Part I. (Do 3 problems)

1. Suppose \( f_n \to f \) uniformly in \( E \) where \( f_n \) are continuous. Prove that if \( x_0 \in E \) and \( x_n \to x_0 \) with \( x_n \in E \), then \( f_n(x_n) \to f(x_0) \).

2. Let \( f_n(x) = n \sin \left( \frac{x}{n} \right) \). Prove that:
   (a) \( f_n \) converges uniformly on any finite interval. Hint: \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \) for all \( x \).
   (b) \( f_n \) does not converge uniformly on \( \mathbb{R} \).
   (c) \( f_n \) does not converge in measure on \( \mathbb{R} \). Hint: the interval \((n\pi,(n+1)\pi)\) is contained in the set \(|f_n(x) - x| > \epsilon\).

3. Prove that the upper lim is sub additive and lower lim is super additive:
   \[
   \limsup_{k \to \infty} (a_k + b_k) \leq \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k
   \]
   \[
   \liminf_{k \to \infty} (a_k + b_k) \geq \liminf_{k \to \infty} a_k + \liminf_{k \to \infty} b_k.
   \]
   To avoid operations with \( \pm \infty \) assume the sequences are bounded.

4. Prove that on \( C[0,1] \) the norms \( \|f\|_\infty = \max_{x \in [0,1]} |f(x)| \) and \( \|f\|_1 = \int_0^1 |f(x)| \, dx \) are not equivalent.

Part II. (Do 2 problems)

1. Let \( f \in L^p(E, \mu) \), \( 1 \leq p < \infty \), and \( E = \bigcup_{j=1}^\infty E_j \) with \( E_j \subset E_{j+1} \). Prove that \( f \chi_{E_j} \to f \) in \( L^p(E, \mu) \).

2. Let \( \mu \) be a Borel measure in \( \mathbb{R} \) with \( \mu(\mathbb{R}) < \infty \). Define \( f(x) = \mu((a,x]) \) for \( x \in \mathbb{R} \). Prove that
   (a) \( f \) is monotone increasing
   (b) \( \mu([a,b]) = f(b) - f(a) \); for \( a < b \)
   (c) \( f \) is continuous from the right
   (d) \( \lim_{x \to -\infty} f(x) = 0 \).

3. Let \( f : [a,b] \to \mathbb{R} \) integrable. Prove that the functions \( f_n(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) \, dt \) are well defined for \( a \leq x \leq b \), \( n = 1, 2, \cdots \) and satisfy \( \int_a^x f_n(t) \, dt = f_{n+1}(x) \).