Part I. (Do 3 problems)

1. Let $a_n$ and $\varepsilon_n$ be sequences of real numbers satisfying $|a_{n+1} - a_n| \leq \varepsilon_n$ for all $n$ with $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Prove that $a_n$ converges to some $a$ and $|a - a_n| \leq \sum_{k=n}^{\infty} \varepsilon_k$.

2. Given two sets $A, B \subset \mathbb{R}^n$ define $A + B = \{x + y, x \in A, y \in B\}$. Prove that
   (a) if $A$ is open or $B$ is open, then $A + B$ is open,
   (b) if $A$ is compact and $B$ is closed, then $A + B$ is closed.
   (c) in $\mathbb{R}^2$ take $A = \{(x, 0) : x \in \mathbb{R}\}$ and $B = \{(y, 1/y) : y > 0\}$, show $A$ and $B$ are both closed and $A + B$ is not.

3. Let $f_n(x) = n x e^{-n x^2}$ on $[0, +\infty)$. Prove that
   (a) $f_n$ converges to zero pointwise in $[0, +\infty)$
   (b) $f_n$ does not converge uniformly in $[0, +\infty)$
   (c) $f_n$ converges in measure on $[0, +\infty)$
   (d) $\int_0^{\infty} f_n(x) \, dx = \frac{1}{2}$.

   HINT for (c): may use that $e^z \geq z^2/2$ for all $z \geq 0$.

4. Suppose $f_n \to f$ a.e. on $\mathbb{R}^n$, $f_n$ measurable. Prove that for each $\epsilon > 0$ there exist a sequence of disjoint measurable sets $E_j$ of finite measure such that $|\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} E_j| < \epsilon$ and $f_n \to f$ uniformly on each $E_j$.

Part II. (Do 2 problems)

1. Let $b > 0$, $f \in L^1(0, b)$ and let $g(x) = \int_x^b \frac{f(t)}{t} \, dt$ for $0 < x < b$. Prove that $g \in L^1(0, b)$ and
   $$\int_0^b g(x) \, dx = \int_0^b f(t) \, dt.$$

2. Let $1 \leq p < \infty$, $f_k, f \in L^p(E)$, and let $a_k = \int_E |f_k(x) - f(x)|^p \, dx$. Suppose that $\sum_{k=1}^{\infty} a_k < \infty$. Prove that $f_k \to f$ a.e. in $E$ as $k \to \infty$.

3. Let $f$ be a continuous function in $[-1, 2]$. For $0 \leq x \leq 1$ and $k \geq 1$ let
   $$f_k(x) = \frac{k}{2} \int_{x - \frac{1}{k}}^{x + \frac{1}{k}} f(t) \, dt.$$

   Prove that $f_k$ is continuous in $[0, 1]$ and $f_k \to f$ uniformly in $[0, 1]$. 