## Mathematics Real Analysis Ph.D. Qualifying Exam    Friday, January 14, 2006

All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable, all integrals are against Lebesgue measure, and $L^p = L^p(\mathbf{R}^d)$. You may not use or refer to the Riemann integral in any of your answers.

**Part I. (Select 3 questions.)**

1. Let $f$ be a nonnegative measurable function defined on a measurable set $E$. Show that there exist a sequence $(f_n)$ of simple measurable functions that increases pointwise to $f$ as $n \to \infty$.

2. Let
$$F(y) = \int_{-\infty}^{\infty} e^{-x^2} \cos(2xy)dx, \quad y \in \mathbf{R}.$$
Show that $F$ satisfies the differential equation
$$F'(y) + 2yF(y) = 0.$$
Solve for $F$ and justify all the steps.

3. Let $f_n \to f$ in $L^p$, $1 \leq p < \infty$, and let $(g_n)$ be a sequence of measurable functions such that $|g_n| \leq M < \infty$, for all $n$, and $g_n \to g$ a.e. Show that $g_n f_n \to gf$ in $L^p$.

4. Let $E$ be a measurable subset of $\mathbf{R}^d$. Let $f_n$ and $f$ be measurable functions defined on $E$. Suppose that $f_n$ and $f$ are finite a.e., for all $n$.
   (a) Suppose the Lebesgue measure $|E| < \infty$. Show that if $f_n \to f$, a.e., as $n \to \infty$, then $f_n \to f$, in measure, as $n \to \infty$.
   (b) Give a counter example if the condition $|E| < \infty$ in (a) is omitted.

**Part II. (Select 2 questions.)**

1. Let $\mathcal{B}$ be the class of all Borel sets in $\mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be Borel measurable if $\{f > \alpha\} \in \mathcal{B}$, for all $\alpha \in \mathbf{R}$.
   (a) Show that $f$ is Borel measurable if and only if $f^{-1}(B) \in \mathcal{B}$, for all $B \in \mathcal{B}$.
   (b) Show that if $f$ and $g$ are Borel measurable then $f \circ g$ is Borel measurable.
   (c) Show that a continuous function is Borel measurable.

2. Let $(a_n)$ be a positive sequence and let $F(t) = \sum_{n=1}^{\infty} e^{-tn} a_n$, $t \geq 0$.
   (a) Suppose $\sum_{n=1}^{\infty} n a_n < \infty$. Show that the right-hand derivative
   $$F'_+(0) = \lim_{t \to 0+} \frac{F(t) - F(0)}{t}$$
   exists and is finite.
   (b) Conversely, suppose that $F(t)$ is finite for every $t \geq 0$ and that $F'_+(0)$ exists and is finite. Show that $\sum_{n=1}^{\infty} n a_n < \infty$. (Hint: Apply Fatou's Lemma)

3. Let $a_{nm} \in \mathbf{R}$, for $n = 1, 2, 3, ...$ and $m = 1, 2, 3, ...$ Suppose $|a_{nm}| \leq 1$, for all $n, m$. Show that there exists a subsequence $n_k$ such that $\lim_{k \to \infty} a_{n_k m} = b_m$ exists for all $m$.