Mathematics Real Analysis Ph.D. Qualifying Exam
Temple University
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All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable and all integrals are against Lebesgue measure. Justify your answers.

Part I. (Select 3 questions.)

1. Let $f_n(x) = \frac{1}{n} e^{-n^2 x^2}$ for $x \in \mathbf{R}$. Prove that
    1. f_n converges to 0 uniformly in $\mathbf{R}$;
    2. f_n' does not converge uniformly on any interval containing 0.
2. Let $f_n(x) = n^{3/2} x e^{-n^2 x^2}$ for $-1 \le x \le 1$. Prove that
    1. f_n converges to zero pointwise in $[-1, 1]$;
    2. $\int_{-1}^{1} |f_n(x)|^2\, dx \not\to 0$.
3. Let $f_n(x) = \sin \sqrt{x + 4n^2\pi^2}$ on $[0, +\infty)$. Prove that
    1. f_n is equicontinuous on $[0, +\infty)$.
    2. f_n is uniformly bounded.
    3. $f_n \to 0$ pointwise on $[0, +\infty)$.
    4. There is no subsequence of f_n that converges to 0 uniformly.
    5. Compare with Arzelà-Ascoli.
4. Show that the series
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$
converges for each $x \in \mathbf{R}$ and the sum is a Lipschitz function.
5. Let $f(x) = x^2 \sin(1/x^2)$ for $x \in [-1, 1]$, $x \neq 0$, and $f(0) = 0$. Show that f is differentiable on $[-1, 1]$ but f' is unbounded on $[-1, 1]$.

Part II. (Select 2 questions.)

1. If $|f_k| \le g$ a.e. with g integrable in E, and $f_k \to f$ in measure in E, then prove that
$$\int_E f(x)\, dx = \lim_{k\to\infty} \int_E f_k(x)\, dx.$$
2. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbf{R}^n$ such that $|E_j \cap E_i| = 0$ for $j \neq i$. Prove that
$$|\bigcup_{j=1}^{\infty} E_j| = \sum_{j=1}^{\infty} |E_j|.$$
3. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is Lipschitz, i.e., there exists $K > 0$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in \mathbf{R}^n$. Prove that if N is a set of measure zero, then $f(N)$ has measure zero.