left(\pi_V(h), \rho_V^U(A_\pm)\right) > R_2 \geq D$$

$$d_{\mathscr{C}hV}\left(\pi_{hV}(h), \rho_{hV}^V\right) \geq d_{\mathscr{C}hV}\left(\pi_{hV}(h), \rho_{hV}^U(A_\pm)\right) > R_2 \geq D.$$

The above inequalities, however, contradict the transversality axiom (Axiom 4) applied to h projected to V and hV, which states that

$$\min\left\{d_{\mathscr{C}V}\left(\pi_V(h), \rho_V^{hV}\right), d_{\mathscr{C}hV}\left(\pi_{hV}(h), \rho_{hV}^V\right)\right\} \leq D.$$

It remains to prove the two claims.

Proof of Claim 1. Let $L > 0$ be fixed. Let $A'_\pm$ be the neighborhoods of α_+ and α_- defined with respect to V'. If $d_{\mathscr{C}V'}\left(\pi_{V'}(1), \rho^U_{V'}(A'_\pm)\right) > L$ then we are done by taking $V = V'$. Otherwise, $d_{\mathscr{C}V'}\left(\pi_{V'}(1), \rho^U_{V'}(A'_\pm)\right) \leq L$. Since $\rho^U_V(A'_\pm)$ is bounded and $\pi_{V'}$ is D–coarsely surjective there is an element $g^{-1} \in G$ so that $d_{\mathscr{C}V'}\left(\pi_{V'}(g^{-1}), \rho^U_{V'}(A_\pm)\right) > L + D$. By equivariance, we can apply g to get $d_{\mathscr{C}gV'}\left(\pi_{gV'}(1), \rho^U_{gV'}(gA'_\pm)\right) > L + D$. Taking $V = gV'$ completes the claim.

Proof of Claim 2. Let $L > D$ be fixed, exceeding the hierarchy constant, and $t \in G$ be an element acting by translation on $\mathscr{C}U$, which exists by assumption. Let γ be any isometry of $\mathscr{C}U$ that fixes the endpoints and moves some point $x_0 \in \mathscr{C}U$ less than L. Then there is a constant $\bar{L} \geq L$ depending only on the quasi-line constants of $\mathscr{C}U$ (and not on the choice of γ) such that γ moves every point of $\mathscr{C}U$ by at most $\bar{L}$.

Let $\widehat{G} \leq G$ be the index 2 subgroup of G that fixes $\partial\mathscr{C}U$ pointwise. Note that t acts as translation, and so $t \in \widehat{G}$. Moreover, since G coarsely surjects onto $\mathscr{C}U$, so does $\widehat{G}$. Pick $M > 0$ so that $M\tau_0 > 2\bar{L} + D$, where τ_0 is as in Lemma 4.4.21. As before, coarse surjectivity guarantees the existence of an element $h' \in \widehat{G}$ satisifying

$$d_{\mathscr{C}U}(\pi_V(h'), \rho^U_V(A_\pm)) > \bar{L} + KM|t| + K,$$

where π_V is K–coarsely Lipschitz and $|t|$ is the word length of t in the fixed generating set. If $d_{\mathscr{C}U}(\rho_U^{h'V}, \rho_U^V) > L$ then we are done by taking $h = h'$, so assume $d_{\mathscr{C}U}(\rho_U^{h'V}, \rho_U^V) \leq L$.

Consider $h = h't^M$. Using the fact that π_V is Lipschitz and the triangle inequality, we have

$$\begin{aligned} d_{\mathscr{C}V}\left(\pi_V(h), \rho_V^U(A_\pm)\right) &\geq d_{\mathscr{C}V}\left(\pi_V(h'), \rho_V^U(A_\pm)\right) - d_{\mathscr{C}V}\left(\pi_V(h't^M), \pi_V(h')\right) \\ &\geq (\bar{L} + KM|t| + K) - (KM|t| + K) \\ &\geq \bar{L} \geq L. \end{aligned}$$

Thus the first statement of the claim holds. By the choice of $\bar{L}$, we have that $d_{\mathscr{C}U}(x, h'x) \leq \bar{L}$ for all $x \in \mathscr{C}U$. Thus

$$\begin{aligned} d_{\mathscr{C}U}\left(\rho_U^V, \rho_U^{hV}\right) &\geq d_{\mathscr{C}U}\left(\rho_U^V, h't^M\rho_U^V\right) - D \\ &\geq d_{\mathscr{C}U}\left(\rho_U^V, t^M\rho_U^V\right) - d_{\mathscr{C}U}\left(t^M\rho_U^V, h't^M\rho_U^V\right) - D \\ &\geq (2\bar{L} + D) - \bar{L} - D \\ &\geq \bar{L} \geq L, \end{aligned}$$

completing the proof. □

Next, we give a sufficient condition for a collection of pairwise orthogonal domains to be quasi-lines.

