DEFORMATION QUANTIZATION OVER A Z-GRADED BASE

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Elif Altinay-Ozaslan
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Examining Committee Members:

Prof. Vasily A. Dolgushev, Advisory Chair, Dept. of Mathematics
Prof. Edward S. Letzter, TU Dept. of Mathematics
Prof. Martin W. Lorenz, TU Dept. of Mathematics
Prof. Mathieu Stienon, External Member, PSU Dept. of Mathematics

ABSTRACT

DEFORMATION QUANTIZATION OVER A Z-GRADED BASE

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We investigate the problem how to describe the equivalence classes of formal deformations of a symplectic manifold M in the case when we have several deformation parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_g$ of non-positive degrees. We define formal deformations of M over the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ as Maurer-Cartan elements of the differential graded Lie algebra $(\varepsilon, \varepsilon_1, \ldots, \varepsilon_g)\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ where $\mathsf{PD}^\bullet(M)$ denotes the algebra of polydifferential operators on M. The interesting feature of such deformations is that, if at least one formal parameter carries a non-zero degree, then the resulting Maurer-Cartan element corresponds to a $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$-multilinear A_∞-structure on the graded vector space $\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ with the zero differential, where $\mathcal{O}(M)$ is the algebra of smooth complex-valued functions M.

This dissertation focuses on formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ such that corresponding MC elements μ satisfy these two conditions: The Kodaira-Spencer class of μ is $\varepsilon\alpha$ and μ satisfies the equation $\mu|_{\varepsilon=0} = 0$. The main result of this study gives us a bijection between the set of isomorphism classes of such deformations and the set of all degree 2 vectors of the graded vector space $\bigoplus_{q\ge 0} (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g)\, H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ where $H^\bullet(M, \mathbb{C})$ is the singular cohomology of M with coefficients in $\mathbb{C}$ and every vector of $H^q(M, \mathbb{C})$ carries degree q.

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To my parents, Kamil and Yildiz

And my husband, Salih Tolga

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CHAPTER 1

INTRODUCTION

Let M be a real manifold M, $\mathcal{O}(M)$ be the algebra of smooth complex-valued functions on M, and $\varepsilon, \varepsilon_1, \dots, \varepsilon_g$ be formal variables of degrees

$$\deg(\varepsilon) = 0\,, \quad \deg(\varepsilon_1) = d_1, \quad \deg(\varepsilon_2) = d_2, \;\; \dots, \;\deg(\varepsilon_g) = d_g\,, \tag{1.0.1}$$

where $d_1, d_2, \dots, d_g$ are non-positive integers.

In this dissertation, we investigate the problem of deformation quantization [3], [4], [5], [6], [8], [15], [24] of M in the setting when we have several formal deformation parameters $\varepsilon, \varepsilon_1, \dots, \varepsilon_g$ and some of the (non-positive) integers $d_1, d_2, \dots, d_g$ are actually non-zero.

A formal deformation of $\mathcal{O}(M)$, in this setting, is a (non-curved) $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$-linear A_∞-structure on $\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$ with the multiplications $\{\mathsf{m}_n\}_{n\geq 2}$ of the form

$$\mathsf{m}_n(a_1, \dots, a_n) = \begin{cases} a_1 a_2 + \displaystyle\sum_{k_0 d_0 + \dots + k_g d_g = 0} \varepsilon^{k_0} \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g} \mu_{k_0, k_1, \dots, k_g}(a_1, a_2) & \text{if } \; n = 2, \\ \displaystyle\sum_{k_0 d_0 + \dots + k_g d_g = 2-n} \varepsilon^{k_0} \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g} \mu_{k_0, k_1, \dots, k_g}(a_1, \dots, a_n) & \text{if } \; n > 2, \end{cases} \tag{1.0.2}$$

where each $\mu_{k_0, k_1, \dots, k_g}$ is a polydifferential operator on M (with complex coefficients) acting on $2 - k_0 d_0 - \dots - k_g d_g$ arguments. Moreover, $\mu_{k_0, k_1, \dots, k_g} \equiv 0$ if at least one $k_i < 0$ or $k_0 + k_1 + \dots + k_g = 0$.

Such A_∞-structures are in bijection with Maurer-Cartan (MC) elements of the dg Lie algebra

$$(\varepsilon, \varepsilon_1, \dots, \varepsilon_g)\,\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]], \tag{1.0.3}$$

where $\mathsf{PD}^\bullet(M)$ denotes the algebra of polydifferential operators[1] on M.

Let us denote by $\mathfrak{G}$ the group which is obtained by exponentiating the Lie algebra of degree zero elements of (1.0.3) and recall [7], [18] that this group acts in a natural way on the set of MC elements of (1.0.3).

By analogy with 1-parameter ("ungraded") formal deformations, we declare that two such formal deformations are equivalent if the corresponding MC elements of (1.0.3) belong to the same orbit of the action of $\mathfrak{G}$.

Let us observe that, for every MC element μ of (1.0.3), the coset of μ in

$$(\varepsilon, \varepsilon_1, \dots, \varepsilon_g)\,\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \;/\; (\varepsilon, \varepsilon_1, \dots, \varepsilon_g)^2\,\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \tag{1.0.4}$$

is closed with respect to the Hochschild differential ∂^{Hoch}. Moreover, if two MC elements μ_1 and μ_2 lie on the same orbit of $\mathfrak{G}$, then the corresponding cosets in (1.0.4) are ∂^{Hoch}-cohomologous. By analogy with "ungraded" formal deformations, we call the ∂^{Hoch}-cohomology class of the coset of μ in (1.0.4) the Kodaira-Spencer class of μ.

Let us recall that the cohomology space of $\mathsf{PD}^\bullet(M)$ (with respect to ∂^{Hoch}) is isomorphic to the space $\mathsf{PV}^\bullet(M)$ of polyvector fields on M. So the Kodaira-Spencer class of every MC element of (1.0.3) can be identified with a degree 1 vector in the graded space

$$\varepsilon\mathsf{PV}^\bullet(M) \oplus \varepsilon_1\mathsf{PV}^\bullet(M) \oplus \dots \oplus \varepsilon_g\mathsf{PV}^\bullet(M).$$

Let us now assume that M has a symplectic structure[2] ω and denote by $\alpha \in \mathsf{PV}^1(M)$ the Poisson structure corresponding to ω.

In this dissertation, we consider formal deformations (1.0.2) of $\mathcal{O}(M)$ which satisfy these two conditions:

[1]See Section 2.5.

[2]In particular, it means that the dimension of M is even.

1. the Kodaira-Spencer class of this deformation is $\varepsilon\,\alpha$ and

2. $$\mathsf{m}_n\big|_{\varepsilon=0} = \begin{cases} a_1 a_2 & \text{if} \quad n=2, \\ 0 & \text{if} \quad n>2. \end{cases}$$

We denote by TL the set of equivalence classes of formal deformations (1.0.2) satisfying the above conditions and call TL the *topological locus* of the triple $(M, \omega, \{\varepsilon, \varepsilon_1, \dots, \varepsilon_g\})$. Using Kontsevich's formality [24] and a construction inspired by paper [32] due to Sharygin and Talalaev we give a description of the topological locus TL in terms of the singular cohomology of M. More precisely,

Theorem 1.0.1 *For every symplectic manifold (M,ω), the equivalence classes of formal deformations (1.0.2) of $\mathcal{O}(M)$ satisfying the above conditions are in bijections with degree 2 vectors of the graded vector space*

$$\bigoplus_{q\ge 0} (\varepsilon, \varepsilon_1, \dots, \varepsilon_g)\, H^q(M,\mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]],$$

where $H^\bullet(M,\mathbb{C})$ is the singular cohomology of M with coefficients in $\mathbb{C}$ and every vector of $H^q(M,\mathbb{C})$ carries degree q.

Remark 1.0.2 In the "ungraded" case (i.e. $g=0$), this result reproduces the classical theorem [6], [8] of Bertelson, Deligne, Cahen, and Gutt on the description of the equivalence classes of star products on a symplectic manifold. In this respect, Theorem 1.0.1 may be viewed as a generalization of this classification theorem to the case of a $\mathbb{Z}$-graded base.

Remark 1.0.3 Deformations over a $\mathbb{Z}$-graded (and even differential graded) base were considered in the literature. See, for example, paper [2] by Barannikov and Kontsevich or J. Lurie's ICM address [30] in which even more sophisticated examples of bases for deformation problems were considered. We should also mention that deformations over a differential graded base naturally show up in the construction of rational homotopy models for classifying spaces of fibrations. For more details, we refer the reader to paper [28] by Lazarev.

The organization of this dissertation is as follows.

The first chapter is devoted to the preliminaries. At first, we give basic notation and conventions we use throughout the thesis. Here, we also recall algebraic structures on Hochschild complexes of an associative algebra. In the second section, we recall the basic tools of differential graded (dg) Lie algebras. We give a review of Maurer-Cartan (MC) elements of dg Lie algebras, Goldman-Millson groupoid and Deligne-Getzler-Hinich ∞-groupoid. In Section 2.3, we recall some required notions of L_∞-morphisms and L_∞-quasi-isomorphisms of dg Lie algebras. We also introduce the twisting procedure to obtain a new dg Lie algebra structures and L_∞-morphisms via a MC element. Section 2.4 is a reminder of 1-parameter formal deformations of an associative algebra A (over $\mathbb{C}$). We describe the formal deformations of A as MC elements of the dg Lie algebra $C^\bullet[[\varepsilon]](A)$ of shifted Hochschild cochains. In the last section of this chapter, we review the sheaf $\mathsf{PD}^\bullet$ of polydifferential operators and the sheaf $\mathsf{PV}^\bullet$ of polyvector fields on a smooth manifold M and recall Kontsevich's Formality Theorem for M in Theorem 2.5.3.

Chapter 3 is the main part of this dissertation. We begin by proposing a natural generalization of the 1-parameter formal deformations to the case when the base ring of the deformation problem is the completion of the free graded[3] commutative algebra $\mathbb{C}[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]$. Next, we consider the case when A is the algebra of functions $\mathcal{O}(M)$ on a smooth real manifold M. Finally, we assume that M has a symplectic structure and formulate the main result of this thesis (see Theorem 3.1.5).

The proof of Theorem 3.1.5 is given in Section 3.4 and it is based on two auxiliary constructions which are presented in Sections 3.2 and 3.3, respectively. In Section 3.2, by applying Kontsevich's Formality Theorem, we construct a bijection between the topological locus TL and the set of isomorphism

[3] $\mathbb{C}[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]$ should not be confused with the polynomial algebra because $\varepsilon_i \varepsilon_j = (-1)^{d_i d_j} \varepsilon_j \varepsilon_i$.

classes of MC elements of a dg Lie algebra which is obtained by twisting the extended dg Lie algebra of polyvector fields on M. In Section 3.3, we describe this set in terms of the singular cohomology of M with the help of a degree zero coderivation Π of the cocommutative coalgebra $\underline{S}(\mathbf{s}^{-1}\,\tilde{\mathcal{L}}_{\Omega})$.

In the last chapter, we propose two conjectures related to Theorem 3.1.5.

Appendix A contains the proof of a very useful lemma on a quasi-isomorphism between filtered complexes which we use in the proof of Claim 3.2.3.

Finally, Appendix B is devoted to the proof of the technical Proposition 3.3.3.

This dissertation is based on joint paper [1] with my advisor.

CHAPTER 2

PRELIMINARIES

In this chapter, we firstly introduce notation and conventions which will be used throughout this thesis. We give a review of Maurer-Cartan elements of a differential graded Lie algebra, Goldman-Millson groupoid and Deligne-Getzler-Hinich ∞-groupoid in Section 2.2. Next, we recall L_∞-morphisms of differential graded Lie algebras and twisting procedure via a Maurer-Cartan element. In Section 2.4 we review 1-parameter formal deformations of an associative algebra. In the last section of this chapter, we recall the sheaf of polydifferential operators and the sheaf of polyvector fields on a smooth manifold and give the statement of well-known Kontsevich's Formality Theorem.

2.1 Notation and conventions

Throughout, we assume that the ground field is the field of complex numbers $\mathbb{C}$ and set $\otimes := \otimes_{\mathbb{C}}$, $\mathrm{Hom} := \mathrm{Hom}_{\mathbb{C}}$. For a cochain complex V we denote by $\mathbf{s}V$ (resp. by $\mathbf{s}^{-1}V$) the suspension (resp. the desuspension) of V. In other words,

$$\big(\mathbf{s}V\big)^\bullet = V^{\bullet-1}, \qquad \big(\mathbf{s}^{-1}V\big)^\bullet = V^{\bullet+1}.$$

For a homogeneous vector v in a cochain complex or a graded vector space the notation $|v|$ is reserved for its degree.

The notation S_n is reserved for the symmetric group on n letters and $\mathrm{Sh}_{p_1,\dots,p_k}$ denotes the subset of $(p_1,\dots,p_k)$-shuffles in S_n, i.e. $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \dots + p_k$ such that

$$\sigma(1) < \sigma(2) < \dots < \sigma(p_1),$$

$$\sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2),$$

$$\dots$$

$$\sigma(n-p_k+1) < \sigma(n-p_k+2) < \dots < \sigma(n)\,.$$

For a groupoid $\mathcal{G}$, $\pi_0(\mathcal{G})$ denotes the set of isomorphism classes of objects of $\mathcal{G}$. For a graded vector space (or a cochain complex) V the notation $S(V)$ (resp. $\underline{S}(V)$) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of V:

$$S(V) = \mathbb{C} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \dots\,,$$

$$\underline{S}(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \dots\,,$$

where

$$S^n(V) = \big(V^{\otimes n}\big)_{S_n}$$

is the subspace of coinvariants with respect to the obvious action of S_n.

Recall that $\underline{S}(V)$ is the vector space of the cofree cocommutative coalgebra (without counit) cogenerated by V. The comultiplication on $\underline{S}(V)$

$$\Delta : \underline{S}(V) \to \underline{S}(V) \otimes \underline{S}(V) \tag{2.1.1}$$

is given by the formula:

$$\Delta(v_1) := 0\,,$$

$$\Delta(v_1, v_2, \dots, v_n) := \sum_{p=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{p,n-p}} (-1)^{\varepsilon(\sigma; v_1,\dots,v_n)} (v_{\sigma(1)}, \dots, v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}, \dots, v_{\sigma(n)})\,,$$

where $(-1)^{\varepsilon(\sigma; v_1,\dots,v_n)}$ is the Koszul sign factor

$$(-1)^{\varepsilon(\sigma; v_1,\dots,v_n)} := \prod_{(i<j)} (-1)^{|v_i||v_j|} \tag{2.1.2}$$

and the product in (2.1.2) is taken over all inversions $(i < j)$ of $\sigma \in S_n$.

For an associative algebra A, we denote by $C^\bullet(A)$ the graded vector space of Hochschild cochains with the shifted grading:

$$C^\bullet(A) = \bigoplus_{k \ge -1} C^k(A), \qquad C^k(A) := \mathrm{Hom}(A^{\otimes (k+1)}, A). \tag{2.1.3}$$

We denote by $[\ ,\ ]_G$ the well-known Gerstenhaber bracket [17] on $C^\bullet(A)$:

$$[P_1, P_2]_G(a_0, \dots, a_{k_1+k_2}) := \tag{2.1.4}$$

$$\sum_{i=0} (-1)^{ik_2} P_1(a_0, \dots, a_{i-1}, P_2(a_i, \dots, a_{i+k_2}), a_{i+k_2+1}, \dots, a_{k_1+k_2}) - (-1)^{k_1 k_2}(1 \leftrightarrow 2),$$

where $P_j \in C^{k_j}(A)$. Indeed, (2.1.4) is a Lie bracket.

