Real Analysis Ph D Qualifying Exam
Temple University
January 2020

Part I (Do three problems)

I.1. Give an example of an \( f \) which is not Lebesgue integrable on \( \mathbb{R} \), but whose improper Riemann integral exists and is finite. Prove your answer.

I.2. Show that if \( g : \mathbb{R} \to \mathbb{R} \) is continuous and compactly supported, and \( f, \{ f_n \} \in L^1(\mathbb{R}) \) are such that \( \int_{\mathbb{R}} |f_n - f| \, dx \to 0 \) as \( n \to \infty \), then \( gf \in L^1(\mathbb{R}) \) and \( \int_{\mathbb{R}} |gf_n - gf| \, dx \to 0 \) as \( n \to \infty \).

I.3. Let \( f : [a, b] \to \mathbb{R} \) be continuous. Prove that
\[
\lim_{n \to \infty} \left( \int_{a}^{b} |f(x)|^n \, dx \right)^{1/n} = \sup_{x \in [a, b]} |f(x)|.
\]

I.4. Consider a sequence of real-valued functions \( \{f_n(x)\} \). Recall that \( f_n \) converges in measure to \( f \) if for every \( \delta > 0 \),
\[
\lim_{n \to \infty} \left| \{ \{x \in \mathbb{R} : |f_n(x) - f(x)| > \delta \} \right| = 0.
\]

(a) Prove that if \( f_n \) converges to \( f \) in \( L^1 \) then \( f_n \) converges to \( f \) in measure.
(b) Does convergence in measure imply convergence in \( L^1 \)? Justify your answer.
(c) Do there exist sequences \( f_n \) defined on \([0, 1]\) that converge in measure, to the function 0, but do not converge a.e.?

Part II (Do two problems)

II.1. Let \( A \subset [0, 1] \) be a Borel set such that \( 0 < m(A \cap I) < m(I) \) for every interval \( I \) in \([0, 1]\). Here \( m \) denotes the Lebesgue measure. Let \( F(x) = m([0, x] \cap A) \). Prove that \( F \) is absolutely continuous and strictly increasing on \([0, 1]\) and that \( F' = 0 \) on a set of positive measure.

II.2. Let \( f(x, y), 0 \leq x, y \leq 1 \), satisfy the following conditions: for each \( x \), \( f(x, y) \) is an integrable function of \( y \), and \( (\partial f(x, y)/\partial x) \) is a bounded function of \( (x, y) \). Show that \((\partial f(x, y)/\partial x)\) is a measurable function of \( y \) for each \( x \) and \( \frac{d}{dx} \int_{0}^{1} f(x, y) \, dy = \int_{0}^{1} \frac{\partial}{\partial x} f(x, y) \, dy \).

II.3. Let \( \Gamma(y) = \int_{0}^{\infty} e^{-x} x^{y-1} \, dx \), \( y > 0 \).

(a) Show that \( \Gamma \) is continuous on \((0, \infty)\), without using Part (b).
(b) Show that \( \Gamma'(y) = \int_{0}^{\infty} e^{-x} x^{y-1} \ln x \, dx \), \( y > 0 \).