Proposition 4.5.4 (Orthogonal quasi-lines [ANS19, Proposition 3.3])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and suppose that there is a collection $\{U_1, \ldots, U_n\}$ of pairwise orthogonal domains such that for each U_i there is a pair of points $\alpha_i, \beta_i \in \partial\mathscr{C}U_i$ which is preserved by all generators. Then all the $\mathscr{C}U_i$ are uniformly quasi-lines.

Proof. We assume that all the generators fix each U_i. Fix some i, and to simplify notation we set $U = U_i$. Let γ be a geodesic between the points $\alpha, \beta \in \partial\mathscr{C}U$, and let $h \in G$. We want to uniformly bound $d_U(h, \gamma)$. Since there exists $C = C(\mathfrak{S})$ such that π_U is C–coarsely surjective, this would prove the result. Let $g \in G$ be such that $d_U(g, \gamma) \leq C$, and consider $hg^{-1}\gamma$. Since all the generators fix $\alpha, \beta \in \partial\mathscr{C}U$, we have that $hg^{-1}\gamma$ is a geodesic of $\mathscr{C}U$ with the same endpoints as γ. By the hyperbolicity of $\mathscr{C}U$, the Hausdorff distance between γ and $hg^{-1}\gamma$ is uniformly bounded. Moreover, by equivariance of the map π_U we have $d_U(h, hg^{-1}\gamma) = d_U(g, \gamma) \leq C$, which implies that $d_U(h, \gamma)$ is uniformly bounded, concluding the proof. □

We end this section by describing domains which are transverse to a G–invariant domain whose associated hyperbolic space has infinite diameter.

Proposition 4.5.5 (Invariant domains transverse to bounded diameter [ANS19, Proposition 3.4])**.** Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and suppose there is a G–invariant domain $U \in \mathfrak{S}$ such that $\operatorname{diam}(\mathscr{C}U) = \infty$. For any $W \in \mathfrak{S}$ satisfying $W \pitchfork U$, the space $\mathscr{C}W$ has uniformly bounded diameter.

Proof. Let $\Omega_\kappa \subset \Pi_{W \in \mathfrak{S}} 2^{\mathscr{C}W}$ and $\Phi \colon \Omega_\kappa \to 2^{\mathcal{X}}$ be as in Section 4.4.2. Let $\kappa \geq \kappa_1$ and let Y be the subset of Ω_κ consisting of all tuples whose W–coordinate is

ρ_W^U for each $W \pitchfork U$. Since $\mathscr{C}U$ has infinite diameter, $\Phi(Y)$ is an infinite diameter subset of G. Moreover, since U is G–invariant, so are Y and $\Phi(Y)$. Since G acts coboundedly on itself, we have that $\Phi(Y)$ coarsely coincides with G. Since Φ is a quasi-isometry, we conclude that Y coarsely coincides with Ω_κ. Thus, the spaces $\mathscr{C}W$ are uniformly bounded for every $W \pitchfork U$. □

4.6 Building free subgroups of hierarchically hyperbolic groups

In this section, we use the structure of big sets of infinite order hierarchical automorphisms developed in Section 4.4 to determine when we can apply the ping-pong lemma and show uniform exponential growth in Theorem 1.2.8. Our proof follows a similar outline to work of Mangahas for the mapping class group [Man10].

Mangahas showed that any finitely generated subgroup of the mapping class group is either virtually abelian or contains an N–short free subgroup where N is a function of the complexity of the surface [Man10]. The first step in her proof is to pass to a particular finite index subgroup that is torsion-free. From this point, she needs only consider rank-2 subgroups generated by pseudo-Anosov or reducible elements by the Nielsen–Thurston classification [Thu88, Theorem 4]. The pseudo-Anosov case is immediate from [Fuj15], and the reducible case requires a careful analysis of subsurface projection and the Behrstock inequality. The Behrstock inequality is precisely the transvserality inequality from the the proof that the mapping class group is a hierarchically hyperbolic space.

In the more general setting of hierarchically hyperbolic groups, one needs to handle certain difficult behavior not present in the action of the mapping class group on the hierarchy of curve graphs. In particular, a general hierarchically hyperbolic group does not contain a *pure subgroup* (in the sense of [Iva92]), that is, a finite index torsion-free subgroup such that every element stabilizes some collection of subsurfaces on which it acts either trivially or loxodromically on the associated curve graph. Indiscrete BMW groups (see Definition 3.2.3) give one class of examples of such phenomena. Indeed, Caprace, Kropholler, Reid, and Wesolek [CKRW19, Corollary 32(i), (iv)] show that in these groups every finite index subgroup contains infinite order elements which are non-trivial elliptic isometries with respect to the action on one of the tree factors.

The goal of this section is to prove Theorem 1.2.8. This follows immediately from Proposition 2.7.7 and the following stronger theorem.

Theorem 4.6.1 (Uniform length free semigroup basis [ANS19, Theorem 4.1])**.** Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group. Then there exists a constant $M > 0$ depending only on $(G, \mathfrak{S})$ such that one of the following occurs.

(1) G contains uniformly M-short free semigroups.

(2) G is virtually abelian.