It is convenient to think of the multiplication on A as the Hochschild cochain $m_A \in C^1(A)$ and define the Hochschild differential ∂^{Hoch} on $C^\bullet(A)$ as

$$\partial^{\mathrm{Hoch}} := [m_A,\ ]_G\,. \tag{2.1.5}$$

Notice that the Hochschild differential (2.1.5) is exactly the usual Hochschild coboundary operator ∂ on $C^\bullet(A)$ up to a sign factor. Namely, for any $P \in C^{k-1}(A)$

$$\partial^{\mathrm{Hoch}} P = [m_A, P]_G = (-1)^{k+1} \partial P$$

where the Hochschild coboundary operator ∂ is given by the formula

$$\begin{aligned}(\partial P)(a_0, a_1 \dots, a_k) = a_0 P(a_1, \dots, a_k) - P(a_0 a_1, a_2, \dots, a_k) + \\ P(a_0, a_1 a_2, a_3, \dots, a_k) - \dots + (-1)^k P(a_0, \dots, a_{k-2}, a_{k-1} a_k) + \\ (-1)^{k+1} P(a_0, \dots, a_{k-2}, a_{k-1}) a_k\,,\end{aligned} \tag{2.1.6}$$

for all elements $a_i \in A$.

We also recall that the graded vector space $C^{\bullet -1}(A)$ is equipped with the *cup product*

$$\cup : C^{k_1-1}(A) \otimes C^{k_2-1}(A) \to C^{k_1+k_2-1}(A)$$

$$P_1 \cup P_2(a_1, \dots, a_{k_1+k_2}) = P_1(a_1, \dots, a_{k_1}) \cdot P_2(a_{k_1+1}, \dots, a_{k_1+k_2}) \tag{2.1.7}$$

where $P_i \in C^{k_i-1}$, $a_i \in A$ and " $\cdot$ " denotes the usual multiplication in the algebra A.

The cup-product is compatible with the Hochschild differential in the following sense

$$\partial^{\rm Hoch}(P_1 \cup P_2) = P_1 \cup \partial^{\rm Hoch}(P_2) + (-1)^{k_2} \partial^{\rm Hoch}(P_1) \cup P_2 \,, \qquad P_i \in C^{k_i-1} \,.$$

We reserve the notation $HH^\bullet(A)$ for the Hochschild cohomology of A with coefficients in A, i.e.

$$HH^\bullet(A) := H^\bullet\big(C^\bullet(A)\big).$$

For example, if A is the polynomial algebra $\mathbb{C}[x^1, \dots, x^m]$ in m variables then [29, Section 3.2]

$$HH^\bullet(A) \cong S_A\big(\mathbf{s}{\rm Der}(A)\big), \tag{2.1.8}$$

where $S_A(E)$ denotes the symmetric algebra of an A-module E and ${\rm Der}(A)$ is the A-module of $\mathbb{C}$-linear derivations.

If $\varepsilon, \varepsilon_1, \dots, \varepsilon_g$ are variables of degrees

$$\deg(\varepsilon) = d_0 \,, \quad \deg(\varepsilon_1) = d_1, \quad \deg(\varepsilon_2) = d_2, \;\; \dots, \;\; \deg(\varepsilon_g) = d_g \,,$$

then the notation

$$\mathbb{C}[\varepsilon, \varepsilon_1, \dots, \varepsilon_g] \tag{2.1.9}$$

is reserved for the free graded commutative algebra over $\mathbb{C}$ generated by $\varepsilon, \varepsilon_1, \dots, \varepsilon_g$. Furthermore,

$$\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$$

denotes the completion of (2.1.9) with respect to the ideal $\mathfrak{m} = (\varepsilon, \varepsilon_1, \dots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]$. For example, if ε_1 carries an odd degree then $\varepsilon_1^2 = 0$ in (2.1.9) and in its completion.

For a smooth real manifold M, the notation $\mathcal{O}_M$ (resp. $\mathcal{O}(M)$) is reserved for the sheaf (resp. the algebra) of smooth complex valued functions on M. The symbols $x^1, x^2, \dots, x^m$ are often reserved for coordinates on an open subset $U \subset M$. TM (resp. T^*M) denotes the tangent (resp. cotangent) bundle of M.

Moreover, $\{dx^1, dx^2, \ldots, dx^m\}$ and $\{\theta_1, \theta_2, \ldots, \theta_m\}$ will be the standard local frames[1] for $T^*M\big|_U$ and $TM\big|_U$ corresponding to coordinates $x^1, x^2, \ldots, x^m$, respectively. In particular, the graded commutative algebra $\Omega^\bullet(U)$ (resp. $\wedge^\bullet TM(U)$) of exterior forms (resp. polyvector fields) on U will be tacitly identified with[2] $\mathcal{O}_M(U)[dx^1, dx^2, \ldots, dx^m]$ (resp. $\mathcal{O}_M(U)[\theta_1, \theta_2, \ldots, \theta_m]$). Occasionally, we will use the (left) "partial derivative"

$$\frac{\partial}{\partial\, dx^i} : \Omega^\bullet(U) \to \Omega^{\bullet-1}(U)$$

with respect to the degree 1 symbol dx^i. This operation is defined by the formula

$$\frac{\partial}{\partial\, dx^i}\, \eta_{i_1\ldots i_k}(x) dx^{i_1} dx^{i_2} \ldots dx^{i_k} \;:=\; k\eta_{i i_2\ldots i_k}(x) dx^{i_2} dx^{i_3} \ldots dx^{i_k}.$$

Equivalently, $\dfrac{\partial\,\eta}{\partial\, dx^i}$ is the contraction of an exterior form η with the local vector field $\partial/\partial x^i$.

2.2 Maurer-Cartan equation in differential graded Lie algebras

In this section we recall some basic concepts of differential graded Lie algebras which are very useful for our purposes in the next chapters. Let L be a $\mathbb{Z}$-graded vector space

$$L = \bigoplus_{k\in\mathbb{Z}} L^k.$$

Definition 2.2.1 *A differential graded Lie algebra (or dg Lie algebra for short) is a graded vector space L equipped with a linear map of degree one $\partial : L^k \to L^{k+1}$, called differential, and a bilinear map of degree zero $[\ ,\ ] : L^k \bigotimes L^l \to L^{k+l}$, called bracket, which satisfy the following axioms:*

---

[1] In other words, $\theta_i := \partial/\partial x^i$.

[2] Recall that, since the symbols dx^i and θ_i carry degree 1, we have $dx^i dx^j = -dx^j dx^i$ and $\theta_i\theta_j = -\theta_j\theta_i$.

(i) $\partial^2 = 0$

(ii) $\partial[a,b] = [\partial a, b] + (-1)^{|a|}[a, \partial b]$ *(graded Leibniz rule)*

(iii) $[a,b] = -(-1)^{|a||b|}[b,a]$ *(graded skew-symmetry)*

(iv) $\big[a,[b,c]\big] + (-1)^{|a|(|b|+|c|)}\big[b,[c,a]\big] + (-1)^{|c|(|a|+|b|)}\big[c,[a,b]\big] = 0$
(graded Jacoby identity)

For dg Lie algebras, we will use the notation

$$\big(L, \partial, [\ ,\ ]\big)\,.$$

Example 2.2.2 For an associative algebra A, the Hochschild cochain complex $\big(C^\bullet(A), \partial^{\mathrm{Hoch}}\big)$ is a dg Lie algebra with the Gerstenhaber bracket (2.1.4).

Definition 2.2.3 *We call a dg Lie algebra filtered if the underlying cochain complex* (L,∂) *is equipped with a complete descending filtration*

$$\cdots \supset \mathcal{F}_0 L \supset \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \mathcal{F}_3 L \supset \dots \qquad L \cong \lim_k L\big/\mathcal{F}_k L. \tag{2.2.1}$$

For every filtered dg Lie algebra $(L, \partial, [\ ,\ ])$, we assume that the differential and the bracket are compatible with the filtration in the following sense

$$\partial \mathcal{F}_k L \subset \mathcal{F}_k L\,, \quad [\mathcal{F}_{k_1} L, \mathcal{F}_{k_2} L] \subset \mathcal{F}_{k_1+k_2} L\,. \tag{2.2.2}$$

Most of the dg Lie algebras $(L, \partial, [\ ,\ ])$, we consider, are filtered.

Definition 2.2.4 *A Maurer-Cartan element of* L *is a degree degree* 1 *element* $\mu \in \mathcal{F}_1 L$ *satisfying the equation*

$$\partial \mu + \frac{1}{2}[\mu, \mu] = 0\,. \tag{2.2.3}$$

In this work, the abbreviation "MC" is reserved for the term "Maurer-Cartan" and the set of Maurer-Cartan elements of a dg Lie algebra L is denoted by

$$\mathrm{MC}(L)\,.$$

Since the filtration is complete the subalgebra $\mathcal{F}_1 L^0$ of degree zero elements in $\mathcal{F}_1 L$ is a pro-nilpotent Lie algebra (in the category of $\mathbb{C}$-vector spaces). Hence, $\mathcal{F}_1 L^0$ can be exponentiated to a group which we denote by $\mathfrak{G}$.

As a set $\mathfrak{G} = \mathcal{F}_1 L^0$. The multiplication in the group $\mathfrak{G}$ is defined by Campbell-Hausdorff series:

$$\mathrm{CH}(\xi,\eta) = \log(e^{\xi} e^{\eta}) = \xi + \eta + \frac{1}{2}[\xi,\eta] + \dots \tag{2.2.4}$$

for all $\xi,\eta \in \mathcal{F}_1 L^0$.

For a degree zero element ξ and a Maurer-Cartan element μ of L we consider the new degree 1 element $\tilde{\mu} \in L$ which is given by the formula

$$e^{\xi}(\mu) := e^{[\xi,\ ]}\mu - \frac{e^{[\xi,\ ]}-1}{[\xi,\ ]}(\partial\,\xi), \qquad \xi \in \mathcal{F}_1 L^0 \tag{2.2.5}$$

In (2.2.5), the expressions $e^{[\xi,\ ]}$ and

$$\frac{e^{[\xi,\ ]}-1}{[\xi,\ ]}$$

are defined by the Taylor expansion of the functions e^x and $(e^x-1)/x$, respectively, around the point $x=0$. Both terms on the right-hand side of (2.2.5) are well defined because the filtration on L is complete.

Let us recall [7, Appendix B], [18] that for every degree zero element $\xi \in \mathcal{F}_1 L^0$ and for every MC element μ of L, the degree 1 element of $\mathcal{F}_1 L$ obtained by the formula (2.2.5) is also a MC element of L. Moreover, the formula (2.2.5) defines a left action of the group $\mathfrak{G}$ on the set of MC elements of L, i.e. for all $\xi,\eta \in \mathcal{F}_1 L^0$ we have

$$e^{\xi}\big(e^{\eta}(\mu)\big) = e^{\mathrm{CH}(\xi,\eta)}(\mu)\,.$$

Definition 2.2.5 *The transformation groupoid corresponding to the action* (2.2.5) *is called the Goldman-Millson groupoid of the dg Lie algebra L. It is denoted by $\mathcal{G}(L)$.*

The objects of the Goldman-Millson groupoid are MC elements of L and its morphisms between two MC elements μ and $\tilde{\mu}$ are elements of the group $\mathfrak{G}$ which transform μ to $\tilde{\mu}$.

We denote by

$$\pi_0\big(\mathcal{G}(L)\big)$$

the set of isomorphism classes of MC elements of L. For a MC element $\mu \in \mathcal{F}_1 L\big($ i.e. an object of $\mathcal{G}(L)\big)$ we denote by $[\mu]$ the isomorphism class of μ.

Definition 2.2.6 *Let $(L, \partial, [\ ,\ ])$ and $(\tilde{L}, \tilde{\partial}, [\ ,\ ]^{\tilde{}})$ be dg Lie algebras. A morphism of dg Lie algebras is a degree zero linear map $F : L \to \tilde{L}$ such that*

$$F\partial = \tilde{\partial}F \quad \textit{and} \quad F([a,b]) = [F(a), F(b)]^{\tilde{}} \quad a, b \in L.$$

We consider morphisms of filtered dg Lie algebras which are compatible with the filtrations, i.e.

$$F(\mathcal{F}_{k_1} L) \subset \mathcal{F}_{k_1} \tilde{L}\,. \tag{2.2.6}$$

We will also need L_∞-morphisms and L_∞-quasi-isomorphisms of dg Lie algebras. We recall them in Section 2.3.

Now, for every dg Lie algebra L (equipped with a complete filtration (2.2.1)), we introduce a very useful simplicial set $\mathbf{MC}_\bullet(L)$ [19], [20] with

$$\mathbf{MC}_n(L) := \mathrm{MC}(L \,\hat{\otimes}\, \Omega_n), \tag{2.2.7}$$

where

$$L \,\hat{\otimes}\, \Omega_n := \varprojlim_k \big((L/\mathcal{F}_k L) \otimes \Omega_n\big)$$

and Ω_n is the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex Δ^n (with coefficients in $\mathbb{C}$).

As the graded commutative algebra (with 1), Ω_n is generated by $n+1$ symbols $t_0, t_1, \dots, t_n$ of degree 0 and $n+1$ symbols $dt_0, dt_1, \dots, dt_n$ of degree 1 subject to the relations

$$t_0 + t_1 + \dots + t_n = 1, \qquad dt_0 + dt_1 + \dots + dt_n = 0.$$

Furthermore, the differential d_t on Ω_n is defined by the formulas

$$d_t(t_i) := dt_i, \qquad d_t(dt_i) := 0.$$

For example, $\Omega_0 = \mathbb{C}$ and $\Omega_1 \cong \mathbb{C}[t] \oplus \mathbb{C}[t]\, dt$.

Since Ω_n is a dg commutative algebra for every n, the completed tensor product

$$L \,\hat{\otimes}\, \Omega_n$$

is naturally a dg Lie algebra with the differential

$$\partial(v \otimes \omega) = \partial(v) \otimes \omega + (-1)^{|v|} v \otimes d_t \omega\,,$$

and, with the bracket

$$[v_1 \otimes \omega_1, v_2 \otimes \omega_2] = (-1)^{|\omega_1||v_2|}[v_1, v_2] \otimes \omega_1 \omega_2\,,$$

where $v, v_1, v_2 \in L$ and $\omega, \omega_1, \omega_2 \in \Omega_n$.

Moreover, the simplicial structure on the collection $\{\Omega_n\}_{n \geq 0}$ gives us the simplicial structure of a simplicial set on the collection

$$\mathbf{MC}_n(L) := \mathrm{MC}(L \,\hat{\otimes}\, \Omega_n)\,.$$

Due to [13, Proposition 4.1], the simplicial set $\mathbf{MC}_\bullet(L)$ is a Kan complex (a.k.a. an ∞-groupoid).

Definition 2.2.7 *The Kan complex (a.k.a. a fibrant simplicial set)* $\mathbf{MC}_\bullet(L)$ *is called the Deligne-Getzler-Hinich (DGH)* ∞*-groupoid.*

We naturally view MC elements of L as 0-cells of $\mathbf{MC}_\bullet(L)$. 1-cells of $\mathbf{MC}_\bullet(L)$ are elements

$$\eta = \eta_0(t) + dt\eta_1(t), \qquad \eta_0(t) \in L^1 \,\hat{\otimes}\, \mathbb{C}[t], \qquad \eta_1(t) \in L^0 \,\hat{\otimes}\, \mathbb{C}[t]$$

satisfying equations

$$\partial\big(\eta_0(t)\big) + \frac{1}{2}\big[\eta_0(t), \eta_0(t)\big] = 0, \text{ and} \tag{2.2.8}$$

$$\frac{d}{dt}\eta_0(t) = \partial^{\eta_0(t)}\eta_1(t) \tag{2.2.9}$$

where $\partial^{\eta_0(t)}$ denotes the differential on $L \hat{\otimes} \mathbb{C}[t]$ twisted by the MC element $\eta_0(t)$.

