All functions on $\mathbb{R}^d$ are assumed Lebesgue measurable and all integrals are against Lebesgue measure. Justify your answers.

Part I. (Select 3 questions.)

1. Let $f_n(x) = \frac{1}{n}e^{-n^2x^2}$ for $x \in \mathbb{R}$. Prove that
   1. $f_n$ converges to 0 uniformly in $\mathbb{R}$;
   2. $f'_n$ does not converge uniformly on any interval containing 0.

2. Let $f_n(x) = n^{3/2}xe^{-n^2x^2}$ for $-1 \leq x \leq 1$. Prove that
   1. $f_n$ converges to zero pointwise in $[-1, 1]$;
   2. $\int_{-1}^{1} |f_n(x)|^2 dx \not\to 0$.

3. Let $f_n(x) = \sin \sqrt{x} + 4n^2\pi^2$ on $[0, +\infty)$. Prove that
   1. $f_n$ is equicontinuous on $[0, +\infty)$.
   2. $f_n$ is uniformly bounded.
   3. $f_n \to 0$ pointwise on $[0, +\infty)$.
   4. There is no subsequence of $f_n$ that converges to 0 uniformly.
   5. Compare with Arzelà-Ascoli.

4. Show that the series
   $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$
   converges for each $x \in \mathbb{R}$ and the sum is a Lipschitz function.

5. Let $f(x) = x^2 \sin(1/x^2)$ for $x \in [-1, 1], \ x \neq 0$, and $f(0) = 0$. Show that $f$ is differentiable on $[-1, 1]$ but $f'$ is unbounded on $[-1, 1]$.

Part II. (Select 2 questions.)

1. If $|f_k| \leq g$ a.e. with $g$ integrable in $E$, and $f_k \to f$ in measure in $E$, then prove that
   $$\int_E f(x) \, dx = \lim_{k \to \infty} \int_E f_k(x) \, dx.$$

2. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbb{R}^n$ such that $|E_j \cap E_i| = 0$ for $j \neq i$. Prove that
   $$|\bigcup_{j=1}^{\infty} E_j| = \sum_{j=1}^{\infty} |E_j|.$$

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, i.e., there exists $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}^n$. Prove that if $N$ is a set of measure zero, then $f(N)$ has measure zero.