(3) There is a G–invariant collection $\overline{\mathcal{B}}$ of pairwise orthogonal domains such that G is quasi-isometric to $\mathbb{Z}^{|\overline{\mathcal{B}}|} \times E$, where E is a non-elementary space. Moreover, G has a generating set all of whose members act elliptically on E.

Let us temporarily fix a torsion-free hierarchically hyperbolic group $(G, \mathfrak{S})$ and finite generating set S for G, with the convention that S contains the identity. Recall that N is the maximal number of pairwise orthogonal domains of H. Let

$$\mathcal{B} = \bigcup_{s \in S} \operatorname{Big}(s)$$

be the collection of domains onto whose associated hyperbolic spaces the axes of the generators have unbounded projection and let

$$\bar{\mathcal{B}} = S^N . \mathcal{B} \tag{4.5}$$

be the set of images of these domains under words of length at most N. Note that since S is finite and $|\operatorname{Big}(s)| \leq N$ for all $s \in S$, it is always the case that $\overline{\mathcal{B}}$ is a finite set. Moreover, since G is torsion-free, $\operatorname{Big}(s)$ is non-empty for every $s \in S$ (see Remark 4.4.20), and therefore $\mathcal{B} \neq \varnothing$.

The proof of Theorem 4.6.1 will be divided into two main cases using the following proposition.

Proposition 4.6.2 (Big orthogonality witnesses invariant domains [ANS19, Proposition 4.2])**.** Let $(G, \mathfrak{S})$ be a torsion-free hierarchically hyperbolic group, S a finite generating set for G containing the identity, and N the maximal number of pairwise orthogonal domains. Then one of the following holds.

(1) There are elements $s, t \in S^{2N+1}$ such that $\operatorname{Big}(s)$ and $\operatorname{Big}(t)$ contain two non-orthogonal elements;

(2) The set $\bar{\mathscr{B}}$ defined in (4.5) is a finite collection of pairwise orthogonal domains stabilized by G in the action on $\mathfrak{S}$.

Moreover, if (2) holds, then there is a finite index subgroup, $\widehat{G} \leq G$, of index at most $N!$ that fixes $\bar{\mathscr{B}}$ pointwise.

Proof. Suppose that (2) does not hold. Then either $\overline{\mathscr{B}}$ contains two non-orthogonal elements or $\overline{\mathscr{B}}$ is not a G–invariant set. Suppose that $\overline{\mathscr{B}}$ is not G–invariant. Then S does not fix $\overline{\mathscr{B}} = S^N.\mathscr{B}$ setwise, and thus S does not fix $S^k.\mathscr{B}$ setwise for any $1 \leq k \leq N$. Hence for each $1 \leq k \leq N$,

$$S^k.\mathscr{B} \neq S^{k+1}.\mathscr{B}.$$

Since the identity is contained in S, we have

$$S^k.\mathscr{B} \subseteq S^{k+1}.\mathscr{B}.$$

In particular, since $\mathscr{B} \neq \emptyset$, this implies that $|S^N.\mathscr{B}| \geq N+1$. However, this is a contradiction, as there can be at most N pairwise orthogonal elements. We conclude that if (2) does not hold, then there must be non-orthogonal domains $V_1, V_2 \in \overline{\mathcal{B}}$. Thus for $i = 1, 2$ there are generators $s_i \in S$, domains $U_i \in \mathrm{Big}\,(s_i)$ and elements $g_i \in S^N$ such that

$$V_i = g_i U_i.$$

This implies that $V_i \in \mathrm{Big}\left(g_i s_i g_i^{-1}\right)$. If we denote the word length with respect to the generating set S by $|\cdot|_S$, we have

$$|g_i s_i g_i^{-1}|_S \leq |g_i|_S + |s_i|_S + |g_i^{-1}|_S \leq 2N + 1,$$

and (1) follows by setting $s = g_1 s_1 g_1^{-1}$ and $t = g_2 s_2 g_2^{-1}$.

Finally, suppose (2) holds, that is, suppose that $\bar{\mathcal{B}}$ is a finite collection of pairwise orthogonal domains stabilized by G in the action on $\mathfrak{S}$. By definition of N, we have $|\bar{\mathcal{B}}| \leq N$. This induces a map to the symmetric group $G \to \mathrm{Sym}(N)$ whose kernel is a subgroup of G of index at most $N!$ fixing $\bar{\mathcal{B}}$ pointwise, which establishes the final statement of the proposition. □

We address the two cases of Proposition 4.6.2 in separate subsections.

4.6.1 Case 1: Big transversality or nesting

Assume that (1) of Proposition 4.6.2 holds, that is, there exist elements $s, t \in S^{2N+1}$ and domains $U \in \mathrm{Big}\,(s)$ and $V \in \mathrm{Big}\,(t)$ such that $U \not\perp V$. There are two possibilities in this case: either $U \pitchfork V$ or $U \subsetneq V$ (the case $V \subsetneq U$ is completely analogous). We deal with each possibility in a separate proposition and will demonstrate that in each case there are uniform powers of s and t which generate a free subgroup.