Moreover, we have the following proposition :

Proposition 2.2.8 *Two MC elements μ and $\tilde{\mu}$ of a dg Lie algebra L (i.e. 0-cells of $\mathbf{MC}_\bullet(L)$) are connected by a 1-cell if and only if they belong to the same orbit of the action (2.2.5). In other words, we have the identification:*

$$\pi_0\big(\mathcal{G}(L)\big) \cong \pi_0\big(\mathbf{MC}_\bullet(L)\big). \tag{2.2.10}$$

Proof. Suppose that μ and $\tilde{\mu}$ belong to the same orbit of the action (2.2.5). Namely, there exists a degree zero element $\xi \in \mathcal{F}_1 L^0$ satisfying the equation

$$\tilde{\mu} = e^{[\xi,\ ]}\mu - \frac{e^{[\xi,\ ]} - 1}{[\xi,\ ]}(\partial\,\xi)\,. \tag{2.2.11}$$

Set

$$\eta := \eta_0(t) - dt\xi \tag{2.2.12}$$

where

$$\eta_0(t) := e^{t[\xi,\ ]}\mu - \frac{e^{t[\xi,\ ]} - 1}{[\xi,\ ]}(\partial\,\xi)\,.$$

Hence, we get that

$$\eta_0(t)|_{t=0} = \mu \quad \text{and} \quad \eta_0(t)|_{t=1} = \tilde{\mu}\,.$$

A direct computation shows that

$$\frac{d}{dt}\eta_0(t) = -\partial\xi + [\xi, \eta_0(t)] = -\partial\xi + [\eta_0(t), -\xi] = \partial^{\eta_0(t)}(-\xi)$$

and $\eta_0(t)$ satisfies the MC equation (2.2.8). Thus the proposition is proved from the following lemma. □

Lemma 2.2.9 (Lemma B.2, [12]) *If two 0-cells are connected in $\mathbf{MC}_\bullet(L)$ then they can be connected by a rectified 1-cell, which are of the form*

$$\eta = \eta_0(t) + dt\eta_1\,, \tag{2.2.13}$$

where η_1 *is a vector in* $L^0 \subset L^0 \hat{\otimes} \mathbb{C}[t]$*, i.e. the component* η_1 *does not "depend" on* t*.*

Remark 2.2.10 In this dissertation, we mostly use the "truncation" of $\mathbf{MC}_{\bullet}(L)$, i.e. the honest (transformation) groupoid $\mathcal{G}(L)$.

2.3 L_∞-morphisms of differential graded Lie algebras

For every dg Lie algebra $(L, \partial, [\ ,\ ])$, the cofree cocommutative coalgebra

$$\underline{S}(\mathbf{s}^{-1} L) \tag{2.3.1}$$

is equipped with a degree 1 coderivation Q which satisfies $Q^2 = 0$.

We recall that every coderivation Q of a cofree cocommutative coalgebra (2.3.1) is uniquely determined by the composition

$$p_{\mathbf{s}^{-1} L} \circ Q : \underline{S}(\mathbf{s}^{-1} L) \to \mathbf{s}^{-1} L. \tag{2.3.2}$$

with the canonical projection $p_{\mathbf{s}^{-1} L} : \underline{S}(\mathbf{s}^{-1} L) \to \mathbf{s}^{-1} L$. The composition (2.3.2) is expressed in terms of ∂ and $[\ ,\ ]$ as follows:

$$p_{\mathbf{s}^{-1} L} \circ Q(\mathbf{s}^{-1} v_1 \dots \mathbf{s}^{-1} v_n) := \begin{cases} \mathbf{s}^{-1} (\partial v_1) & \text{if} \quad n = 1, \\ (-1)^{|v_1|-1} \mathbf{s}^{-1} [v_1, v_2] & \text{if} \quad n = 2, \\ 0 & \text{if} \quad n \ge 3\,, \end{cases} \tag{2.3.3}$$

where $v_1, v_2, \dots, v_n$ are homogeneous elements of L. The assignment

$$(L, \partial, [\ ,\ ]) \mapsto \big(\underline{S}(\mathbf{s}^{-1} L), Q, \Delta\big)$$

is often called *the Chevalley-Eilenberg construction.*

Definition 2.3.1 *An* L_∞*-morphism* F *from a dg Lie algebra* $(L, \partial, [\ ,\ ])$ *to a dg Lie algebra* $(\tilde{L}, \tilde{\partial}, [\ ,\ ]^{\sim})$ *is a homomorphism of the corresponding dg* *cocommutative coalgebras*

$$F : (\underline{S}(\mathbf{s}^{-1} L), Q) \to (\underline{S}(\mathbf{s}^{-1} \tilde{L}), \tilde{Q}), \tag{2.3.4}$$

compatible with the coderivations Q and $\tilde{Q}$

$$F \circ Q = \tilde{Q} \circ F. \tag{2.3.5}$$

In this dissertation, we denote by F_n the restriction of $p_{\mathbf{s}^{-1} \tilde{L}} \circ F$ onto $S^n(\mathbf{s}^{-1} L)$:

$$F_n := p_{\mathbf{s}^{-1} \tilde{L}} \circ F\Big|_{S^n(\mathbf{s}^{-1} L)} : S^n(\mathbf{s}^{-1} L) \to \mathbf{s}^{-1} \tilde{L}$$

and call F_1 the *linear term* of the L_∞-morphism F.

Since the compatibility condition (2.3.5) holds for every L_∞-morphism F, its linear term F_1 is a cochain map $(L, \partial) \to (\tilde{L}, \tilde{\partial})$. That is,

$$F_1(\partial v) = \tilde{\partial} F_1(v)\,, \qquad v \in L\,.$$

This motivates the following definition:

Definition 2.3.2 *An L_∞-quasi-isomorphism F from the dg Lie algebra $(L, \partial, [\ ,\ ])$ to the dg Lie algebra $(\tilde{L}, \tilde{\partial}, [\ ,\ ]^{\tilde{}})$ is an L_∞-morphism from L to $\tilde{L}$ whose linear term*

$$F_1 : L \to \tilde{L}$$

gives an isomorphism of the spaces of cohomologies $H^\bullet(L, \partial)$ and $H^\bullet(\tilde{L}, \tilde{\partial})$.

In what follows the notation

$$F : L \rightsquigarrow \tilde{L}$$

means that F is an L_∞-morphism from the dg Lie algebra $(L, \partial, [\ ,\ ])$ to the dg Lie algebra $(\tilde{L}, \tilde{\partial}, [\ ,\ ]^{\tilde{}})$.

Example 2.3.3 An example of an L_∞-quasi-isomorphism from a dg Lie algebra $(L, \partial, [\ ,\ ])$ to a dg Lie algebra $(\tilde{L}, \tilde{\partial}, [\ ,\ ]^{\tilde{}})$ is given by a morphism of dg Lie algebras

$$F : L \to \tilde{L}$$

which induces an isomorphism on the spaces of cohomologies $H^\bullet(L, \partial)$ and $H^\bullet(\tilde{L}, \tilde{\partial})$. In this case, the only nonzero structure map is F_1:

$$F_1 := F \quad \text{and} \quad F_2 = F_3 = \ldots = 0\,.$$

Moreover, the compatibility of an L_∞-morphism F with the coderivations Q and $\tilde{Q}$ (2.3.5) yields to the relation on F_2 :

$$\tilde{\partial} F_2(v_1, v_2) + F_2(\partial v_1, v_2) + (-1)^{|v_1|} F_2(v_1, \partial v_2) = F_1\big([v_1, v_2]\big) - \big[F_1(v_1), F_1(v_2)\big]^{\,\tilde{}}\,,$$

where $v_1, v_2 \in L$. Hence, F_1 is compatible with the brackets up to homotopy.

For every L_∞-morphism $F : L \rightsquigarrow \tilde{L}$ of filtered dg Lie algebras we tacitly assume that our F is compatible with the filtrations in the following sense

$$F_n\big(\mathbf{s}^{-1}\,\mathcal{F}_{k_1} L \otimes \cdots \otimes \mathbf{s}^{-1}\,\mathcal{F}_{k_n} L\big) \ \subset\ \mathbf{s}^{-1}\,\mathcal{F}_{k_1+\cdots+k_n} \tilde{L}. \tag{2.3.6}$$

If $F : L \rightsquigarrow \tilde{L}$ is an L_∞-morphism of filtered dg Lie algebras and $\mu \in \mathcal{F}_1 L$ is a MC element of L then

$$\sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{s} F_n\big((\mathbf{s}^{-1}\,\mu)^n\big) \tag{2.3.7}$$

is a MC element of $\tilde{L}$. The infinite sum in (2.3.7) makes sense because $\tilde{L}$ is assumed to be complete with the respect to the corresponding filtration. Thus, for every L_∞-morphism $F : L \rightsquigarrow \tilde{L}$, the formula

$$F_*(\mu) := \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{s} F_n\big((\mathbf{s}^{-1}\,\mu)^n\big)$$

defines a map of sets

$$F_* : \mathrm{MC}(L) \to \mathrm{MC}(\tilde{L}). \tag{2.3.8}$$

Furthermore, any L_∞-morphism of filtered dg Lie algebras $F : L \rightsquigarrow \tilde{L}$ gives us the collection of L_∞-morphisms of dg Lie algebras

$$F^{(n)} : L \,\hat{\otimes}\, \Omega_n \rightsquigarrow \tilde{L} \,\hat{\otimes}\, \Omega_n$$

$$F^{(n)}(v_1 \otimes \omega_1, v_2 \otimes \omega_2, \ldots, v_m \otimes \omega_m) = \pm F(v_1, v_2, \ldots, v_m) \otimes \omega_1 \omega_2 \ldots \omega_m\,,$$

where $v_i \in L\,, \omega_i \in \Omega_n$, and $\pm$ is the usual Koszul sign. This collection is obviously compatible with the all faces and all the degeneracies. Hence, the map F_* naturally upgrades to the morphism of simplicial sets

$$F_* : \mathbf{MC}_\bullet(L) \to \mathbf{MC}_\bullet(\tilde{L}) \tag{2.3.9}$$

for which we use the same notation and F_* (2.3.9) is given by the formula

$$(F_*)_n(\mu) := (F^{(n)})_*(\mu)\,.$$

Therefore F_* gives us a map of sets

$$\pi_0(F_*) : \pi_0\big(\mathcal{G}(L)\big) \to \pi_0\big(\mathcal{G}(\tilde{L})\big) \tag{2.3.10}$$

thanks to the identification (2.2.10).

Remark 2.3.4 It is not hard to see that the assignments $L \mapsto \mathbf{MC}_\bullet(L)$ and $F \mapsto F_*$ define a functor from the category of filtered dg Lie algebras to the category of simplicial sets.

Now, we give the following version of Goldman-Millson Theorem which will play an important role in the proof of our main result in the next chapter:

Theorem 2.3.5 (Goldman-Millson Theorem for filtered L_∞-morphisms [24]) *Let L and $\tilde{L}$ be filtered dg Lie algebras and $F : L \rightsquigarrow \tilde{L}$ be an L_∞-morphism compatible with the filtrations on L and $\tilde{L}$. If the linear term $F_1 : L \to \tilde{L}$ of F induces a quasi-isomorphism of cochain complexes*

$$F_1|_{\mathcal{F}_n L} : \mathcal{F}_n L \to \mathcal{F}_n \tilde{L} \tag{2.3.11}$$

for every n, then the map (2.3.9) *induces a bijection*

$$\pi_0\big(\mathbf{MC}_\bullet(L)\big) \to \pi_0\big(\mathbf{MC}_\bullet(\tilde{L})\big). \tag{2.3.12}$$

Finally, we introduce a procedure to construct a new filtered dg Lie algebra L^μ from a filtered dg Lie algebra L using a MC element μ of L. As a graded vector space with a filtration, $L^\mu = L$. The Lie bracket of the dg Lie algebra L^μ is unchanged and the differential is defined by

$$\partial^\mu = \partial + [\mu, \ ]. \tag{2.3.13}$$

Thanks to the MC equation (2.2.3), ∂^μ is indeed a 2-nilpotent derivation.

Definition 2.3.6 *This procedure of changing the initial dg Lie algebra structure on L is called twisting of the dg Lie algebra L by the* MC *element μ.*

A twisting procedure can be also defined for L_∞-morphisms. For every L_∞-morphism $F : L \rightsquigarrow \tilde{L}$ of filtered dg Lie algebras and for every MC element $\mu \in L$, we can construct a new L_∞-morphism

$$F^\mu : L^\mu \rightsquigarrow \tilde{L}^{F_*(\mu)}$$

with the structure maps

$$F^\mu_m(\mathbf{s}^{-1}\, v_1 \mathbf{s}^{-1}\, v_2 \ldots \mathbf{s}^{-1}\, v_m) := \sum_{n=0}^{\infty} \frac{1}{n!} F_{m+n}\big((\mathbf{s}^{-1}\, \mu)^n \mathbf{s}^{-1}\, v_1 \mathbf{s}^{-1}\, v_2 \ldots \mathbf{s}^{-1}\, v_m\big). \tag{2.3.14}$$

2.4 1-parameter formal deformations

Before we start talking about deformation quantization, the deformations of the algebra of smooth functions on a smooth manifold, we review the deformations of any associative algebra over $\mathbb{C}$.

Let A be an associative algebra over[3] $\mathbb{C}$. Let us consider the ring $\mathbb{C}[[\varepsilon]]$ of formal power series in variable ε and the algebra $A[[\varepsilon]]$ of formal power series over $\mathbb{C}[[\varepsilon]]$ with coefficients in A.

Definition 2.4.1 *A formal deformation (so-called a star product) of A is an associative $\mathbb{C}[[\varepsilon]]$-linear product*

$$* : A[[\varepsilon]] \otimes_{\mathbb{C}[[\varepsilon]]} A[[\varepsilon]] \to A[[\varepsilon]] \tag{2.4.1}$$

[3] The general deformation theory works for any ground field of characteristic zero.

which is given by the following formula for $a, b \in A \subset A[[\varepsilon]]$ *:*

$$a * b = a.b + \varepsilon B_1(a,b) + \varepsilon^2 B_2(a,b) + \dots \tag{2.4.2}$$

where B_i *are bilinear maps* $A \otimes A \to A$*. The product of arbitrary elements of* $A[[\varepsilon]]$ *is defined by the condition of* $\mathbb{C}$*-linearity and* ε*-adic continuity:*

$$\Big(\sum_{k\geq 0} a_k \varepsilon^k\Big) * \Big(\sum_{l\geq 0} b_l \varepsilon^l\Big) = \sum_{k,l\geq 0} a_k b_l \varepsilon^{k+l} + \sum_{k,l\geq 0, m\geq 1} B_m(a_k, b_l)\varepsilon^{k+l+m}. \tag{2.4.3}$$

We can define an equivalence relation on the set of star-products of an associative algebra A.

Definition 2.4.2 *Let* $*$ *and* $\tilde{*}$ *be two star-products on* A*.* $*$ *and* $\tilde{*}$ *are called equivalent if there exists a* $\mathbb{C}[[\varepsilon]]$*-linear map*

$$T : A[[\varepsilon]] \to A[[\varepsilon]] \tag{2.4.4}$$

such that

$$T = Id + \varepsilon T_1 + \varepsilon^2 T_2 + \dots$$

where $T_k : A \to A$ *are* $\mathbb{C}$*-linear maps and,*

$$T(a * b) = T(a)\tilde{*}T(b) \tag{2.4.5}$$

for all $a, b \in A[[\varepsilon]]$*.*

Since T starts with the identity it is indeed an isomorphism of associative algebras:

$$\big(A[[\varepsilon]], *\big) \to \big(A[[\varepsilon]], \tilde{*}\big)$$

Let us recall the dg Lie algebra

$$\big(C^\bullet(A), \partial^{\mathrm{Hoch}}, [\ ,\ ]_G\big) \tag{2.4.6}$$

of Hochschild cochains from Example (2.2.2). We can construct a filtered dg Lie algebra $C^\bullet(A)[[\varepsilon]]$ over the ring $\mathbb{C}[[\varepsilon]]$ by extending the Hochschild differential and the Gerstenhaber bracket by $\mathbb{C}$-linearity. Then $C^\bullet(A)[[\varepsilon]]$ has the following obvious descending filtration

$$\mathcal{F}_k C^\bullet(A) = \varepsilon^k C^\bullet(A)[[\varepsilon]]. \tag{2.4.7}$$

The new dg Lie algebra with extended Hochschild differential and extended Gerstenhaber bracket

$$\big(C^\bullet(A)[[\varepsilon]], \partial^{\mathrm{Hoch}}, [\ ,\ ]_G\big) \tag{2.4.8}$$

is clearly complete with respect to this filtration.

