Real Analysis Ph.D. Qualifying Exam
Temple University
January, 2016

- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let X, d be a compact metric space and μ an arbitrary Borel measure on X. Use the latter to define $L^\infty(X)$ with norm $\|f\|_\infty$, $f \in L^\infty(X)$.

(a) Show that

$$\|f\|_\infty \leq \sup_{x \in X} |f(x)|, \quad f \in C(X) \tag{$\dagger$}$$

(b) Show that if $\mu(B) > 0$ for every (nonempty) ball in X, then equality holds in (†).

I.2. Give an example of a non-negative measurable function on $R = [-1,1] \times [-1,1]$ such that $\int_R f(x,y)\,dx\,dy < \infty$ but

$$\int_{[-1,1]} f(x,y)\,dx = \infty \quad \forall y \in \mathbb{Q} \cap [0,1].$$

I.3. Let $f \in L^1(0,\infty)$ be nonnegative. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x) dx \to 0.$$

I.4. Suppose $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \leq M < \infty$ for all k. Prove that $f_k g_k \to fg$ in L^p.

Part II (Do 2 problems)

II.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} x & \text{if } |x| > 1 \\ 0 & \text{if } |x| \leq 1 \end{cases}$$

Let μ be the Lebesgue measure on $\mathbb{R}$ (on the Borel σ-algebra $\mathcal{B}$) and let $\nu = f_* \mu$ be the measure defined by

$$\nu(E) = \mu(f^{-1}(E)), \quad E \in \mathcal{B}.$$

Find the Lebesgue decomposition of ν with respect to μ, i.e., find measures λ and ρ such that $\nu = \lambda + \rho$, $\lambda \perp \mu$, $\rho \ll \mu$. Here $\lambda \perp \mu$ means that there exist disjoint sets $A, B \in \mathcal{B}$ such that $\mathbb{R} = A \cup B$ and $\lambda(A) = \mu(B) = 0$. $\rho \ll \mu$ means that $\rho(E) = 0$ for each set E for which $\mu(E) = 0$.

II.2. Suppose f and xf are Lebesgue integrable functions $\mathbb{R} \to \mathbb{C}$. Show that the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\, dx$$

is of class C^1 (i.e., $\hat{f}$ is differentiable everywhere with $\hat{f}'$ continuous).

II.3. Let $a_n > 0$ with $\sum_n a_n < \infty$. Let $x_n \in R$ for all n. Let $f(x) = \sum_{n=1}^{\infty} a_n \chi_{[x_n,\infty)}(x)$. Show that

(a) f is uniformly convergent on R.

(b) f is continuous at $x \neq x_n$.

(c) f is right continuous with left limit at x_n.

(d) $f' = 0$ a.e.