• Justify your answers thoroughly.
• You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
• For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let \( \{f_n\} \) be a sequence of Lebesgue measurable functions on \( \mathbb{R}^n \). Suppose you have an estimate of the form
\[
\int_{\mathbb{R}^n} |f_n| \leq c_n \text{ where } c_n \downarrow 0.
\]
Can you conclude that \( f_n \to 0 \) a.e.? If not, what additional condition(s) on \( \{c_n\} \) would guarantee this?

I.2. Let \( f \in L^\infty[0, 1] \) and assume that \( f \) is not identically zero. Show that the limit,
\[
\lim_{p \to \infty} \frac{\int_0^1 |f|^p + 1 \, dx}{\int_0^1 |f|^p \, dx},
\]
exists and compute it.

I.3. Let \( F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x} \, dx \), \( y > 0 \).
   (a) Show that \( F \) is continuous on \((0, \infty)\).
   (b) Prove \( F'(y) = -\int_0^\infty e^{-xy} \sin x \, dx \), \( y > 0 \).

I.4. Let \( A \triangle B = (A \setminus B) \cup (B \setminus A) \) denote the symmetric difference of sets \( A \) and \( B \). Let \( A_n \) and \( B_n \) be measurable subsets of \( \mathbb{R} \). Suppose \( \lambda(A_n \triangle B_n) = 0 \), for all \( n \), where \( \lambda \) is the Lebesgue measure.
   (a) Show that \( \lambda[(\bigcup_{n=1}^\infty A_n) \triangle (\bigcup_{n=1}^\infty B_n)] = 0 \).
   (b) Show that
\[
\lambda[(\limsup_{n \to \infty} A_n) \triangle (\limsup_{n \to \infty} B_n)] = 0,
\]
where \( \limsup_{n \to \infty} A_n = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n \).

Part II (Do 2 problems)

II.1. Let \( S \) be a measurable subset of \( \mathbb{R}^2 \). Assume for every \( x \in S \) there exists a sequence of cubes \( \{Q_k(x)\} \) centered at \( x \) with side lengths tending to zero such that
\[
|S \cap Q_k(x)| \leq \frac{1}{2} |Q_k(x)|.
\]
Show that \( |S| = 0 \).
II.2. A sequence of functions \( \{ f_n \} \in L^1[0, 1] \) is said to be uniformly integrable if

\[
\lim_{C \to \infty} \sup_{n \geq 1} \int_{\{ x \in [0,1] : |f_n(x)| > C \}} |f_n(x)| \, dx = 0.
\]

If for such a sequence it holds that \( f_n \to f \) almost everywhere for some measurable \( f \), prove that \( f_n \to f \) in \( L^1[0,1] \) norm.

II.3. Prove that

\[
\int_0^\infty \frac{\sin t}{e^t - x} \, dt = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2 + 1},
\]

for \(-1 < x < 1\).