MC elements of the dg Lie algebra (2.4.8) are elements of the form

$$\mu = \sum_{k \ge 1} \varepsilon^k \mu_k \in \varepsilon C^1(A)[[\varepsilon]] \tag{2.4.9}$$

satisfying the MC equation

$$\partial^{\mathrm{Hoch}} \mu + \frac{1}{2}[\mu, \mu]_G = 0. \tag{2.4.10}$$

Every MC element $\mu \in \varepsilon C^1(A)[[\varepsilon]]$ gives us an associative multiplication on $A[[\varepsilon]]$

$$a \bullet_\mu b := ab + \mu(a, b) \tag{2.4.11}$$

because the MC equation (2.4.10) is equivalent to the associativity condition for the multiplication (2.4.11).

Therefore, formal deformations of A with the base ring $\mathbb{C}[[\varepsilon]]$ are in bijection with MC elements (2.4.9).

Furthermore, if μ and $\tilde{\mu}$ are isomorphic MC elements of $C^\bullet(A)[[\varepsilon]]$, that is, if

$$\tilde{\mu} = e^{[\xi,\ ]_G} \mu - \frac{e^{[\xi,\ ]_G} - 1}{[\xi,\ ]_G} (\partial^{\mathrm{Hoch}} \xi)$$

for some $\xi \in \varepsilon C^0(A)[[\varepsilon]]$ then the operator

$$T_\xi : A[[\varepsilon]] \to A[[\varepsilon]], \qquad T_\xi(a) := a + \sum_{k=1}^{\infty} \frac{1}{k!} \xi^k(a)$$

intertwines the multiplications $\bullet_\mu$ and $\bullet_{\tilde{\mu}}$:

$$T_\xi(a \bullet_\mu b) = T_\xi(a) \bullet_{\tilde{\mu}} T_\xi(b), \qquad \forall\, a, b \in A[[\varepsilon]].$$

Thus equivalence classes of 1-parameter formal deformations of A are in bijection with isomorphism classes of MC elements in $C^\bullet(A)[[\varepsilon]]$, i.e. elements in

$$\pi_0\big(\mathcal{G}(\varepsilon C^\bullet(A)[[\varepsilon]])\big).$$

MC equation (2.2.3) implies that the first term μ_1 of μ in (2.4.9) is necessarily a degree 1 cocycle in $C^\bullet(A)$. For isomorphic MC elements μ and $\tilde{\mu}$, since there exists an operator $T : A[[\varepsilon]] \to A[[\varepsilon]]$ such that

$$T(a) = a + \sum_{k=1} \varepsilon^k T_k(a) \quad \text{and} \quad T(a \bullet_\mu b) = T(a) \bullet_{\tilde{\mu}} T(b),$$

we get that

$$\mu_1(a,b) - \tilde{\mu}_1(a,b) = \partial^{\text{Hoch}} T_1(a,b)\,.$$

Therefore, the cohomology class κ of this cocycle in $HH^1(A)$ depends only on the isomorphism class of the MC element.

Definition 2.4.3 *Let κ be the cohomology class in $HH^1(A)$ corresponding to the cocycle μ_1. It is traditionally [17], [23] called the Kodaira-Spencer class of μ.*

MC equation (2.2.3) also implies that the Kodaira-Spencer class κ of any formal deformation satisfies the "integrability" condition

$$[\kappa, \kappa] = 0, \tag{2.4.12}$$

where $[\ ,\ ]$ is the induced Lie bracket on $HH^\bullet(A)$.

The general story presented above applies to the case $A = \mathcal{O}(M)$ with the minor amendment: instead of the full Hochschild cochain complex, we use the sub- dg Lie algebra

$$\mathsf{PD}^\bullet(M) \subset C^\bullet(\mathcal{O}(M)), \tag{2.4.13}$$

where $\mathsf{PD}^\bullet$ is the sheaf of polydifferential operators on M.

Thus, 1-parameter formal deformations of $A = \mathcal{O}(M)$ are defined as the MC elements of the dg Lie algebra

$$\varepsilon \mathsf{PD}^\bullet(M)[[\varepsilon]]$$

We define the sheaf of polydifferential operators on M in the next section.

2.5 Kontsevich's Formality Theorem

We begin with a review of the sheaf $\mathsf{PD}^\bullet$ of polydifferential operators and the sheaf $\mathsf{PV}^\bullet$ of polyvector fields on M.

Let $U \subset M$ be an open coordinate subset of a manifold M with coordinates $x^1, x^2, \dots, x^m$.

For every $k \ge -1$, the space $\mathsf{PD}^k(U)$ consists of $\mathbb{C}$-multilinear maps

$$P : \mathcal{O}_M(U)^{\otimes k+1} \to \mathcal{O}_M(U)$$

which can be written (in local coordinates) as finite sums

$$P \;=\; \sum_{\alpha_0, \alpha_1, \dots, \alpha_k} P^{\alpha_0, \alpha_1, \dots, \alpha_k}(x)\, \partial_{x^{\alpha_0}} \otimes \partial_{x^{\alpha_1}} \otimes \dots \otimes \partial_{x^{\alpha_k}}\,, \tag{2.5.1}$$

where α_j are multi-indices, $P^{\alpha_0, \alpha_1, \dots, \alpha_k}(x) \in \mathcal{O}_M(U)$ and, if $\alpha = (i_1, \dots, i_s)$, then

$$\partial_{x^\alpha} = \partial_{x^{i_1}} \partial_{x^{i_2}} \dots \partial_{x^{i_s}}\,.$$

For example, $\mathsf{PD}^{-1} := \mathcal{O}_M$ and PD^0 is the sheaf of differential operators on M.

We do consider polydifferential operators which do not necessarily annihilate constant functions. For example, the usual (commutative) multiplication $m_{\mathcal{O}_M}$ can be viewed as the global section of PD^1.

It is easy to see that the Gerstenhaber bracket (2.1.4) is defined on sections of the sheaf

$$\mathsf{PD}^\bullet := \bigoplus_{k \ge -1} \mathsf{PD}^k\,. \tag{2.5.2}$$

Thus $\mathsf{PD}^\bullet$ is a sheaf of graded Lie algebras.

It is also easy to see that the formula

$$\partial^{\mathrm{Hoch}} := [m_{\mathcal{O}_M}\,,\;] \tag{2.5.3}$$

defines a differential on $\mathsf{PD}^\bullet$ which is compatible with the Lie bracket $[\ ,\ ]_G$. So we view $\mathsf{PD}^\bullet$ as a sheaf of dg Lie algebras.

Let us denote by PV^k the sheaf of local sections of $\wedge^{k+1} TM$ and set

$$\mathsf{PV}^\bullet := \bigoplus_{k \ge -1} \mathsf{PV}^k\,.$$

We call $\mathsf{PV}^\bullet$ the sheaf of polyvector fields on M.

Since the graded commutative algebra $\wedge^\bullet TM(U)$ is identified with

$$\mathcal{O}_M(U)[\theta_1, \theta_2, \dots, \theta_m],$$

where $\theta_1, \theta_2, \dots, \theta_m$ are degree 1 symbols, every polyvector field v of degree k has the unique expansion

$$v = \sum v^{i_0 i_1 \dots i_k}(x) \theta_{i_0} \theta_{i_1} \dots \theta_{i_k}\,, \qquad v^{i_0 i_1 \dots i_k}(x) \in \mathcal{O}_M(U),$$

$$v^{i_0 \dots i_t i_{t+1} \dots i_k}(x) = -v^{i_0 \dots i_{t+1} i_t \dots i_k}(x).$$

The functions $v^{i_0 i_1 \dots i_k}(x)$ are called components of $v \in \mathsf{PV}^k(U)$.

Recall [27] that $\mathsf{PV}^\bullet$ is a sheaf of Lie algebras. The Lie bracket $[\ ,\ ]_S$ (known as the Schouten bracket) is defined locally by the equations

$$[x^i, x^j]_S = [\theta_i, \theta_j]_S = 0\,, \qquad [\theta_i, x^j]_S = -[x_j, \theta_i]_S = \delta_i^j \tag{2.5.4}$$

and the Leibniz compatibility condition with the multiplication

$$[v, v_1 v_2]_S = [v, v_1]_S\, v_2 + (-1)^{(|v_1|+1)|v|} v_1\, [v, v_2]_S\,.$$

Let us also recall the every polyvector field $v \in \mathsf{PV}^k(U)$ can be identified with the polydifferential operator in $\mathsf{PD}^k(U)$ which acts as

$$a_0 \otimes a_1 \otimes \dots \otimes a_k \ \mapsto\ \sum v^{i_0 i_1 \dots i_k}(x) (\partial_{x^{i_k}} a_0)(\partial_{x^{i_{k-1}}} a_1) \dots (\partial_{x^{i_0}} a_k),$$

where $v^{i_0 i_1 \dots i_k}(x)$ are components of v and $x^1, \dots, x^m$ are coordinates on U.

This embedding of sheaves

$$\mathsf{PV}^\bullet \hookrightarrow \mathsf{PD}^\bullet \tag{2.5.5}$$

is often called the Hochschild-Kostant-Rosenberg (HKR) map [21].

Clearly, every polyvector field is a ∂^{Hoch}-closed polydifferential operator. So (2.5.5) is a chain map from the graded sheaf $\mathsf{PV}^\bullet$ with the zero differential to the graded sheaf $\mathsf{PD}^\bullet$ with the Hochschild differential ∂^{Hoch}.

We claim that

Proposition 2.5.1 (M. Kontsevich, Section 4.6.1.1, [24]) *The embedding* (2.5.5) *gives us a quasi-isomorphism of cochain complexes*

$$\big(\mathsf{PV}^\bullet(M), 0\big) \xrightarrow{\sim} \big(\mathsf{PD}^\bullet(M), \partial^{\mathrm{Hoch}}\big). \tag{2.5.6}$$

In other words, $H^k\big(\mathsf{PD}^\bullet(M), \partial^{\mathrm{Hoch}}\big) \cong \mathsf{PV}^k(M)$ *for all* k*.*

Remark 2.5.2 A version of Proposition 2.5.1 in the setting of algebraic geometry is known as the Hochschild-Kostant-Rosenberg theorem [21].

One has to be careful with the above identification of $\mathsf{PV}^\bullet$ (resp. $\mathsf{PV}^\bullet(M)$) with the corresponding subsheaf of $\mathsf{PD}^\bullet$ (resp. subspace of $\mathsf{PD}^\bullet(M)$) because the embedding (2.5.5) is compatible with the Lie brackets only *up to homotopy*. So in general,

$$[v, w]_S \neq [v, w]_G\,, \qquad v, w \in \mathsf{PV}^\bullet(M).$$

On the other hand, we have celebrated Kontsevich's formality theorem which states that

Theorem 2.5.3 (M. Kontsevich, Section 7.3.5, [24]) *There exists an* L_∞*-quasi-isomorphism from the dg Lie algebra* $\big(\mathsf{PV}^\bullet(M), 0, [\ ,\ ]_S\big)$ *to the dg Lie algebra* $\big(\mathsf{PD}^\bullet(M), \partial^{\mathrm{Hoch}}, [\ ,\ ]_G\big)$*.*

Remark 2.5.4 Paper [24] gives us an explicit sequence of quasi-isomorphisms of dg Lie algebra connecting $\big(\mathsf{PV}^\bullet(\mathbb{R}^m), 0, [\ ,\ ]_S\big)$ with $\big(\mathsf{PD}^\bullet(\mathbb{R}^m), \partial^{\mathrm{Hoch}}, [\ ,\ ]_G\big)$. For a detailed proof of Theorem 2.5.3 for an arbitrary smooth manifold, we refer the reader to [9], [25, Appendix A.3].

CHAPTER 3

DEFORMATION QUANTIZATION OVER A $\mathbb{Z}$-GRADED BASE

Now, we begin the generalization of deformation quantization to the case when we have several formal deformation parameters of non-positive degrees.

Let $\varepsilon, \varepsilon_1, \dots, \varepsilon_g$ be formal variables of degrees

$$\deg(\varepsilon) = 0\,, \quad \deg(\varepsilon_1) = d_1, \quad \deg(\varepsilon_2) = d_2, \ \dots, \ \deg(\varepsilon_g) = d_g\,, \qquad (3.0.1)$$

where $d_1, d_2, \dots, d_g$ are non-positive integers.

First, we start with the case A is any associative algebra over $\mathbb{C}$. By analogy with the section (2.4) "classical" 1-parameter case, we define formal deformations of A with the base ring

$$\Bbbk := \mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$$

as MC elements

$$\mu = \sum_{k_0 + k_1 + \dots + k_g \ge 1} \varepsilon^{k_0} \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g} \mu_{k_0, k_1, \dots, k_g}\,, \qquad \mu_{k_0, k_1, \dots, k_g} \in C^{1 - (k_0 d_0 + \dots + k_g d_g)}(A) \qquad (3.0.2)$$

of the dg Lie algebra

$$\mathfrak{m}C^{\bullet}(A)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]], \qquad (3.0.3)$$

where $\mathfrak{m}$ is the maximal ideal

$$\mathfrak{m} = (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]$$

Furthermore, we declare that two such deformations are equivalent if the corresponding MC elements are isomorphic.

An interesting feature of such deformations is that, if at least one formal parameter carries a non-zero degree, then the resulting MC element μ corresponds to (a $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$-linear) A_∞-structure on the graded vector space:

$$A[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \tag{3.0.4}$$

with the multiplications:

$$\mathsf{m}^\mu_n(a_1, \ldots, a_n) := \begin{cases} a_1 a_2 + \sum\limits_{k_0 d_0 + \cdots + k_g d_g = 0} \varepsilon^{k_0} \varepsilon_1^{k_1} \ldots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, a_2) & \text{if } \; n = 2, \\ \sum\limits_{k_0 d_0 + \cdots + k_g d_g = 2-n} \varepsilon^{k_0} \varepsilon_1^{k_1} \ldots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, \ldots, a_n) & \text{if } \; n > 2\,. \end{cases}$$

Since the degrees of all formal parameters are non-positive, all non-zero A_∞-multiplications have ≥ 2 inputs. In other words, μ gives us a usual (i.e. non-curved) A_∞-structure on (3.0.4) with the zero differential.

Just as in the 1-parameter case, the MC equation for μ implies that the element

$$\varepsilon \mu_{1,0,\ldots,0} + \varepsilon_1 \mu_{0,1,0,\ldots,0} + \cdots + \varepsilon_g \mu_{0,\ldots,0,1} \tag{3.0.5}$$

is a degree 1-cocycle in

$$\varepsilon C^\bullet(A) \oplus \varepsilon_1 C^\bullet(A) \oplus \cdots \oplus \varepsilon_g C^\bullet(A) \;\cong\; \mathfrak{m} C^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \,\big/\, \mathfrak{m}^2 C^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \tag{3.0.6}$$

Furthermore, isomorphic MC elements have cohomologous cocycles in (3.0.6). As in the 1-parameter case, we call the cohomology class κ of (3.0.5) the *Kodaira-Spencer class* of μ.

Example 3.0.1 (The Penkava-Schwarz example) Let A be the polynomial algebra $\mathbb{C}[x^1, \ldots, x^{2n-1}]$ in $2n - 1$ variables (of degree zero) and ε_1 be formal parameter of degree $3-2n$. Then the element

$$\mu := \varepsilon_1 \partial_{x^1} \cup \partial_{x^2} \cup \cdots \cup \partial_{x^{2n-1}} \ \in \ \mathfrak{m}C^\bullet(A)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \tag{3.0.7}$$

is ∂^{Hoch}-closed. Furthermore,

$$[\mu, \mu]_G = 0$$

since ε_1 is an odd variable and hence $\varepsilon_1^2 = 0$.

Thus μ is a MC element of (3.0.3) which gives an example of a deformation of A in "the A_∞-direction". It is easy to see that the Kodaira-Spencer class of $[\mu]$ is non-zero. So this is an example of non-trivial deformation[1].

