Justify your answers thoroughly.
You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let $X, \mu$ be a compact metric space and $\mu$ an arbitrary Borel measure on $X$. Use the latter to define $L^\infty(X)$ with norm $\|f\|_\infty$, $f \in L^\infty(X)$.

(a) Show that $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $f \in C(X)$ (†)

(b) Show that if $\mu(B) > 0$ for every (nonempty) ball in $X$, then equality holds in (†).

I.2. Give an example of a non-negative measurable function on $R = [-1, 1] \times [-1, 1]$ such that $\int_R f(x, y) \, dx \, dy < \infty$ but $\int_{[-1,1]} f(x, y) \, dx = \infty$ $\forall y \in Q \cap [0,1]$.

I.3. Let $f \in L^1(0, \infty)$ be nonnegative. Prove that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n xf(x) \, dx \to 0.$$ 

I.4. Suppose $f_k \to f$ in $L^p$, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \leq M < \infty$ for all $k$. Prove that $f_k g_k \to fg$ in $L^p$.

Part II (Do 2 problems)

II.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} x & \text{if } |x| > 1 \\ 0 & \text{if } |x| \leq 1 \end{cases}$$

Let $\mu$ be the Lebesgue measure on $\mathbb{R}$ (on the Borel $\sigma$-algebra $\mathcal{B}$) and let $\nu = f^* \mu$ be the measure defined by

$$\nu(E) = \mu(f^{-1}(E)), \quad E \in \mathcal{B}.$$ 

Find the Lebesgue decomposition of $\nu$ with respect to $\mu$, i.e., find measures $\lambda$ and $\rho$ such that $\nu = \lambda + \rho$, $\lambda \perp \mu$, $\rho \ll \mu$. Here $\lambda \perp \mu$ means that there exist disjoint sets $A, B \in \mathcal{B}$ such that $\mathbb{R} = A \cup B$ and $\lambda(A) = \mu(B) = 0$. $\rho \ll \mu$ means that $\rho(E) = 0$ for each set $E$ for which $\mu(E) = 0$. 

II.2. Suppose \( f \) and \( xf \) are Lebesgue integrable functions \( \mathbb{R} \to \mathbb{C} \). Show that the function \( \hat{f} : \mathbb{R} \to \mathbb{C} \) defined by
\[
\hat{f}(\xi) = \int e^{-ix\xi} f(x) \, dx
\]
is of class \( C^1 \) (i.e., \( \hat{f} \) is differentiable everywhere with \( \hat{f}' \) continuous).

II.3. Let \( a_n > 0 \) with \( \sum_n a_n < \infty \). Let \( x_n \in \mathbb{R} \) for all \( n \). Let \( f(x) = \sum_{n=1}^{\infty} a_n \chi_{(x_n, \infty)}(x) \). Show that
(a) \( f \) is uniformly convergent on \( \mathbb{R} \).
(b) \( f \) is continuous at \( x \neq x_n \).
(c) \( f \) is right continuous with left limit at \( x_n \).
(d) \( f' = 0 \) a.e.