Real Analysis Ph.D. Qualifying Exam
Department of Mathematics, Temple University
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• Justify your answers thoroughly.
• You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
• For any theorem that you wish to cite, you should either give its name or a statement of the theorem.
• \( \mathbb{N} \) and \( \mathbb{R} \) stand for the set of natural numbers, and the set of real numbers, respectively.

Part I (Do three problems)

I.1.

(a) Show that there exists a constant \( c > 0 \) such that \( (1 + x/n)^n \geq 1 + cx^2 \) for all \( x \geq 0 \) and all \( n \in \mathbb{N}, n \geq 2 \).
(b) Compute \( \lim_{n \to \infty} \int_{0}^{\infty} (1 + x/n)^{-n} \sin(x/n) \, dx \) and justify the calculation. Here \( dx \) denotes integration with respect to the Lebesgue measure in \( \mathbb{R} \).

I.2. Let \( a, b \in \mathbb{R} \) such that \( a < b \), and consider a function \( f : [a, b] \to \mathbb{R} \). Show that \( V(f; [a, b]) \), the total variation of \( f \) on \( [a, b] \), satisfies \( V(f; [a, b]) = f(b) - f(a) \) if and only if \( f \) is monotonically increasing on \( [a, b] \).

I.3. Let \( (X, \mathcal{M}, \mu) \) be a measure space (with the typical convention that \( \mu \) is positive). Suppose \( f_n, g_n \), for \( n \in \mathbb{N} \), and \( f, g \) are real valued integrable functions on \( X \) satisfying

(a) \( f_n \to f \) and \( g_n \to g \), as \( n \to \infty \), \( \mu \)-a.e. on \( X \),
(b) \( |f_n| \leq g_n \) on \( X \) for each \( n \in \mathbb{N} \),
(c) \( \int_X g_n \, d\mu \to \int_X g \, d\mu \) as \( n \to \infty \).

Prove that \( \int_X f_n \, d\mu \to \int_X f \, d\mu \). Hint: Use Fatou’s Lemma for \( f_n + g_n \) and for \(-f_n + g_n\).

I.4. Let \( (X, \mathcal{M}, \mu) \) be a measure space (with the typical convention that \( \mu \) is positive). A collection of functions \( \{f_\alpha\}_{\alpha \in A} \) in \( L^1(X, \mathcal{M}, \mu) \) is called uniformly integrable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \left| \int_E f_\alpha \, d\mu \right| < \varepsilon \) for all \( \alpha \in A \) whenever \( E \in \mathcal{M} \) satisfies \( \mu(E) < \delta \). Show that
(a) Any finite subset of $L^1(X, \mathcal{M}, \mu)$ is uniformly integrable.

(b) Prove that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^1(X, \mathcal{M}, \mu)$ that converges in the $L^1$ metric to $f \in L^1(X, \mathcal{M}, \mu)$, then the collection $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

Part II (Do two problems)

II.1. Let $(X, \mathcal{M}, \mu)$ be a measure space (with the typical convention that $\mu$ is positive) and assume that $1 < p < \infty$, and that $f \in L^p(X, \mathcal{M}, \mu)$. Suppose $D$ is a dense subset of $L^q(X, \mathcal{M}, \mu)$, where $1/p + 1/q = 1$. Prove that $f = 0$ $\mu$-a.e. on $X$ if and only if $\int_X fg \, d\mu = 0$ for all $g \in D$.

II.2. Consider the measure space $(\mathbb{R}, \mathcal{M}, \mathcal{L}^1)$, where $\mathcal{M}$ is the $\sigma$-algebra of Lebesgue measurable sets in $\mathbb{R}$ and $\mathcal{L}^1$ is the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$.

(a) Prove that if $f \in L^p(\mathbb{R}, \mathcal{M}, \mathcal{L}^1)$ and $g \in L^q(\mathbb{R}, \mathcal{M}, \mathcal{L}^1)$ for $1 \leq p < \infty$ with $1/p + 1/q = 1$, then the function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) = (f \ast g)(x) = \int_{\mathbb{R}} f(x-y)g(y) \, d\mathcal{L}^1(y)$$

is uniformly continuous.

(b) Show that if $A \in \mathcal{M}$ is such that $\mathcal{L}^1(A) > 0$ then $1_A \ast 1_A \neq 0$ $\mathcal{L}^1$-a.e. by computing its integral (you are allowed to change order of integration without having to justify it.) Here $1_A$ denotes the characteristic function of the set $A$.

(c) Let $A \in \mathcal{M}$ be such that $\mathcal{L}^1(A) > 0$. Prove that the set

$$A + A = \{x \mid \exists a \in A, b \in A, x = a + b\}$$

contains an open interval.

II.3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $C^1$ bijective function. Define $\nu(A) := |\varphi^{-1}(A)|$ for each Borel measurable set $A \subseteq \mathbb{R}$. Prove that $\nu$ is a Borel measure in $\mathbb{R}$ and that, with $\mathcal{L}^1$ denoting the Lebesgue measure in $\mathbb{R}$, the Radon-Nikodym derivative $d\nu/d\mathcal{L}^1$ equals $|\varphi'|$. 