Now, we start investigating the problem of deformation quantization over a $\mathbb{Z}$-graded base. Let M be a real manifold M, $\mathcal{O}(M)$ be the algebra of smooth complex-valued functions on M. By analogy with the 1-parameter "ungraded" case (when $g=0$) we define formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$:

Definition 3.0.2 *Formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$ are defined as MC elements*

$$\mu = \sum_{k_0+k_1+\cdots+k_g \ge 1} \varepsilon^{k_0} \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g} \mu_{k_0,k_1,\dots,k_g}\,, \qquad \mu_{k_0,k_1,\dots,k_g} \in \mathsf{PD}^{1-(k_0 d_0+\cdots+k_g d_g)}(M) \tag{3.0.8}$$

of the dg Lie algebra

$$\mathcal{L}_{\mathsf{PD}} := \mathfrak{m}\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.0.9}$$

Furthermore, the equivalence classes of such deformations are elements of

$$\pi_0\big(\mathcal{G}(\mathcal{L}_{\mathsf{PD}})\big),$$

where the dg Lie algebra $\mathcal{L}_{\mathsf{PD}}$ is considered with the filtration

$$\mathcal{F}_k \mathcal{L}_{\mathsf{PD}} := \mathfrak{m}^k \mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.0.10}$$

[1] A very similar example is described in [22, Section 3.2] and [31].

The MC equation for μ (3.0.8)

$$\partial^{\mathrm{Hoch}}\mu + \frac{1}{2}[\mu,\mu]_G = 0 \tag{3.0.11}$$

implies that the coset of μ in

$$\mathcal{F}_1\mathcal{L}_{\mathsf{PD}} \,\big/\, \mathcal{F}_2\mathcal{L}_{\mathsf{PD}} \;\cong\; \varepsilon\mathsf{PD}^\bullet(M)\oplus\varepsilon_1\mathsf{PD}^\bullet(M)\oplus\varepsilon_2\mathsf{PD}^\bullet(M)\oplus\cdots\oplus\varepsilon_g\mathsf{PD}^\bullet(M)$$

is ∂^{Hoch}-closed and the corresponding vector in

$$\varepsilon\mathsf{PV}^1(M)\oplus\varepsilon_1\mathsf{PV}^{1-d_1}(M)\oplus\cdots\oplus\varepsilon_g\mathsf{PV}^{1-d_g}(M) \tag{3.0.12}$$

does not depend on the choice of a representative in the equivalence class of the deformation. This motivates the following definition:

Definition 3.0.3 *We call the corresponding vector in* (3.0.12) *the Kodaira-Spencer class of* μ*.*

Since the degrees of all formal parameters are non-positive, the A_∞-structure on

$$\mathcal{O}(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]] \tag{3.0.13}$$

corresponding to μ has the following $\mathbb{C}[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]]$-multilinear multiplications:

$$\mathsf{m}_n^\mu(a_1,\dots,a_n) := \begin{cases} a_1a_2 + \displaystyle\sum_{k_0d_0+\cdots+k_gd_g=0} \varepsilon^{k_0}\varepsilon_1^{k_1}\dots\varepsilon_g^{k_g}\mu_{k_0,k_1,\dots,k_g}(a_1,a_2) & \text{if } \; n=2, \\ \displaystyle\sum_{k_0d_0+\cdots+k_gd_g=2-n} \varepsilon^{k_0}\varepsilon_1^{k_1}\dots\varepsilon_g^{k_g}\mu_{k_0,k_1,\dots,k_g}(a_1,\dots,a_n) & \text{if } \; n>2\,, \end{cases}$$

where $a_i \in \mathcal{O}(M)$ and $\mu_{k_0,k_1,\dots,k_g}$ are the coefficients in the expansion of μ

$$\mu \;=\; \sum_{k_0+k_1+\cdots+k_g\ge 1} \varepsilon^{k_0}\varepsilon_1^{k_1}\dots\varepsilon_g^{k_g}\mu_{k_0,k_1,\dots,k_g}\,.$$

3.1 The main result

Let M be a smooth real manifold equipped with a symplectic structure ω and $\alpha \in \mathsf{PV}^1(M)$ be the corresponding (non-degenerate) Poisson structure. In other words, for every open coordinate subset $U \subset M$, we have

$$\alpha^{ij}(x)\omega_{jk}(x) = \delta^i_k\,,$$

where $\alpha^{ij}(x)$ (resp. $\omega_{ij}(x)$) are components of $\alpha\big|_U$ (resp. $\omega\big|_U$).

Let us consider MC elements μ in (3.0.9) satisfying these two conditions:

Condition 3.1.1 *The Kodaira-Spencer class of μ equals $\varepsilon\alpha$.*

Condition 3.1.2 *The MC element μ satisfies the equation*

$$\mu\big|_{\varepsilon=0} \;=\; 0. \tag{3.1.1}$$

We denote by $\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}})$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_{\mathsf{PD}})$ whose objects are MC elements μ satisfying Conditions 3.1.1 and 3.1.2. Furthermore, we denote by TL the set of isomorphism classes of the subgroupoid $\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}})$, i.e.

$$\mathsf{TL} := \pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}})\big) \tag{3.1.2}$$

Definition 3.1.3 *We call TL the topological locus of $\pi_0\big(\mathcal{G}(\mathcal{L}_{\mathsf{PD}})\big)$.*

Remark 3.1.4 Note that every MC element μ satisfying Condition 3.1.2 is isomorphic to infinitely many MC elements of $\mathcal{L}_{\mathsf{PD}}$ which do not satisfy this condition. Indeed, consider a MC element μ which satisfies (3.1.1) and a polydifferential operator $P \in \mathsf{PD}^{-d_1}(M)$ for which $\partial^{\mathrm{Hoch}} P \neq 0$. Then the MC element

$$e^{[\varepsilon_1 P,\ ]_G}\,\mu - \frac{e^{[\varepsilon_1 P,\ ]_G} - 1}{[\varepsilon_1 P,\ ]_G}\,(\partial^{\mathrm{Hoch}}\,\varepsilon_1 P)$$

does not satisfy equation (3.1.1).

Equation (3.1.1) guarantees that the A_∞-multiplications $\{\mathsf{m}_n\}_{n\geq 2}$ corresponding to the MC element μ satisfy the property

$$\mathsf{m}_n(a_1,\dots,a_n)\big|_{\varepsilon=0} = \begin{cases} a_1 a_2 & \text{if} \;\; n=2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

In other words, the A_∞-algebra

$$\big(\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]], \{\mathsf{m}_n\}_{n\ge 2}\big)$$

can be viewed as a 1-parameter formal deformation of the graded commutative algebra $\mathcal{O}(M)[[\varepsilon_1, \dots, \varepsilon_g]]$.

The goal of this dissertation is to describe the topological locus TL of equivalence classes of formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$:

Theorem 3.1.5 *For every symplectic manifold M, the equivalence classes of formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$ satisfying Conditions 3.1.1 and 3.1.2 are in bijection with degree 2 elements of the graded vector space*

$$\bigoplus_{q\ge 0} \frac{1}{\varepsilon^{q-1}} \left(\mathfrak{m}\, \mathbf{s}^q\, H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]\right)^2 . \tag{3.1.4}$$

Here $\mathfrak{m}$ is the maximal ideal $(\varepsilon, \varepsilon_1, \dots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]$, $H^\bullet(M, \mathbb{C})$ is the singular cohomology of M with coefficients in $\mathbb{C}$, and $\left(\mathfrak{m}\, \mathbf{s}^q\, H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]\right)^2$ is the subspace of degree 2 elements in

$$\mathfrak{m}\, \mathbf{s}^q\, H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] .$$

Remark 3.1.6 In the "ungraded" case (i.e. $g = 0$), this result reproduces the classical theorem [6], [8] of Bertelson, Deligne, Cahen, and Gutt on the description of the equivalence classes of star products on a symplectic manifold. In this respect, Theorem 3.1.5 may be viewed as a generalization of this classification theorem to the case of a $\mathbb{Z}$-graded base.

The rest of the chapter is organized as follows. The proof of Theorem 3.1.5 is given in Section 3.4 and it is based on two auxiliary constructions. The first construction is presented in Section 3.2 and its main ingredient is Kontsevich's formality quasi-isomorphism [24] for polydifferential operators. The second construction is presented in Section 3.3 and it is inspired by a result [32] due to G. Sharygin and D. Talalaev.

3.2 Applying Kontsevich's formality theorem

Let us fix an L_∞-quasi-isomorphism

$$\mathcal{U} : \mathsf{PV}^\bullet(M) \rightsquigarrow \mathsf{PD}^\bullet(M) \tag{3.2.1}$$

whose linear term coincides with the embedding (2.5.6). Let us also extend it by $\mathbb{C}[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$-linearity to

$$\mathcal{U} : \mathfrak{m}\,\mathsf{PV}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \rightsquigarrow \mathfrak{m}\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \tag{3.2.2}$$

and denote by μ_α the following MC element of $\mathcal{L}_{\mathsf{PD}}$

$$\mu_\alpha \ := \ \mathcal{U}_*(\varepsilon\alpha). \tag{3.2.3}$$

Twisting (3.2.2) by α, we get an L_∞-morphism

$$\mathcal{U}^\alpha : \big(\mathfrak{m}\mathsf{PV}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]], [\varepsilon\alpha, \ ]_S, [\ ,\ ]_S\big) \rightsquigarrow \mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}\,, \tag{3.2.4}$$

where the dg Lie algebra $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ is obtained from $\mathcal{L}_{\mathsf{PD}}$ (3.0.9) via twisting MC element μ_α and replacing the differential ∂^{Hoch} by

$$\partial^{\mathrm{Hoch}} + [\mu_\alpha, \ ]_G\,.$$

Notice that

$$\partial^{\mathrm{Hoch}} + [\mu_\alpha, \ ]_G = \partial^{\mathrm{Hoch}}_*\,,$$

where $\partial^{\mathrm{Hoch}}_*$ is the Hochschild differential corresponding to the star product

$$a * b := ab + \mu_\alpha(a, b). \tag{3.2.5}$$

Furthermore, the dg Lie algebra $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ carries the same descending filtration as $\mathcal{L}_{\mathsf{PD}}$

$$\mathcal{F}_k\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha} := \mathfrak{m}^k\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.2.6}$$

We will need the following technical lemma :

Lemma 3.2.1 *Let* μ_α *be the* MC *element of the dg Lie algebra* $\mathcal{L}_{\mathsf{PD}}$ *defined in* (3.2.3) *and* $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ *be the twisted dg Lie algebra obtained from* $\mathcal{L}_{\mathsf{PD}}$ *via the MC element* μ_α*. Then the following assignment*

$$\mathrm{Shift}_{\mu_\alpha}(\tilde{\mu}) := \mu_\alpha + \tilde{\mu} \tag{3.2.7}$$

defines a bijection from the set of MC elements of $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ *to the set of MC elements of* $\mathcal{L}_{\mathsf{PD}}$ (3.0.9)*. Furthermore,* $\mathrm{Shift}_{\mu_\alpha}$ *upgrades to a functor*

$$\mathrm{Shift}_{\mu_\alpha} : \mathcal{G}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}) \to \mathcal{G}(\mathcal{L}_{\mathsf{PD}}) \tag{3.2.8}$$

which acts "as identity" on the set of morphisms and (3.2.8) *is actually a* <u>*strict*</u> *isomorphism of groupoids.*

Proof. Let $\tilde{\mu}$ be a MC element of the twisted dg Lie algebra $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$. Then, $\tilde{\mu}$ satisfies the MC equation in $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$:

$$\partial_*^{\mathrm{Hoch}} \tilde{\mu} + \frac{1}{2}[\tilde{\mu}, \tilde{\mu}]_G = 0 \tag{3.2.9}$$

Also, $\mu_\alpha \in \mathrm{MC}(\mathcal{L}_{\mathsf{PD}})$ satisfies MC equation in dg Lie algebra $\mathcal{L}_{\mathsf{PD}}$:

$$\partial^{\mathrm{Hoch}} \mu_\alpha + \frac{1}{2}[\mu_\alpha, \mu_\alpha]_G = 0. \tag{3.2.10}$$

(3.2.9) and (3.2.10) imply that

$$\begin{aligned} 0 &= \partial_*^{\mathrm{Hoch}} \tilde{\mu} + \frac{1}{2}[\tilde{\mu}, \tilde{\mu}]_G + \partial^{\mathrm{Hoch}} \mu_\alpha + \frac{1}{2}[\mu_\alpha, \mu_\alpha]_G \\ &= \partial^{\mathrm{Hoch}}(\mu_\alpha + \tilde{\mu}) + \frac{1}{2}[\mu_\alpha + \tilde{\mu}, \mu_\alpha + \tilde{\mu}]_G \,. \end{aligned}$$

Namely, $\mu_\alpha + \tilde{\mu}$ is a MC element of $\mathcal{L}_{\mathsf{PD}}$. Thus the assignment (3.2.7) gives us a map

$$\mathrm{Shift}_{\mu_\alpha} : \mathrm{MC}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}) \to \mathrm{MC}(\mathcal{L}_{\mathsf{PD}}) \tag{3.2.11}$$

from the set of MC elements of $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ to the set of MC elements of $\mathcal{L}_{\mathsf{PD}}$ (3.0.9).

We can directly compute [2] that for every degree zero element $\xi \in \mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$

$$\begin{aligned}
\mathrm{Shift}_{\mu_\alpha}\Big(e^{\xi}(\tilde{\mu})\Big) &= \mu_\alpha + e^{[\xi,\ ]_G}\tilde{\mu} - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}_{*}\xi) \\
&= \mu_\alpha + e^{[\xi,\ ]_G}\tilde{\mu} - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\xi) + \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}([\mu_\alpha,\xi]_G) \\
&= \mu_\alpha - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\xi) + e^{[\xi,\ ]_G}(\mu_\alpha+\tilde{\mu}) - \mu_\alpha \\
&= e^{[\xi,\ ]_G}(\mu_\alpha+\tilde{\mu}) - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\xi) \\
&= e^{\xi}(\mu_\alpha+\tilde{\mu})\,.
\end{aligned}$$

Therefore, $\mathrm{Shift}_{\mu_\alpha}$ upgrades to a functor

$$\mathrm{Shift}_{\mu_\alpha} : \mathcal{G}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}) \to \mathcal{G}(\mathcal{L}_{\mathsf{PD}}) \tag{3.2.12}$$

which acts "as identity" on the set of morphisms. It is not hard to see that (3.2.8) is actually a strict isomorphism of groupoids and the inverse functor operates on objects as

$$\mu \mapsto \mu - \mu_\alpha \;:\; \mathrm{MC}(\mathcal{L}_{\mathsf{PD}}) \to \mathrm{MC}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}). \tag{3.2.13}$$

□

Let us denote by $\tilde{\mathcal{L}}_{\mathsf{PD}}$ the following sub- dg Lie algebra of $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$

$$\tilde{\mathcal{L}}_{\mathsf{PD}} := \big(\varepsilon\,\mathfrak{m}\,\mathsf{PD}^{\bullet}(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]],\, \partial^{\mathrm{Hoch}}_{*},\, [\ ,\ ]_G\big) \tag{3.2.14}$$

and observe that the filtration from $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ induces the descending filtration on $\tilde{\mathcal{L}}_{\mathsf{PD}}$:

$$\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PD}} := \mathcal{F}_k\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha} \cap \tilde{\mathcal{L}}_{\mathsf{PD}} = \varepsilon\,\mathfrak{m}^{k-1}\,\mathsf{PD}^{\bullet}(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]]. \tag{3.2.15}$$

Next, we denote by $\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha})$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha})$ whose set of objects is $\mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PD}})$. Moreover, we set

$$\mathsf{TL}^{\mathrm{tw}} := \pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha})\big). \tag{3.2.16}$$

Let us prove that

[2] For the version of this statement in the setting of L_∞-algebras and the corresponding DGH ∞-groupoids, we refer the reader to [13, Lemma 4.3].

Proposition 3.2.2 *The restriction of the functor* (3.2.8) *to the subgroupoid* $\tilde{\mathcal{G}}(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}})$ *induces a bijection*

$$\mathsf{TL}^{\mathrm{tw}} \cong \mathsf{TL}\,.$$

Proof. Let $\tilde{\mu}$ be a MC element of $\tilde{\mathcal{L}}_{\mathsf{PD}}$. It is clear that the Kodaira-Spencer class of $\mu_\alpha + \tilde{\mu}$ coincides with the Kodaira-Spencer class of μ_α. Hence $\mu_\alpha + \tilde{\mu}$ satisfies Condition 3.1.1.

Since

$$\mu_\alpha\big|_{\varepsilon=0} \;=\; \tilde{\mu}\big|_{\varepsilon=0} \;=\; 0,$$

the MC element $\mu_\alpha + \tilde{\mu}$ also satisfies Condition 3.1.2.