Proposition 4.6.3 (Big transversality witnesses short free subgroup [ANS19, Proposition 4.3])**.** Let s and t by axial hierarchical automorphisms with domains $U \in \operatorname{Big}(s)$ and $V \in \operatorname{Big}(t)$ such that $U \pitchfork V$. There exists a constant $k_1 = k_1(\mathfrak{S})$ such that $\left\langle s^{k_1}, t^{k_1} \right\rangle \cong F_2$.

Proof. By passing to a uniform power $(2N+1)!$, we may assume that $\operatorname{Big}(s)$ and $\operatorname{Big}(t)$ are fixed pointwise by s and t, respectively.

Let κ_0 be the constant from transversality (Axiom 4 of Definition 4.3.1). We will apply the ping-pong lemma to the following subsets of G:

$$Y_s = \left\{x \in G : d_U\left(\pi_U(x), \rho_U^V\right) > \kappa_0\right\} \text{ and } Y_t = \left\{x \in G : d_V\left(\pi_V(x), \rho_V^U\right) > \kappa_0\right\}.$$

The transversality and consistency inequalities (Axiom 4) imply that these sets are disjoint. Note that for all $W, T \in \mathfrak{S}$, the projection map $\pi_W : G \to \mathscr{C}W$ is coarsely surjective and ρ_W^T is a bounded subset of $\mathscr{C}W$ whenever $T \pitchfork W$. Since $\mathscr{C}U$ and $\mathscr{C}V$ are infinite diameter, this implies that Y_s and Y_t are non-empty.

Let τ_0 be the minimal translation length from Lemma 4.4.21. Fix a constant $k \geq 2\kappa_0\tau_0^{-1}$ and a point $x \in Y_s$. By transversality and consistency, we have $d_V(x, \rho_V^U) \leq \kappa_0$. Using this fact in addition to Lemma 4.4.21 and the triangle inequality, we have

$$\begin{aligned} d_V\left(\rho_V^U, t^{k(2N+1)!}.x\right) &\geq d_V\left(x, t^{k(2N+1)!}.x\right) - d_V\left(x, \rho_V^U\right) \\ &\geq \tau_0 \left|k\right| - d_V\left(x, \rho_V^U\right) \\ &\geq 2\kappa_0 - \kappa_0 \\ &= \kappa_0 \end{aligned}$$

Thus $t^{k(2N+1)!}.x \in Y_t$, and so $t^{k(2N+1)!}(Y_s) \subseteq Y_t$. By a symmetric argument, it follows that $s^{k(2N+1)!}(Y_t) \subseteq Y_s$. Thus, by the ping-pong lemma $\left\langle s^{k(2N+1)!}, t^{k(2N+1)!}\right\rangle \cong F_2$. Setting $k_1 = 2\kappa_0\tau_0^{-1}(2N+1)!$ completes the proof. $\square$

We note that in the previous proposition (and in many of the later results), if we allow s and t to have different exponents, then we can find smaller constants $k_{1,s}$ and $k_{1,t}$ such that $\left\langle s^{k_{1,s}}, t^{k_{1,t}}\right\rangle \cong \mathbb{F}_2$. In particular, we may take $k_{1,s} = 2\kappa_0\tau_0^{-1}m_s$ and $k_{1,t} = 2\kappa_0\tau_0^{-1}m_t$, for some $m_s, m_t \leq N$. Also, the stabilization power $(2N+1)!$ is not optimal since it is given by the kernel of a map from a copy of $\mathbb{Z}$ to a cyclic subgroup of $\mathrm{Sym}(2N+1)$, which can have size at most $\mathrm{LCM}(1, 2, \ldots, 2N+1)$, which grows slower than factorial. For ease of notation, however, we choose to use the larger uniform exponent.

We now turn to the second possibility in Case 1.

Proposition 4.6.4 (Nesting of big sets witnesses transversality [ANS19, Proposition 4.4])**.** Let s and t be a pair of axial hierarchical automorphisms with domains $U \in \text{Big}(s)$ and $V \in \text{Big}(t)$ such that U is properly nested in V. Then there exist constants $k_2 = k_2(\mathfrak{S})$ and $n_0 = n_0(\mathfrak{S})$ such that $\left\langle s^{k_2}, t^{n_0} s^{k_2} t^{-n_0} \right\rangle \cong \mathbb{F}_2$.

Proof. Since U is properly nested in V, the projection ρ_V^U in $\mathscr{C}V$ satisfies $\text{diam}_{\mathscr{C}V}\left(\rho_V^U\right) \leq D$. Recall that $d_V(t^i.\rho_V^U, \rho_V^{t^i.U}) \leq \kappa_0$ for all i. By Lemma 4.4.21, there is a uniform power n_0 of t such that $d_V(\rho_V^{t^{n_0}.U}, \rho_V^U) \geq 10D$. In particular, we can take any $n_0 \geq 10D\tau_0^{-1}$. By Lemma 4.5.2, this implies that $(t^{n_0}.U) \pitchfork U$. Applying Proposition 4.6.3 to the pair $s, t^{n_0} s t^{-n_0}$ and replacing $2N + 1$ with $2n_0 + 1$ yields the desired constant k_2, which completes the proof. □

The following is immediate from Definition 4.3.1, Proposition 4.6.3, and Proposition 4.6.4 by taking $K = \max\{k_1, k_2 + 2n_0\}$.

