Real Analysis Ph.D. Qualifying Exam
Temple University
January 14, 2021

Part I (Do three problems)

I.1. Prove that

(a) the integral $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$ converges; and

(b) $\lim_{n\to\infty} n \int_{1/n}^1 \frac{\cos\left(x+\frac{1}{n}\right) - \cos x}{x^{3/2}}\,dx = -\int_0^1 \frac{\sin x}{x^{3/2}}\,dx.$

I.2. Suppose $\{f_n\}$ and f are nonnegative measurable functions on a measure space (X, Σ, μ) with $f_n, f \in L^1(d\mu)$. Prove that if $f_n \to f$ a.e. and

$$\int_X f_n d\mu \to \int_X f\,d\mu,$$

then for every measurable function g bounded

$$\int_X f_n\, g\, d\mu \to \int_X f\, g\, d\mu.$$

Hint: Use Fatou's Lemma for $f_n(g + M)$ and for $f_n(-g + M)$ where M is such that $|g| \leq M$ in X.

I.3. Prove that E is Lebesgue measurable if and only if $\forall\, \epsilon > 0\ \exists\, F$ Borel measurable such that $F \subset E$ and $|E \setminus F|_e < \epsilon$; $|\cdot|_e$ denotes Lebesgue outer measure.

I.4. Let $E \subset \mathbb{R}^n$ be a measurable set with $|E| < \infty$ and let $E_k \subset E$ be measurable sets such that $|E_k| \to |E|$ as $k \to \infty$. Prove that there is a subsequence E_{k_j} such that $\chi_{E_{k_j}}(x) \to \chi_E(x)$ as $j \to \infty$ for a.e. x.

Hint: consider $\int_{\mathbb{R}^n} \left(\chi_{E_k}(x) - \chi_E(x)\right)^2\, dx$.

Part II (Do two problems)

II.1. Let f be absolutely continuous on $[a,b]$ and assume that $f' \in L^p([a,b])$ for some p, $1 < p \leq \infty$. Prove that f is Hölder continuous with exponent $\alpha = 1 - \frac{1}{p}$.

II.2. Take for granted the following fact: if $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the convolution function $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ is uniformly continuous.

Prove that if $A \subset \mathbb{R}^n$ is a measurable set with Lebesgue measure $0 < |A| < \infty$, then the set

$$A + A = \{x : \ \exists\, a, b \in A,\ x = a + b\}$$

contains an open ball.

Hint: Take $f = g = \chi_A$, and show that $\displaystyle\int_{\mathbb{R}^n} (\chi_A * \chi_A)(x)\, dx > 0$; χ_A denotes the characteristic function of A. Use the granted fact to conclude.

II.3. We say that a sequence of functions $f_n \in L^2([0,1])$ converges to zero weakly if

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = 0$$

for every $g \in L^2([0,1])$.

(a) If $\lim_{n\to\infty} \|f_n\|_{L^2([0,1])} = 0$, then prove that f_n converges to zero weakly.

(b) Give an example of a sequence of functions in $L^2([0,1])$ that converges to zero almost everywhere but does not converge to zero weakly.

(c) Show that $\sin(2\pi n x)$ converges to zero weakly.