Thus restricting $\mathrm{Shift}_{\mu_\alpha}$ to the full sub-groupoid $\tilde{\mathcal{G}}(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}})$, we get a functor

$$\tilde{\mathcal{G}}(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}) \to \tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PD}}). \tag{3.2.17}$$

Hence we get a map

$$\mathsf{TL}^{\mathrm{tw}} \to \mathsf{TL}. \tag{3.2.18}$$

Using the functor (3.2.13) from $\mathcal{G}(\mathcal{L}_{\mathsf{PD}})$ to $\mathcal{G}(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}})$, it is easy show that the map (3.2.18) is one-to-one. So it remains to show that, for every $\mu \in \mathrm{MC}(\mathcal{L}_{\mathsf{PD}})$ satisfying Conditions 3.1.1 and 3.1.2, there exists $\tilde{\mu} \in \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PD}})$ such that μ is isomorphic to $\mu_\alpha + \tilde{\mu}$.

Since the Kodaira-Spencer classes of μ and μ_α coincide, the coset of the difference $\mu - \mu_\alpha$ in

$$\mathfrak{m}\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \;/\; \mathfrak{m}^2\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$$

is ∂^{Hoch}-exact.

Hence there exists a degree zero vector $\xi \in \mathfrak{m}\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$ such that

$$\Big(e^{[\xi,\ ]_G}\mu - \frac{e^{[\xi,\ ]_G} - 1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\,\xi)\Big) \;-\; \mu_\alpha \in \mathfrak{m}^2\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.2.19}$$

Moreover, since $\mu\big|_{\varepsilon=0} = 0$, the vector ξ, for which (3.2.19) holds, can be found in

$$(\varepsilon)\mathsf{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]].$$

Therefore,

$$\Big(e^{[\xi,\ ]_G}\mu - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\,\xi)\Big) \ - \ \mu_\alpha \in \varepsilon\,\mathfrak{m}\mathsf{PD}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]].$$

In other words, the MC element

$$\tilde{\mu} := \Big(e^{[\xi,\ ]_G}\mu - \frac{e^{[\xi,\ ]_G}-1}{[\xi,\ ]_G}(\partial^{\mathrm{Hoch}}\,\xi)\Big) \ - \ \mu_\alpha$$

of $\mathcal{L}_{\mathsf{PD}}^{\mu_\alpha}$ belongs to the sub- dg Lie algebra $\tilde{\mathcal{L}}_{\mathsf{PD}}$.

The MC element $\mu_\alpha + \tilde{\mu}$ of $\mathcal{L}_{\mathsf{PD}}$ is isomorphic to μ by construction.

Thus the proposition is proved. □

Now, we would like to describe the set $\mathsf{TL}^{\mathrm{tw}}$. For this reason, we need to define the following dg Lie algebras :

Let $\mathcal{L}_{\mathsf{PV}}$ be the dg Lie algebra obtained from

$$\big(\mathfrak{m}\,\mathsf{PV}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]],\, 0\,[\ ,\ ]_S\big)$$

via twisting $\varepsilon\alpha$, i.e.

$$\mathcal{L}_{\mathsf{PV}} := \big(\mathfrak{m}\,\mathsf{PV}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]],\, [\varepsilon\alpha,\ ]_S,\, [\ ,\ ]_S\big)$$

We denote by $\tilde{\mathcal{L}}_{\mathsf{PV}}$ its sub- dg Lie algebra

$$\tilde{\mathcal{L}}_{\mathsf{PV}} := \varepsilon\,\mathfrak{m}\,\mathsf{PV}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]] \ \subset \ \mathcal{L}_{\mathsf{PV}}\,. \tag{3.2.20}$$

We consider $\mathcal{L}_{\mathsf{PV}}$ and $\tilde{\mathcal{L}}_{\mathsf{PV}}$ with the following descending filtrations:

$$\mathcal{F}_k\mathcal{L}_{\mathsf{PV}} := \mathfrak{m}^k\,\mathsf{PV}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]], \tag{3.2.21}$$

$$\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PV}} \ := \ \tilde{\mathcal{L}}_{\mathsf{PV}} \cap \mathcal{F}_k\mathcal{L}_{\mathsf{PV}} \ = \ \varepsilon\,\mathfrak{m}^{k-1}\,\mathsf{PV}^\bullet(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]]. \tag{3.2.22}$$

Furthermore, we denote by $\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PV}})$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_{\mathsf{PV}})$ whose set of objects is $\mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PV}})$ and set

$$\mathsf{TL}_{\mathsf{PV}} := \pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PV}})\big). \tag{3.2.23}$$

Restricting $\mathcal{U}^\alpha$ (3.2.4) to the sub- dg Lie algebra $\tilde{\mathcal{L}}_{\mathsf{PV}}$, we get an L_∞-morphism

$$\mathcal{U}^\alpha : \tilde{\mathcal{L}}_{\mathsf{PV}} \rightsquigarrow \tilde{\mathcal{L}}_{\mathsf{PD}}\,. \tag{3.2.24}$$

Let us prove that

Claim 3.2.3 *The L_∞-morphism* (3.2.4) *(resp.* (3.2.24)*) is compatible with the filtrations* (3.2.6), (3.2.21) *(resp.* (3.2.15), (3.2.22)*) in the sense of* (2.3.6). *Moreover the linear terms of the L_∞-morphisms* (3.2.4) *and* (3.2.24) *give us quasi-isomorphisms of cochain complexes*

$$\mathcal{F}_k\mathcal{L}_{\mathsf{PV}} \stackrel{\sim}{\longrightarrow} \mathcal{F}_k\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}\,,$$

$$\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PV}} \stackrel{\sim}{\longrightarrow} \mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PD}}$$

for every $k \ge 1$, respectively.

Proof. The L_∞-morphisms (3.2.4) and (3.2.24) are compatible with the filtrations by construction. So we proceed to the second statement.

For every fixed $k \ge 1$ the dg Lie algebras $\mathcal{F}_k\mathcal{L}_{\mathsf{PV}}$, $\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PV}}$, $\mathcal{F}_k\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}$, and $\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PD}}$ are equipped with the complete descending filtrations. For example,

$$\mathcal{F}_k\mathcal{L}_{\mathsf{PV}} \supset \mathcal{F}_{k+1}\mathcal{L}_{\mathsf{PV}} \supset \mathcal{F}_{k+2}\mathcal{L}_{\mathsf{PV}} \supset \dots$$

The linear term $\mathcal{U}^\alpha_1$ gives us chain maps

$$(\mathcal{F}_k\mathcal{L}_{\mathsf{PV}}, [\varepsilon\alpha,\ ]_S) \ \to\ (\mathcal{F}_k\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}, \partial^{\mathrm{Hoch}}_*) \tag{3.2.25}$$

and

$$(\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PV}}, [\varepsilon\alpha,\ ]_S) \ \to\ (\mathcal{F}_k\tilde{\mathcal{L}}_{\mathsf{PD}}, \partial^{\mathrm{Hoch}}_*) \tag{3.2.26}$$

compatible with these filtrations. As above, $\partial^{\mathrm{Hoch}}_*$ is the Hochschild differential corresponding to the star product (3.2.5).

At the level of associated graded complexes, we get the chain maps

$$J_{HKR} : \big(\mathfrak{m}^k\, \mathsf{PV}^\bullet(M)[\varepsilon, \varepsilon_1, \dots, \varepsilon_g], 0\big) \ \to\ \big(\mathfrak{m}^k\, \mathsf{PD}^\bullet(M)[\varepsilon, \varepsilon_1, \dots, \varepsilon_g], \partial^{\mathrm{Hoch}}\big) \tag{3.2.27}$$

and

$$J_{HKR} : \big(\varepsilon\mathfrak{m}^{k-1}\,\mathsf{PV}^{\bullet}(M)[\varepsilon,\varepsilon_1,\dots,\varepsilon_g],0\big) \ \to\ \big(\varepsilon\mathfrak{m}^{k-1}\,\mathsf{PD}^{\bullet}(M)[\varepsilon,\varepsilon_1,\dots,\varepsilon_g],\partial^{\mathrm{Hoch}}\big)\,, \tag{3.2.28}$$

where ∂^{Hoch} is the Hochschild differential corresponding to the usual (commutative) multiplication on $\mathcal{O}_M$ and J_{HKR} is the Hochschild-Kostant-Rosenberg embedding (2.5.5) extended by linearity with respect to $\mathbb{C}[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]$.

Since the functors $\otimes_{\mathbb{C}}\mathfrak{m}^k$ and $\otimes_{\mathbb{C}}\varepsilon\mathfrak{m}^{k-1}$ preserve quasi-isomorphisms of cochain complexes, Proposition 2.5.1 implies that (3.2.27) and (3.2.28) are quasi-isomorphisms.

Hence, by the Lemma A.0.3, the chain maps (3.2.25) and (3.2.26) are quasi-isomorphisms of cochain complexes.

Claim 3.2.3 is proved. □

Let us prove that

Proposition 3.2.4 *The L_∞-morphism* (3.2.24) *induces a map*

$$\mathsf{TL}_{\mathsf{PV}} \to \mathsf{TL}^{\mathrm{tw}}\,. \tag{3.2.29}$$

Furthermore, this map is a bijection.

Proof. To prove the first statement, we observe that the L_∞-morphism (3.2.24) gives a map

$$\mathcal{U}^{\alpha}_{*} : \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PV}}) \to \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PD}}). \tag{3.2.30}$$

Let μ_1 and μ_2 be isomorphic MC elements of $\tilde{\mathcal{L}}_{\mathsf{PV}}$. Due to the identification

$$\pi_0\big(\mathcal{G}(\tilde{\mathcal{L}}_{\mathsf{PV}})\big) \cong \pi_0\big(\mathbf{MC}_{\bullet}((\tilde{\mathcal{L}}_{\mathsf{PV}})\big)$$

μ_1 and μ_2 are connected by a 1-cell

$$\eta \in \mathbf{MC}_1(\mathcal{L}_{\mathsf{PV}}) = \mathrm{MC}\big(\mathcal{L}_{\mathsf{PV}}\,\hat{\otimes}\,\mathbb{C}[t] \oplus \mathcal{L}_{\mathsf{PV}}\,\hat{\otimes}\,\mathbb{C}[t]\,dt\big),$$

i.e.

$$\eta\big|_{t=dt=0} = \mu_1, \qquad \eta\big|_{t=1,dt=0} = \mu_2.$$

Clearly, the 1-cell

$$\mathcal{U}^{\alpha}_{*}(\eta) \ \in \ \mathbf{MC}_1(\mathcal{L}_{\mathsf{PD}})$$

connects the MC elements

$$\mathcal{U}^{\alpha}_{*}(\mu_1),\ \mathcal{U}^{\alpha}_{*}(\mu_2) \ \in \ \tilde{\mathcal{L}}_{\mathsf{PD}}.$$

Thus (3.2.30) descends to the map of sets

$$\mathsf{TL}_{\mathsf{PV}} \to \mathsf{TL}^{\mathrm{tw}}\,.$$

Let us now show that this map is a bijection.

According to Claim 3.2.3, both L_∞-morphisms (3.2.4)

$$\mathcal{U}^{\alpha} : \big(\mathfrak{m}\mathsf{PV}^{\bullet}(M)[[\varepsilon,\varepsilon_1,\dots,\varepsilon_g]], [\varepsilon\alpha,\ ]_S\big) \rightsquigarrow \mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}\,,$$

and (3.2.24)

$$\mathcal{U}^{\alpha} : \tilde{\mathcal{L}}_{\mathsf{PV}} \rightsquigarrow \tilde{\mathcal{L}}_{\mathsf{PD}}$$

satisfy conditions of Theorem 2.3.5. So, applying Theorem 2.3.5 to (3.2.4), we get that

$$\pi_0\big(\mathbf{MC}_{\bullet}(\mathcal{L}_{\mathsf{PV}})\big) \cong \pi_0\big(\mathbf{MC}_{\bullet}(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}})\big)\,. \qquad (3.2.31)$$

Let $\mu_1,\ \mu_2 \in \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PV}})$ such that $\mathcal{U}^{\alpha}_{*}(\mu_1)$ and $\mathcal{U}^{\alpha}_{*}(\mu_2)$ are MC elements of $\tilde{\mathcal{L}}_{\mathsf{PD}} \subset \mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}$ connected by a 1-cell

$$\tilde{\eta} \in \mathbf{MC}_1(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}) = \mathrm{MC}\big(\mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}\,\hat{\otimes}\,\mathbb{C}[t] \oplus \mathcal{L}^{\mu_\alpha}_{\mathsf{PD}}\,\hat{\otimes}\,\mathbb{C}[t]\,dt\big).$$

Due to the isomorphism (3.2.31), there exists a 1-cell

$$\eta \in \mathbf{MC}_1(\mathcal{L}_{\mathsf{PV}}) = \mathrm{MC}\big(\mathcal{L}_{\mathsf{PV}}\,\hat{\otimes}\,\mathbb{C}[t] \oplus \mathcal{L}_{\mathsf{PV}}\,\hat{\otimes}\,\mathbb{C}[t]\,dt\big),$$

connecting the MC elements μ_1 and μ_2. So, we conclude that the map (3.2.29) is one-to-one.

Furthermore, applying this theorem to (3.2.24), we get that

$$\pi_0\big(\mathbf{MC}(\tilde{\mathcal{L}}_{\mathsf{PV}})\big) \cong \pi_0\big(\mathbf{MC}(\tilde{\mathcal{L}}_{\mathsf{PD}})\big)\,. \qquad (3.2.32)$$

Due to (3.2.32), for every $\tilde{\mu} \in \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PD}})$, there exists a MC element $\mu \in \mathrm{MC}(\tilde{\mathcal{L}}_{\mathsf{PV}})$ such that $\mathcal{U}^{\alpha}_{*}(\mu)$ and $\tilde{\mu}$ are connected by a 1-cell in $\mathbf{MC}_1(\tilde{\mathcal{L}}_{\mathsf{PD}})$. Thus, the map (3.2.29) is surjective. □

3.3 Passing to exterior forms

Let $f \in \mathcal{O}(M)$ and $\theta_i := \partial_{x^i}$ be the local vector field defined in a coordinate chart of M. Since the symplectic structure ω is non-degenerate, the formulas

$$\mathsf{J}_\omega(f) := f, \qquad \mathsf{J}_\omega(\theta_i) := \frac{1}{\varepsilon}\omega_{ij}(x) dx^j$$

define an isomorphism of (shifted) graded commutative algebras

$$\mathsf{J}_\omega : \mathfrak{m}\, \mathsf{PV}(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \ \to \ \bigoplus_{q \ge 0} \frac{1}{\varepsilon^q} \mathbf{s}^{-1}\ \mathfrak{m}\, \Omega^q(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \quad (3.3.1)$$

The inverse of J_ω is given by

$$\mathsf{J}_\omega^{-1}(f) = f, \qquad \mathsf{J}_\omega^{-1}(dx^i) = \varepsilon \alpha^{ij}(x) \theta_j\,, \qquad (3.3.2)$$

where $\alpha^{ij}(x)$ are components of the corresponding Poisson structure (in local coordinates).

Here, we tacitly assume that every vector $\eta \in \Omega^q(M)$ carries degree q. The latter means that the vector $\eta\, \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g}$ carries degree $q + k_1 d_1 + \dots + k_g d_g$ and the vector $\mathbf{s}^{-1}\, \eta\, \varepsilon_1^{k_1} \dots \varepsilon_g^{k_g}$ carries degree $q - 1 + k_1 d_1 + \dots + k_g d_g$.

Using this isomorphism and the dg Lie algebra structure $([\varepsilon\alpha, \ ]_S, [\ ,\ ]_S)$ on $\mathcal{L}_{\mathsf{PV}}$, we equip the graded vector space

$$\mathcal{L}_\Omega := \bigoplus_{q \ge 0} \frac{1}{\varepsilon^q} \mathbf{s}^{-1}\ \mathfrak{m}\, \Omega^q(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \qquad (3.3.3)$$

with the structure of a dg Lie algebra.