Corollary 1.2.10 (Not orthogonal implies free)**.** Let $a, b \in G$ be a pair of distinct axial elements of a hierarchically hyperbolic group with domains $A \in \text{Big}(a)$ and $B \in \text{Big}(b)$ such that $A \neq B$ and A and B are not orthogonal. Then there exists a constant $k = k(\mathfrak{S})$ such that $\langle a, b \rangle$ contains a k–short free subgroup.

4.6.2 Case 2: Big orthogonality

Recall that

$$\mathscr{B} = \bigcup_{s \in S} \mathrm{Big}\,(s)\,, \qquad \overline{\mathscr{B}} = S^N.\mathscr{B},$$

and

$$\widehat{G} = \ker\left(G \to \mathrm{Sym}(N)\right).$$

We now suppose that 2 of Proposition 4.6.2 holds, that is, $\overline{\mathscr{B}}$ is a finite collection of pairwise orthogonal domains which is stabilized by the action of G on $\mathfrak{S}$ and fixed pointwise by the action of $\widehat{G}$ on $\mathfrak{S}$. Recall that the index set contains a unique $\sqsubseteq$-maximal domain, which we denote by Σ.

Proposition 4.6.5 (Big top level witnesses short free subgroup)**.** Suppose $\mathscr{C}\Sigma$ has infinite diameter. Then either there exists a constant k_3 depending only on $(G, \mathfrak{S})$ and elements $s, t \in X$ such that $\langle s^{k_3}, ts^{k_3}t^{-1}\rangle \cong F_2$ or G is virtually cyclic.

Proof. Let $U \in \overline{\mathscr{B}}$. Then, by definition, there exists $h \in G$ with $|h|_X \leq N$, a generator $x \in S$, and a domain $W \in \mathrm{Big}\,(x)$ such that $U = h.W$. As $\mathscr{C}W$ has infinite diameter and h acts as an isometry on the associated hyperbolic spaces, $\mathscr{C}U$ must have infinite diameter, as well.

$\mathscr{C}\Sigma$ has infinite diameter by assumption and $U \sqsubseteq \Sigma$ by maximality of Σ. It follows from Lemma 4.5.1 applied with $\mathfrak{U} = \overline{\mathscr{B}}$ that $\Sigma \in \overline{\mathscr{B}}$. By definition, $\Sigma \in \overline{\mathscr{B}}$ implies that $\Sigma = g.V$ for some $g \in G$ with $|g|_S \leq N$ and $V \in \mathrm{Big}\,(s)$

for some $s \in S$. However, hierarchical automorphisms preserve the $\sqsubseteq$–levels of elements of $\mathfrak{S}$, and Σ is the unique $\sqsubseteq$–maximal domain in $\mathfrak{S}$. Thus, $\Sigma = g.V$ if and only if V has the same level as Σ, and we conclude that $V = \Sigma$. This implies that $\Sigma \in \mathrm{Big}\,(s)$. (In fact, this implies that $\Sigma = \mathrm{Big}\,(s)$ by [DHS17, Lemma 6.7], but we will not need this stronger statement.)

The action of G on $\mathscr{C}\Sigma$ is cobounded and acylindrical by [BHS14, Corollary 14.4]. Let $E(s)$ denote the stabilizer of the endpoints of the axis of s in $\partial\mathscr{C}\Sigma$. If for every generator $r \in S$ we have $r \in E(s)$, then G is virtually cyclic by Lemma 3.6.7.

Otherwise, there exists a generator $t \in S \setminus \{s\}$ such that $t \notin E(s)$, hence, t does not stabilize the endpoints of the axis of s in $\partial\mathscr{C}\Sigma$. In particular, $|\partial\mathscr{C}\Sigma| \geq 3$, that is, $\mathscr{C}\Sigma$ is a non-elementary hyperbolic space.

By [DGO16, Corollary 6.6], $t \notin E(s)$ if and only if $ts^n t^{-1} \neq s^{\pm n}$ for any $n \neq 0$. Therefore, with the above choice of s and t, Proposition 4.2.1 produces a constant k_3 such that $\left\langle s^{k_3}, ts^{k_3}t^{-1} \right\rangle \cong F_2$. $\square$

In particular, the proof of Proposition 4.6.5 shows that whenever $\mathscr{C}\Sigma$ has infinite diameter there exist two uniformly short elements which are independent loxodromic elements with respect to the action on $\mathscr{C}\Sigma$.

We are now ready to prove Theorem 4.6.1.

