PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 2002

Part I. Do three (3) of these problems.

I.1. Show that
\[ \lim_{n \to \infty} \sum_{k=0}^{2n} \frac{k}{k^2 + n^2} = \frac{1}{2} \ln 5. \]

I.2. Let \( \{x_k\}_{k=1}^\infty \) be a sequence of real numbers satisfying
(a) \( \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} = L \), and
(b) \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k(x_k - x_{k-1}) = 0 \); (we set \( x_0 = 0 \)).
Prove that \( \lim_{n \to \infty} x_n = L \).

I.3. Suppose \( f \) is Riemann integrable on \([a, b]\) and \( f(x) = 0 \) for all \( x \in [a, b] \cap \mathbb{Q} \). Prove that \( \int_a^b f(x) \, dx = 0 \).

I.4. Let \( f \in C^1[0, 1] \), \( \delta = \min_{[0,1]} |f'(x)| \), and \( \Delta = \max_{[0,1]} |f'(x)| \). Prove that
\[ \frac{1}{12} \delta^2 \leq \int_0^1 f^2(x) \, dx - \left( \int_0^1 f(x) \, dx \right)^2 \leq \frac{1}{12} \Delta^2. \]
Hint: expand \( \int_0^1 \int_0^1 (f(x) - f(y))^2 \, dx \, dy \), Fubini and mean value theorem.
II.1. Let \( f_n(x) = n^{1/2} e^{-nx} \) on \([0, 1]\). Prove that
(a) \( f_n(x) \to 0 \) pointwise on \((0, 1]\),
(b) \( \int_0^1 f_n(x)^2 \, dx \leq C \) for all \( n \),
(c) \( f_n \) does not converge in \( L^2(0, 1) \),
(d) \( \int_0^1 f_n(x) g(x) \, dx \to 0 \) for each \( g \in L^2(0, 1) \).

Hint: for (d) prove it first for simple functions and then use the density of the simple functions in \( L^2(0, 1) \).

II.2. Let \( g : \mathbb{R} \to \mathbb{R} \) be continuous and invertible. Suppose that for each Lebesgue measurable set \( A \), the set \( g(A) \) is Lebesgue measurable and define the measure \( \mu(A) = |g(A)| \). Prove that the measure \( \mu \) is absolutely continuous with respect to Lebesgue measure if and only if \( g \) is an absolutely continuous function, and in that case \( \frac{d\mu}{dx} = g'(x) \) a.e.

II.3. Let \( f_k \) be a sequence of functions in \( L^2(\mathbb{R}^n) \). Suppose that \( \|f_k\|_{L^2(\mathbb{R}^n)} \leq M \) for all \( k \), and \( f_k \to f \) a.e. Prove that
\[
\int_{\mathbb{R}^n} f_k(x) g(x) \, dx \to \int_{\mathbb{R}^n} f(x) g(x) \, dx,
\]
for all \( g \in L^2(\mathbb{R}^n) \).