A direct computation shows that the differential on $\mathcal{L}_\Omega$ corresponding to $[\varepsilon\alpha, \ ]_S$ is

$$-d$$

(where d is the de Rham differential) and the Lie bracket $[\ ,\ ]_\omega$ corresponding to $[\ ,\ ]_S$ is given by the formulas (in local coordinates)

$$[\mathbf{s}^{-1}\, \eta_1, \mathbf{s}^{-1}\, \eta_2]_\omega :=$$

$$\mathbf{s}^{-1}\, \varepsilon dx^k \partial_{x^k} \alpha^{ij}(x) \frac{\partial\, \eta_1}{\partial dx^i} \frac{\partial\, \eta_2}{\partial dx^j} - (-1)^{|\eta_1|} \mathbf{s}^{-1}\, \varepsilon \alpha^{ij}(x) \frac{\partial\, \eta_1}{\partial dx^i}\, \partial_{x^j} \eta_2 + \mathbf{s}^{-1}\, \varepsilon \alpha^{ij}(x) (\partial_{x^i} \eta_1) \frac{\partial\, \eta_2}{\partial dx^j}\,. \qquad (3.3.4)$$

For example,

$$[\mathbf{s}^{-1} f, \mathbf{s}^{-1} g]_\omega := 0, \qquad [\mathbf{s}^{-1}\, dx^i, \mathbf{s}^{-1} f]_\omega := \mathbf{s}^{-1}\, \varepsilon\, \alpha^{ij}(x)(\partial_{x^j} f), \tag{3.3.5}$$

$$[\mathbf{s}^{-1}\, dx^i, \mathbf{s}^{-1}\, dx^j]_\omega := \mathbf{s}^{-1}\, \varepsilon\, dx^k \partial_{x^k} \alpha^{ij}(x),$$

where $f, g \in \mathcal{O}(M)$.

It is clear that, restricting J_ω to the sub- dg Lie algebra (3.2.20), we get an isomorphism of dg Lie algebras

$$\mathsf{J}_\omega : \tilde{\mathcal{L}}_{\mathsf{PV}} \xrightarrow{\cong} \tilde{\mathcal{L}}_\Omega\,, \tag{3.3.6}$$

where $\tilde{\mathcal{L}}_\Omega$ is the sub- dg Lie algebra of $\mathcal{L}_\Omega$:

$$\tilde{\mathcal{L}}_\Omega := \bigoplus_{q \ge 0} \frac{1}{\varepsilon^{q-1}} \mathbf{s}^{-1}\ \mathfrak{m}\, \Omega^q(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.3.7}$$

Both dg Lie algebras (3.3.3) and (3.3.7) are equipped with the obvious complete descending filtrations

$$\mathcal{F}_k \mathcal{L}_\Omega := \bigoplus_{q \ge 0} \frac{1}{\varepsilon^{q}} \mathbf{s}^{-1}\ \mathfrak{m}^k\, \Omega^q(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]], \tag{3.3.8}$$

$$\mathcal{F}_k \tilde{\mathcal{L}}_\Omega := \bigoplus_{q \ge 0} \frac{1}{\varepsilon^{q-1}} \mathbf{s}^{-1}\ \mathfrak{m}^{k-1}\, \Omega^q(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]. \tag{3.3.9}$$

Moreover, the isomorphisms (3.3.1) and (3.3.6) are compatible with these filtrations.

We denote by $\tilde{\mathcal{G}}(\mathcal{L}_\Omega)$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_\Omega)$ whose set of objects is $\mathrm{MC}(\tilde{\mathcal{L}}_\Omega)$ and set

$$\mathsf{TL}_\Omega := \pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_\Omega)\big). \tag{3.3.10}$$

Using the above properties of the isomorphism J_ω we easily deduce that

Claim 3.3.1 *The map J_ω gives us an isomorphism of groupoids*

$$(J_\omega)_* : \tilde{\mathcal{G}}(\mathcal{L}_{\mathsf{PV}}) \xrightarrow{\cong} \tilde{\mathcal{G}}(\mathcal{L}_\Omega)$$

and hence a bijection

$$\mathsf{TL}_{\mathsf{PV}} \xrightarrow{\cong} \mathsf{TL}_\Omega\,.$$

□

Remark 3.3.2 Note that the grading on $\mathcal{L}_\Omega$ and $\tilde{\mathcal{L}}_\Omega$ comes with an additional shift. For example, every degree zero exterior form $\varepsilon\,\eta \in \varepsilon\Omega^0(M) = \varepsilon\mathcal{O}(M)$ gives us the degree -1 vector $\mathbf{s}^{-1}\,\varepsilon\,\eta \in \mathcal{L}_\Omega$. In particular, monomials in $S^n(\mathbf{s}^{-1}\,\tilde{\mathcal{L}}_\Omega)$ will be written as

$$\mathbf{s}^{-2}\eta_1\,\mathbf{s}^{-2}\eta_2\,\ldots\,\mathbf{s}^{-2}\eta_n,$$

where $\eta_1, \eta_2, \ldots, \eta_n$ belong to the space

$$\bigoplus_{q\ge 0} \frac{1}{\varepsilon^{q-1}}\,\mathfrak{m}\,\Omega^q(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \tag{3.3.11}$$

Let us consider the cocommutative coalgebra

$$\underline{S}(\mathbf{s}^{-1}\,\tilde{\mathcal{L}}_\Omega) \tag{3.3.12}$$

with the degree 1 coderivations Q_{-d}, and $Q_{[\ ,\ ]_\omega}$, where Q_{-d} (resp. $Q_{[\ ,\ ]_\omega}$) comes from the dg Lie algebra structure $(-d, [\ ,\ ] = 0)$ (resp. $(\partial = 0, [\ ,\ ]_\omega)$) on $\tilde{\mathcal{L}}_\Omega$ in the sense of (2.3.3). For example,

$$Q_{-d}(\mathbf{s}^{-2}\eta_1\,\mathbf{s}^{-2}\eta_2) = -\mathbf{s}^{-2}\,d\eta_1\,\mathbf{s}^{-2}\,\eta_2 - (-1)^{|\eta_1|}\mathbf{s}^{-2}\,\eta_1\,\mathbf{s}^{-2}\,d\eta_2\,, \tag{3.3.13}$$

and

$$Q_{[\ ,\ ]_\omega}(\mathbf{s}^{-2}\eta_1\,\mathbf{s}^{-2}\eta_2) := (-1)^{|\eta_1|}\mathbf{s}^{-2}\,\varepsilon\,dx^k\partial_{x^k}\alpha^{ij}(x)\frac{\partial\eta_1}{\partial dx^i}\frac{\partial\eta_2}{\partial dx^j} \tag{3.3.14}$$
$$-\mathbf{s}^{-2}\,\varepsilon\,\alpha^{ij}(x)\frac{\partial\eta_1}{\partial dx^i}\partial_{x^j}\eta_2 + (-1)^{|\eta_1|}\mathbf{s}^{-2}\,\varepsilon\,\alpha^{ij}(x)(\partial_{x^i}\eta_1)\frac{\partial\eta_2}{\partial dx^j}\,,$$

where $|\eta_1|$ is the degree of η_1 in (3.3.11).

Using the idea of [32], we consider the coderivation Π of the coalgebra (3.3.12) defined in local coordinates by the formulas

$$p\circ\Pi(\mathbf{s}^{-2}\eta_1\,\mathbf{s}^{-2}\eta_2) \;:=\; (-1)^{|\eta_1|}\mathbf{s}^{-2}\,\varepsilon\alpha^{ij}(x)\Big(\frac{\partial}{\partial dx^i}\eta_1\Big)\Big(\frac{\partial}{\partial dx^j}\eta_2\Big)\,, \tag{3.3.15}$$
$$p\circ\Pi(\mathbf{s}^{-2}\eta_1\,\mathbf{s}^{-2}\eta_2\,\ldots\,\mathbf{s}^{-2}\eta_n) \;:= 0, \qquad \text{if} \quad n\neq 2,$$

where p is the canonical projection $\underline{S}(\mathbf{s}^{-1}\,\tilde{\mathcal{L}}_\Omega) \to \mathbf{s}^{-1}\,\tilde{\mathcal{L}}_\Omega$.

Properties of the coderivation Π are listed in the following proposition:

Proposition 3.3.3 *The coderivation* Π *has degree* 0*. Furthermore,*

$$\Pi \circ Q_{-d} - Q_{-d} \circ \Pi = Q_{[\ ,\ ]_\omega} \tag{3.3.16}$$

and

$$\Pi \circ Q_{[\ ,\ ]_\omega} = Q_{[\ ,\ ]_\omega} \circ \Pi. \tag{3.3.17}$$

□

Remark 3.3.4 By keeping track of terms of negative powers of ε, it is easy to see that, in general,

$$p \circ \Pi\big(\mathbf{s}^{-1}\,\mathcal{L}_\Omega \otimes \mathbf{s}^{-1}\,\mathcal{L}_\Omega\big) \ \not\subset\ \mathbf{s}^{-1}\,\mathcal{L}_\Omega\,.$$

However

$$p \circ \Pi\big(\mathbf{s}^{-1}\,\tilde{\mathcal{L}}_\Omega \otimes \mathbf{s}^{-1}\,\mathcal{L}_\Omega\big) \ \subset\ \mathbf{s}^{-1}\,\mathcal{L}_\Omega \tag{3.3.18}$$

and we will use the inclusion (3.3.18) to extend the coderivation Π to the coalgebra

$$\underline{S}(\mathbf{s}^{-1}\,L), \tag{3.3.19}$$

where L is the following graded vector space

$$L := \tilde{\mathcal{L}}_\Omega\,\hat{\otimes}\,\mathbb{C}[t] \ \oplus\ \mathcal{L}_\Omega\,\hat{\otimes}\,\mathbb{C}[t]dt\,, \tag{3.3.20}$$

$\hat{\otimes}$ is the completed tensor product and $\mathbb{C}[t] \oplus \mathbb{C}[t]dt$ is the algebra of polynomial de Rham forms on the 1-simplex. We will freely use the obvious generalizations of (3.3.16) and (3.3.17) to the corresponding coderivations of the coalgebra (3.3.19).

The proof of Proposition 3.3.3 is given in Appendix B. Here we use this proposition to deduce the following statement:

Corollary 3.3.5 *The formula*

$$\exp(\Pi) := 1 + \sum_{m \ge 1} \frac{1}{m!}\Pi^m \tag{3.3.21}$$

defines a <u>*strictly*</u> *invertible* L_∞*-quasi-isomorphism*

$$(\tilde{\mathcal{L}}_\Omega, -d, 0) \rightsquigarrow (\tilde{\mathcal{L}}_\Omega, -d, [\ ,\ ]_\omega).$$

This isomorphism extends, in the obvious way, to a strictly invertible L_∞*-quasi-isomorphism*

$$(L, -d + d_t, 0) \rightsquigarrow (L, -d + d_t, [\ ,\ ]_\omega),$$

where L *is defined in* (3.3.20) *and* d_t *is the de Rham differential* $dt\partial_t$ *on* $\mathbb{C}[t] \oplus \mathbb{C}[t]\, dt$*.*

Proof. It is straightforward to show that the equation (3.3.21) defines automorphisms of the cocommutative coalgebras (3.3.12) and (3.3.19) (considered with the zero differentials). The inverse of (3.3.21) is given by the formula:

$$\exp(-\Pi) := 1 + \sum_{m \ge 1} \frac{(-1)^m}{m!} \Pi^m\,. \tag{3.3.22}$$

It remains to prove that

$$\exp(\Pi) \circ Q_{-d} = (Q_{-d} + Q_{[\ ,\ ]_\omega}) \circ \exp(\Pi)$$

or equivalently

$$\exp(\Pi) \circ Q_{-d} \circ \exp(-\Pi) = Q_{-d} + Q_{[\ ,\ ]_\omega}\,. \tag{3.3.23}$$

For this purpose we introduce two elements

$$\Psi_L, \Psi_R \in \mathrm{Hom}\big(\underline{S}(\mathbf{s}^{-1}\, \tilde{\mathcal{L}}_\Omega)\,,\, \underline{S}(\mathbf{s}^{-1}\, \tilde{\mathcal{L}}_\Omega)[u]\big)$$

defined by

$$\Psi_L := \exp(u\,\Pi) \circ Q_{-d} \circ \exp(-u\,\Pi), \qquad \Psi_R := Q_{-d} + u\, Q_{[\ ,\ ]_\omega}\,, \tag{3.3.24}$$

where u is an auxiliary variable of degree 0.

Using (3.3.16) and (3.3.17) we get

$$\frac{d}{du}\Psi_L = \exp(u\Pi) \circ \big(\Pi \circ Q_{-d} - Q_{-d} \circ \Pi\big) \circ \exp(-u\Pi)$$

$$= \exp(u\Pi) \circ Q_{[\ ,\ ]_\omega} \circ \exp(-u\Pi) = Q_{[\ ,\ ]_\omega}.$$

Hence both Ψ_L and Ψ_R satisfy the same formal ordinary differential equation

$$\frac{d}{du}\Psi = Q_{[\ ,\ ]_\omega}$$

with the same initial condition $\Psi\big|_{u=0} = Q_{-d}$.

Therefore, $\Psi_L = \Psi_R$ and (3.3.23) follows.

The similar argument shows that the same operator $\exp(\Pi)$ defines an L_∞-quasi-isomorphism

$$(L, -d + d_t, 0) \rightsquigarrow (L, -d + d_t, [\ ,\ ]_\omega).$$

Thus the corollary is proved. □

Now, we can give the proof of the main theorem:

3.4 The proof of Theorem 3.1.5

Let us denote by $\tilde{\mathcal{G}}(\mathcal{L}_\Omega, -d, 0)$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_\Omega, -d, 0)$ whose set of objects is $\mathrm{MC}(\tilde{\mathcal{L}}_\Omega, -d, 0)$.

Clearly,

$$\pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_\Omega, -d, 0)\big) \;\cong\; \bigoplus_{q\ge 0} \frac{1}{\varepsilon^{q-1}} \left(\mathfrak{m}\,\mathbf{s}^q\, H^q(M,\mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]\right)^2, \qquad (3.4.1)$$

where $\left(\mathfrak{m}\,\mathbf{s}^q\, H^q(M,\mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]\right)^2$ is the subspace of degree 2 elements in

$$\mathfrak{m}\,\mathbf{s}^q\, H^q(M,\mathbb{C})[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]\,.$$

Since the L_∞-quasi-isomorphism (3.3.21) is strictly invertible, it induces a bijection of sets

$$\exp(\Pi)_* \,:\, \mathrm{MC}(\tilde{\mathcal{L}}_\Omega, -d, 0) \xrightarrow{\cong} \mathrm{MC}(\tilde{\mathcal{L}}_\Omega, -d, [\ ,\ ]_\omega). \qquad (3.4.2)$$

Moreover, the second part of Corollary 3.3.5 implies that, if $\mu_1, \mu_2 \in \mathrm{MC}(\tilde{\mathcal{L}}_\Omega, -d, 0)$ are connected by a 1-cell in $\mathbf{MC}_\bullet(\mathcal{L}_\Omega, -d, 0)$ then $\exp(\Pi)_*(\mu_1)$ and $\exp(\Pi)_*(\mu_2)$ are connected by a 1-cell in $\mathbf{MC}(\mathcal{L}_\Omega, -d, [\ ,\ ]_\omega)$.

In other words, (3.4.2) gives us a well defined map

$$\Theta_\Pi : \pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_\Omega, -d, 0)\big) \to \mathsf{TL}_\Omega\,. \tag{3.4.3}$$

Let us prove that

Claim 3.4.1 *The map Θ_Π is a bijection of sets.*

Proof. The surjectivity of Θ_Π follows from the fact that the map (3.4.2) is a bijection.

To prove the injectivity, we consider two MC elements $\mu_1, \mu_2 \in \mathrm{MC}(\tilde{\mathcal{L}}_\Omega, -d, 0)$ and assume that the MC elements $\exp(\Pi)_*(\mu_1)$ and $\exp(\Pi)_*(\mu_2)$ are connected by a 1-cell in $\mathbf{MC}(\mathcal{L}_\Omega, -d, [\ ,\ ]_\omega)$. Due to the second part of Corollary 3.3.5, the MC elements

$$\exp(-\Pi)_* \circ \exp(\Pi)_*(\mu_1) = \mu_1 \qquad \text{and} \qquad \exp(-\Pi)_* \circ \exp(\Pi)_*(\mu_2) = \mu_2$$

are connected by a 1-cell in $\mathbf{MC}(\mathcal{L}_\Omega, -d, 0)$.