Proof of Theorem 4.6.1. Consider the finite-index torsion-free subgroup H of G. Then $(H, \mathfrak{S})$ is a normalized HHG by Lemma 4.4.22, and by Lemma 2.7.9, given any generating set S for G there is a generating set for H all of whose elements have word length at most $2d-1$, where $d = [G : H]$. This means that if we can prove the desired trichotomy for H, it will follow for G.

Let k_1 be the constant from Proposition 4.6.3, k_2 and n_0 the constants from Proposition 4.6.4, k_3 the constant from Proposition 4.6.5, δ the hyperbolicity constant of $\mathscr{C}U$ for any $U \in \mathfrak{S}$, and τ_0 the constant from Lemma 4.4.21. Also let

$$k_4 = \left\lceil 10000\delta\tau_0^{-1} \right\rceil,$$

and

$$M \geq \max\{k_1, 2n_0 + k_2, k_3 + 2, 3(k_4 + 2)(N+1)!\}.$$

We recall that our goal is to show that one of the following occurs:

(a) G is virtually abelian;

(b) G contains an M–short free semigroup; or

(c) G is quasi-isometric to a product $\mathbb{Z} \times E$, where E has infinite diameter and is not quasi-isometric to $\mathbb{Z}^n$.

These are all preserved under passing to finite index subgroups (up to multiplying the uniform constant M by a function of the index). Thus, we can and will assume that G is torsion-free.

Let S be an arbitrary generating set for G. One of the two cases of Proposition 4.6.2 must occur. If the hypotheses of Case 1 are satisfied, then (b) holds by either Proposition 4.6.3 or Proposition 4.6.4. So, suppose Case 2 occurs and the set

$$\bar{\mathscr{B}} = S^N.\mathscr{B}$$

defined in (4.5) is fixed setwise by G.

If $\Sigma \in \bar{\mathscr{B}}$, then $\mathscr{C}\Sigma$ has infinite diameter, and so (a) or (b) holds by Proposition 4.6.5. If $\Sigma \notin \bar{\mathscr{B}}$, then $\operatorname{diam}(\mathscr{C}\Sigma) < \infty$ (in particular, it is uniformly bounded) by applying Lemma 4.5.1 with $\mathcal{U} = \overline{\mathscr{B}}$.

By passing to a further finite index subgroup, we can assume that $\bar{\mathscr{B}}$ is fixed pointwise by G. Indeed, consider the subgroup $\widehat{G} = \ker(G \to \operatorname{Sym}(\bar{\mathscr{B}}))$ of index at most $N!$ which fixes $\bar{\mathscr{B}}$ pointwise. As before, since $\widehat{G}$ is finite index, it is enough to prove the desired trichotomy for $\widehat{G}$. Let T' be a generating set for $\widehat{G}$ with $S-$ length at most $2N! - 1$. By definition, every domain $U \in \bar{\mathscr{B}}$ supports the axis of at least one element in X^{2N+1}. Observe also that, by Proposition 4.4.19 there is a constant K between 0 and $N!$ such that $g^K \in \widehat{G}$. Expand the generating set for $\widehat{G}$ to be

$$T = T' \bigcup \left\{ g^K : g \in S^{2N+1} \right\}.$$

Elements of T have S–length at most $(2N+1)N! < 3(N+1)!$. Since each domain of $\bar{\mathscr{B}}$ was in the big set of some element of S^{2N+1}, each domain is also in the big set of some element of T.

For the rest of the proof, we restrict our attention to $\widehat{G}$, which acts on $\mathscr{C}U$ for each $U \in \bar{\mathscr{B}}$. For each $U \in \bar{\mathscr{B}}$, there exists an element $s_U \in G$ with

$|s_U|_S \leq 2N+1$ that acts loxodromically on $\mathscr{C}U$. Thus $(s_U)^K \in \widehat{G}$ also acts loxodromically on $\mathscr{C}U$, and $|(s_U)^K|_T = 1$, by the definition of Y. Let $s_U^{\pm}$ be the fixed point of s_U^K on $\partial\mathscr{C}U$.

We claim that either (b) holds or all the generators fix $\{s_U^+, s_U^-\}$ setwise. Indeed, if t is an element of T that does not fix $\{s_U^+, s_U^-\}$, the conjugate $t^{-1}(s_U)^K t$ is an independent loxodromic with respect to the action on $\mathscr{C}U$. By Lemma 4.4.21 there is a uniform lower bound on the translation length of $(s_U)^K$ (which is equal to the translation length of $t^{-1}s_U^K t$) with respect to the action on $\mathscr{C}U$. Therefore, Proposition 4.2.3 implies that for k_4 defined as above, some pair in $\{(s_U)^{\pm k_4 K}, t^{-1}(s_U)^{\pm k_4 K} t\}$ generates a free semigroup, and hence (b) holds.