Thus Θ_Π is indeed injective. □

By putting all the things together, we can now complete the proof of Theorem 3.1.5.

Indeed, due to Proposition 3.2.2, we have a bijection

$$\mathsf{TL} \cong \mathsf{TL}^{\mathrm{tw}}.$$

The map (3.2.29) induced by the L_∞-quasi-isomorphism (3.2.24) gives us a bijection

$$\mathsf{TL}_{\mathsf{PV}} \cong \mathsf{TL}^{\mathrm{tw}}.$$

The isomorphism J_ω from (3.3.6) gives us a bijection

$$\mathsf{TL}_{\mathsf{PV}} \cong \mathsf{TL}_\Omega$$

and, finally, the map $\exp(\Pi)_*$ induces a bijection

$$\pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_\Omega, -d, 0)\big) \cong \mathsf{TL}_\Omega.$$

Since

$$\pi_0\big(\tilde{\mathcal{G}}(\mathcal{L}_\Omega, -d, 0)\big) \;\cong\; \bigoplus_{q\geq 0} \frac{1}{\varepsilon^{q-1}} \left(\mathfrak{m}\, \mathbf{s}^q \, H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]\right)^2,$$

the proof of Theorem 3.1.5 complete.

CHAPTER 4

FINAL REMARKS

In this concluding section, we would like to pose two open questions.

It is known [11], [24, 4.6.3] and [26] that there are infinitely many homotopy classes of formality quasi-isomorphisms from the dg Lie algebra of polyvector fields to the dg Lie algebra of polydifferential operators. So it would be interesting to determine whether the bijection from Theorem 3.1.5 depends on the homotopy type of the formality quasi-isomorphism for polydifferential operators.

We believe that

Conjecture 4.0.1 *There is a construction of a bijection between* TL *and the set of formal series in* (3.1.4) *which bypasses the use of Kontsevich's formality theorem. This construction comes from an appropriate generalization of the zig-zag of quasi-isomorphisms of dg Lie algebras from paper [10].*

Conjecture 4.0.2 *The bijection between* TL *and the set of formal series in* (3.1.4) *coming from the above conjectural construction coincides with the bijection produced in this paper for any choice of a formality quasi-isomorphism* (3.2.1).

If true, the statement of Conjecture 4.0.2 would imply that the constructed bijection between TL and the set of formal series in (3.1.4) does not depend on the choice of a formality quasi-isomorphism (3.2.1).

We believe that a solution of Conjecture 4.0.1 will allow us to produce explicit examples of A_∞-structures on $\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ corresponding to formal series in (3.1.4) for a large class of symplectic manifolds. To our genuine surprise, Kontsevich's quasi-isomorphism [24] is not very helpful for computing these A_∞-structures even in the case when M is an even dimensional torus with the standard symplectic structure!

We also believe that Conjecture 4.0.2 can be tackled using the ideas from [7].

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APPENDIX A

A LEMMA ON A QUASI-ISOMORPHISM OF FILTERED COMPLEXES

Let us recall that a *cone* $\mathrm{Cone}(f)$ of a morphism of cochain complexes $f : C \to K$ is the cochain complex

$$\mathrm{Cone}(f)^\bullet = C^{\bullet+1} \oplus K^\bullet$$

with the differential

$$\partial^{\mathrm{Cone}}(v_1 + v_2) = \partial(v_1) + f(v_1) - \partial(v_2),$$

where we denote by ∂ the differentials on both complexes C and K . Let us also recall a claim which follows easily from Lemma 3 in Section III. 3. 2 in [16] :

Claim A.0.1 *A morphism $f : C \to K$ of cochain complexes is a quasi-isomorphism if and only if the cochain complex* $\mathrm{Cone}(f)$ *is acyclic.*

Let C be a cochain complex equipped with a complete descending filtration

$$\cdots \supset \mathcal{F}_0 C \supset \mathcal{F}_1 C \supset \mathcal{F}_2 C \supset \mathcal{F}_3 C \supset \dots . \qquad C \cong \lim_k C \big/ \mathcal{F}_k C.$$

Let us denote by $\mathrm{Gr}(C)$ the associated graded cochain complex

$$\mathrm{Gr}(C) := \bigoplus_m \mathcal{F}_m C \big/ \mathcal{F}_{m+1} C \,.$$

We will need the following claim :

Claim A.0.2 *Let C be a cochain complex with a complete descending filtration which is bounded above. I.e.,*

$$C = \mathcal{F}_{-N} C \supset \mathcal{F}_{-N+1} C \supset \cdots \supset \mathcal{F}_0 C \supset \mathcal{F}_1 C \supset \mathcal{F}_2 C \supset \mathcal{F}_3 C \supset \ldots \,. \qquad C \cong \lim_k C \big/ \mathcal{F}_k C.$$

If the associated graded complex $\mathrm{Gr}(C)$ is acyclic then so is C.

Proof. Let v be a cocycle in C of degree d. Our goal is to show that there exists a vector $\omega \in C^{d-1}$ such that

$$v = \partial \omega \,.$$

Since C is bounded above there exists an integer m such that $c \in \mathcal{F}_m C$. Thus, c represents a cocycle in the quotient

$$\mathcal{F}_m C^d / \mathcal{F}_{m+1} C^d \,.$$

Then by acyclicity of the associated graded complex, there is a vector $\omega_m \in \mathcal{F}_m C$ such that $v - \partial \omega_m \in \mathcal{F}_{m+1} C$. Continuing this process, we define vectors $\omega_{m+1}, \omega_{m+2}, \ldots$. Then we set

$$w := \sum_m \omega_m \,.$$

The sum converges, since the filtration on C is complete. Again by completeness, we get that

$$v - \partial \omega = \lim_{N \to \infty} v - \sum_{m \le N} \partial \omega_m = 0 \,.$$

Thus v is exact. $\square$

Lemma A.0.3 *Let C and K be cochain complexes equipped with complete descending filtrations which are bounded above. Let $f : C \to K$ be a morphism of cochain complexes compatible with filtrations. If the induced map of the cochain complexes*

$$\mathrm{Gr}(f) : \mathrm{Gr}(C) \to \mathrm{Gr}(K) \tag{A.0.1}$$

is a quasi-isomorphism then so is f.

Proof. Let us introduce the obvious descending filtration on the cone of f :

$$\dots \supset \mathcal{F}_0\mathrm{Cone}(f) \supset \mathcal{F}_1\mathrm{Cone}(f) \supset \mathcal{F}_2\mathrm{Cone}(f) \supset \mathcal{F}_3\mathrm{Cone}(f) \supset \dots \tag{A.0.2}$$

$$\mathcal{F}_m\mathrm{Cone}(f)^\bullet := \mathcal{F}_m C^{\bullet+1} \oplus \mathcal{F}_m K^\bullet .$$

The differential ∂^{Cone} is compatible with the filtration because f is compatible with the filtrations on C and K. It is obvious to see that the filtration (A.0.2) is complete and bounded above. Furthermore, we observe that

$$\mathrm{Gr}\big(\mathrm{Cone}(f)\big) = \mathrm{Cone}\big(\mathrm{Gr}(f)\big) .$$

Then, since $\mathrm{Gr}(f)$ is a quasi-isomorphism Claim A.0.1 implies that $\mathrm{Gr}\big(\mathrm{Cone}(f)\big)$ is acyclic. Thus, by the Claim A.0.2 we get that $\mathrm{Cone}(f)$ is acyclic. Therefore, applying A.0.1 once again we conclude that f is a quasi-isomorphism. □

APPENDIX B

PROPERTIES OF THE CODERIVATION Π

In this appendix, we prove the properties of the coderivation Π (see (3.3.15)) of the coalgebra (3.3.12) listed in Proposition 3.3.3.

It is straightforward to see that Π has degree zero.

Since

$$p \circ [\Pi, Q_{-d}]\, (\mathbf{s}^{-2}\, \eta_1\, \dots\, \mathbf{s}^{-2}\, \eta_n) = p \circ Q_{[\ ,\ ]_\omega}\, (\mathbf{s}^{-2}\, \eta_1\, \dots\, \mathbf{s}^{-2}\, \eta_n) = 0$$

if $n \neq 2$, to prove (3.3.16), it suffices to show that

$$p \circ (\Pi \circ Q_{-d} - Q_{-d} \circ \Pi)\, (\mathbf{s}^{-2}\, \lambda\, \mathbf{s}^{-2}\, \eta) = p \circ Q_{[\ ,\ ]_\omega}\, (\mathbf{s}^{-2}\, \lambda\, \mathbf{s}^{-2}\, \eta), \qquad \text{(B.0.1)}$$

where λ and η are homogeneous vectors in

$$\frac{1}{\varepsilon^{k-1}}\, \mathfrak{m}\, \Omega^k(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]] \qquad \text{and} \qquad \frac{1}{\varepsilon^{r-1}}\, \mathfrak{m}\, \Omega^r(M)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]]$$

respectively.

Let $U \subset M$ be an open coordinate subset with coordinates $x^1, x^2, \dots, x^m$ and

$$\lambda\big|_U = dx^{i_1} dx^{i_2} \dots dx^{i_k}\, \lambda_{i_1 \dots i_k}\,, \qquad \eta\big|_U = dx^{j_1} dx^{j_2} \dots dx^{j_r}\, \eta_{j_1 \dots j_r}\,,$$

where $\lambda_{i_1 \dots i_k} \in C^\infty(U)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]][\varepsilon^{-1}]$, $\eta_{j_1 \dots j_r} \in C^\infty(U)[[\varepsilon, \varepsilon_1, \dots, \varepsilon_g]][\varepsilon^{-1}]$ and summation over repeated indices is assumed.

The direct computation shows that

$$p \circ \Pi \circ Q_{-d}\,(\mathbf{s}^{-2}\,\lambda\,\mathbf{s}^{-2}\,\eta)\big|_U =$$

$$(-1)^{|\lambda|}\mathbf{s}^{-2}\,\varepsilon\alpha^{ij}(x)dx^{i_1}dx^{i_2}\dots dx^{i_k}\,\partial_{x^i}\lambda_{i_1\dots i_k}\,\frac{\partial\eta}{\partial\,dx^j} \tag{B.0.2}$$

$$-\,\mathbf{s}^{-2}\,\varepsilon\alpha^{ij}(x)\frac{\partial\lambda}{\partial dx^i}\,dx^{j_1}dx^{j_2}\dots dx^{j_r}\,\partial_{x^j}\eta_{j_1\dots j_r} \tag{B.0.3}$$

$$-\,(-1)^{|\lambda|}\mathbf{s}^{-2}\,\varepsilon\,k\,\alpha^{i_1 j}(x)dx^{i}dx^{i_2}\dots dx^{i_k}\,\partial_{x^i}\lambda_{i_1\dots i_k}\frac{\partial\eta}{\partial\,dx^j} \tag{B.0.4}$$

$$+\,\mathbf{s}^{-2}\,\varepsilon\,r\,\alpha^{ij_1}(x)\frac{\partial\lambda}{\partial dx^i}\,dx^{j}dx^{j_2}\dots dx^{j_r}\,\partial_{x^j}\eta_{j_1\dots j_r}\,. \tag{B.0.5}$$

Furthermore,

$$-p \circ Q_{-d} \circ \Pi\,(\mathbf{s}^{-2}\,\lambda\,\mathbf{s}^{-2}\,\eta)\big|_U =$$

$$(-1)^{\lambda}\,\mathbf{s}^{-2}\,\varepsilon\,dx^t\partial_{x^t}\alpha^{ij}(x)\,\frac{\partial\lambda}{\partial dx^i}\,\frac{\partial\eta}{\partial dx^j} \tag{B.0.6}$$

$$(-1)^{|\lambda|}\mathbf{s}^{-2}\,\varepsilon\,k\,\alpha^{i_1 j}(x)dx^{i}dx^{i_2}\dots dx^{i_k}\,\partial_{x^i}\lambda_{i_1\dots i_k}\frac{\partial\eta}{\partial\,dx^j} \tag{B.0.7}$$

$$-\,\mathbf{s}^{-2}\,\varepsilon\,r\,\alpha^{ij_1}(x)\frac{\partial\lambda}{\partial dx^i}\,dx^{j}dx^{j_2}\dots dx^{j_r}\,\partial_{x^j}\eta_{j_1\dots j_r}\,. \tag{B.0.8}$$

Term (B.0.7) (resp. term (B.0.8)) cancels term (B.0.4) (resp. term (B.0.5)) in the left hand side of (B.0.1). Moreover the sum of terms (B.0.2), (B.0.3) and (B.0.6) coincides with

$$p \circ Q_{[\ ,\ ]_\omega}\,(\mathbf{s}^{-2}\,\lambda\,\mathbf{s}^{-2}\,\eta)\big|_U\,.$$

Thus identity (B.0.1) (and hence (3.3.16)) is proved.

To prove (3.3.17), we observe that

$$p \circ \Pi \circ Q_{[\ ,\ ]_\omega}\,(\mathbf{s}^{-2}\,\eta_1\,\dots\,\mathbf{s}^{-2}\,\eta_n) = p \circ Q_{[\ ,\ ]_\omega} \circ \Pi\,(\mathbf{s}^{-2}\,\eta_1\,\dots\,\mathbf{s}^{-2}\,\eta_n) = 0$$

if $n \neq 3$.

For $n = 3$, a direct computation shows that

$$p \circ \Pi \circ Q_{[\ ,\ ]_\omega}\,(\mathbf{s}^{-2}\,\eta_1\,\mathbf{s}^{-2}\,\eta_2\,\mathbf{s}^{-2}\,\eta_3) - p \circ Q_{[\ ,\ ]_\omega} \circ \Pi\,(\mathbf{s}^{-2}\,\eta_1\,\mathbf{s}^{-2}\,\eta_2\,\mathbf{s}^{-2}\,\eta_3) =$$

$$-2\Big(\,(-1)^{|\eta_2|}\,\mathbf{s}^{-2}\,\varepsilon^2\,\alpha^{i_1 j_1}(x)(\partial_{x^{i_1}}\alpha^{ij}(x))\,\frac{\partial\eta_1}{\partial\,dx^i}\,\frac{\partial\eta_2}{\partial\,dx^j}\,\frac{\partial\eta_3}{\partial\,dx^{j_1}}$$

$$+(-1)^{|\eta_3|+|\eta_1|(|\eta_2|+|\eta_3|)}\,\mathbf{s}^{-2}\,\varepsilon^2\,\alpha^{i_1 j_1}(x)(\partial_{x^{i_1}}\alpha^{ij}(x))\,\frac{\partial\eta_2}{\partial\,dx^i}\frac{\partial\eta_3}{\partial\,dx^j}\frac{\partial\eta_1}{\partial\,dx^{j_1}}$$

$$+(-1)^{|\eta_1|+|\eta_3|(|\eta_1|+|\eta_2|)}\,\mathbf{s}^{-2}\,\varepsilon^2\,\alpha^{i_1 j_1}(x)(\partial_{x^{i_1}}\alpha^{ij}(x))\,\frac{\partial\eta_3}{\partial\,dx^i}\frac{\partial\eta_1}{\partial\,dx^j}\frac{\partial\eta_2}{\partial\,dx^{j_1}}\Big) =$$

$$2(-1)^{|\eta_2|+1}\,\mathbf{s}^{-2}\,\varepsilon^2\,\big(\alpha^{i_1 j_1}(x)\partial_{x^{i_1}}\alpha^{ij}(x)+\alpha^{i_1 i}(x)\partial_{x^{i_1}}\alpha^{jj_1}(x)+\alpha^{i_1 j}(x)\partial_{x^{i_1}}\alpha^{j_1 i}(x)\big)\,\frac{\partial\eta_1}{\partial\,dx^i}\frac{\partial\eta_2}{\partial\,dx^j}\frac{\partial\eta_3}{\partial\,dx^{j_1}}\,.$$

Thus (3.3.17) is a consequence of the Jacoby identity for the Poisson structure α. Proposition 3.3.3 is proved. □