Thus, we may assume that for each $U \in \bar{\mathscr{B}}$, the set $\{s_U^+, s_U^-\}$ is $\widehat{G}$–invariant. By Proposition 4.5.4, we conclude that $\mathscr{C}U$ is a quasi-line for each $U \in \bar{\mathscr{B}}$. Let $\overline{\mathscr{B}} = \{U_1, \ldots, U_n\}$ for some n, and let $\bar{\mathfrak{S}} = \{V \in \mathfrak{S} \mid \operatorname{diam}(\mathscr{C}V) = \infty\}$. We claim that $W \perp U_i$ for each $W \in \overline{\mathfrak{S}} - \bar{\mathscr{B}}$ and for all i. To see this, suppose that $\mathscr{C}W$ is unbounded. Then Lemma 4.5.1 and Proposition 4.5.3 imply that for each i, either $W \perp U_i$ or $W \pitchfork U_i$. Since U_i is $\widehat{G}$–invariant, by Proposition 4.5.5, we must have $W \perp U_i$.

Thus, we can partition $\bar{\mathfrak{S}}$ into pairwise orthogonal sets as follows:

$$\bar{\mathfrak{S}} = \{U_1\} \sqcup \cdots \sqcup \{U_n\} \sqcup \left(\bar{\mathfrak{S}} - \bar{\mathscr{B}}\right).$$

Let $\Omega_\kappa^{\mathfrak{S}}$ be as in Section 4.4.2. By Proposition 4.4.7, we conclude that $\widehat{G}$

(and therefore G) is quasi-isometric to $\mathbb{Z}^{|\bar{\mathscr{B}}|} \times \Omega_\kappa^{\bar{\mathfrak{S}}-\bar{\mathscr{B}}}$. If $\Omega_\kappa^{\bar{\mathfrak{S}}-\bar{\mathscr{B}}}$ is quasi-isometric to $\mathbb{Z}^m$ for some m, then (a) holds. Otherwise, (c) holds with respect to the initial generating set, S, and $E = \Omega_\kappa^{\bar{\mathfrak{S}}-\bar{\mathscr{B}}}$, completing the proof. □

Theorem 1.2.8 is an immediate consequence of the proof of Theorem 4.6.1.

4.7 Alternate formulations and applications to uniform exponential growth

Digesting the definition of a hierarchically hyperbolic space (Definition 4.3.1) and the subsequent developments in Section 4.4, and Section 4.6 can be quite challenging without extensive familiarity with hierarchy machinery and coarse geometry. For this reason, we prove several corollaries that provide alternative characterizations of the conditions needed to show uniform exponential growth. At several stages in the proof of Theorem 4.6.1, we produce free subgroups rather than free semigroups. We close this section by giving two conditions under which we can guarantee that our group contains N–short free subgroups.

Our first corollary considers the situation when the asymptotic cone has a cut point.

Corollary 4.7.1 ([ANS19, Corollary 1.2]). Every non-virtually cyclic virtually torsion-free hierarchically hyperbolic group which has an asymptotic cone containing a cut-point has uniform exponential growth.

In particular, if the Cayley graph of a virtually torsion-free hierarchically hyperbolic group G contains an unbounded Morse quasi-geodesic, then G has uniform exponential growth.

Proof. Let G be a non-virtually cyclic virtually torsion-free hierarchically hyperbolic group. It follows from [DMS10, Proposition 1.1] that having a cut-point in an asymptotic cone of G is equivalent to G having super-linear divergence. However, this cannot occur if G is quasi-isometric to a product with unbounded factors, and therefore G has uniform exponential growth by Theorem 1.2.8.

The second statement follows from [DMS10, Proposition 3.24: (1) $\iff$ (2)], which show that if a geodesic metric space X has an unbounded Morse quasi-geodesic, then every asymptotic cone of X has a cut-point. □

Work of Sisto shows that every acylindrically hyperbolic group contains an infinite order Morse element, that is, an infinite order element g such that the quasi-geodesic $\langle g \rangle$ in the Cayley graph of G is Morse [Sis16], and thus we immediately obtain the following result.

Corollary 1.2.11 ([ANS19, Corollary 1.3])**.** Virtually torsion-free hierarchically hyperbolic groups which are acylindrically hyperbolic have uniform exponential growth.

Instead of quasi-geodesics, we may consider quasi-convex subgroups.

Corollary 4.7.2 ([ANS19, Corollary 1.4])**.** Every virtually torsion-free hierarchically hyperbolic group which is not virtually cyclic and contains an infinite quasi-convex subgroup of infinite index has uniform exponential growth.

Proof. Let G be a non-virtually cyclic virtually torsion-free hierarchically hyperbolic group, and let $H \leq G$ be an infinite quasi-convex subgroup of infinite index. If G is quasi-isometric to a product with unbounded factors, then either the inclusion map $H \hookrightarrow G$ is quasi-isometry or H has bounded diameter in the Cayley graph of G. In the first case, we reach a contradiction with the fact that H is infinite index, and in the second case we reach a contradiction with the fact that H is infinite. Then G has uniform exponential growth by Theorem 1.2.8. □

Rather than knowing all of the spaces and maps involved in a hierarchy, we can content ourselves with showing that the top level is associated to an infinite diameter space that is not a quasi-line.

Corollary 4.7.3. Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group such that $\mathscr{C}S$ is a non-elementary hyperbolic space. Then G has uniform exponential growth.

Proof. Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group such that $\mathscr{C}S$ is a non-elementary hyperbolic space. The result follows immediately from [DHS17, Theorem 9.14] and Proposition 4.6.5. □

We now turn our attention to free subgroups. Under the additional assumption that $(G, \mathfrak{S})$ is hierarchically acylindrical our proof of Theorem 4.6.1 can be adjusted to generate free subgroups rather than free semigroups. Hierarchical acylindricity was introduced by Durham, Hagen, and Sisto in [DHS17] to generalize the following property of mapping class groups: for any subsurface $W \subseteq \Sigma$, the subgroup $\mathrm{MCG}(W) \leq \mathrm{MCG}(\Sigma)$ acts acylindrically on domains corresponding to W.

To make this precise in the hirarchically hyperbolic setting, let

$$\mathrm{Stab}(U) = \{g \in G : g^{\diamond}U = U\}\,.$$

By the definition of hierarchically hyperbolic groups, $\mathrm{Stab}(U)$ acts on $\mathscr{C}U$. Let K_U be the kernel of the action, namely the subgroup $\{g \in \mathrm{Stab}\, U \mid g.x = x \quad \forall x \in \mathscr{C}U\}$.

Definition 4.7.4. A hierarchically hyperbolic group is *hierarchically acylindrical* if $\mathrm{Stab}(U)/K_U$ acts acylindrically on $\mathscr{C}U$, for all $U \in \mathfrak{S}$.

Mapping class groups are hierarchically acylindrical because reducible subgroups of the mapping class group act acylindrically on the curve graph corresponding to a subsurface. Similarly, all right-angled Artin groups are also hierarchically acylindrical because parabolic subgroups act acylindrically on the contact graph corresponding to the associated subgraph of the defining graph.

Remark 4.7.5. Not all hierarchically hyperbolic group structures are hierarchically acylindrical. The following example was observed in discussion with Sam Taylor. Let Γ be any BMW group (see Definition 3.2.3). The group Γ admits a hierarchically hyperbolic structure with three domain Σ, U, V. The space $\mathscr{C}\Sigma$ is a point, while $\mathscr{C}U$ and $\mathscr{C}V$ are trees both of which are orthogonal and nest properly into Σ. [BHS14]. The restriction of the action to each tree in the product, however, has trivial kernel and $\mathrm{Stab}(T) = \Gamma$. Γ is not acylindrically hyperbolic because it acts geometrically on a product of trees, so any action on a non-elementary hyperbolic space cannot be acylindrical. In particular, the restricted action of Γ on each tree cannot be acylindrical by Theorem 4.1.6. This example is also described in [DHS18].

Proposition 4.7.6. Let $(G, \mathfrak{S})$ be a virtually torsion-free hierarchically hyperbolic group such that G is not quasi-isometric to $\mathbb{Z} \times E$ for any metric space E. Suppose that either

(1) $\mathscr{C}\Sigma$ is non-elementary; or

(2) G is hierarchically acylindrical.

Then for any generating set S of G, there exists a free subgroup of G generated by two elements whose word length with respect to S is uniformly bounded.

Proof. Fix constants as in the proof of Theorem 4.6.1. The only time that free semigroups are produced in the proof of Theorem 4.6.1 is when 2 of Proposition 4.6.2 holds and $\mathscr{C}\Sigma$ is an elementary hyperbolic space. Equivalently, this occurs when two elements have independent axes in an infinite diameter domain that properly nests into Σ. In this case, we pass to a subgroup $\widehat{G}$ with finite generating set T which fixes $\overline{\mathscr{B}}$ pointwise, and find elements $s, t \in T$ such that s and $t^{-1}st$ are independent loxodromic isometries of $\mathscr{C}U$ for some $U \subsetneq \Sigma$. By hierarchical acylindricity, $\widehat{G}/K_U$ acts nonelementarily and acylindrically on $\mathscr{C}U$. Let $\bar{s}$ and $\bar{t}$ be the images of s and t in the quotient. Applying Proposition 4.2.1, there exists a constant k_5 such that $\left\langle \bar{s}^{k_5}, \bar{t}\bar{s}^{k_5}\bar{t}^{-1} \right\rangle \cong \mathbb{F}_2$ in $\widehat{G}/K_U$. Since free groups are Hopfian, this lifts to a free subgroup of $\widehat{G}$. In particular, the constant M in Theorem 4.6.1 can be updated to be

$$M = M \geq \max\{k_1, 2n_0 + k_2, k_3 + 2, 3(k_5 + 2)(N + 1)!\}.$$ □

Remark 4.7.7. The proof of Proposition 4.7.6 shows that the conclusion of Proposition 4.7.6 also holds in slightly more generality. In particular, it holds for any virtually torsion-free HHG in which 1 of Proposition 4.6.2 holds for every finite generating set S.

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