PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. Let \( \{a_n\} \) be a sequence of real numbers with the following property: there is a constant \( 0 < K < 1 \) such that
\[
|a_{n+2} - a_{n+1}| \leq K|a_{n+1} - a_n| \quad \text{for all } n \geq N_0.
\]
Prove that \( \{a_n\} \) converges.

I.2. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function and \( x_1, \ldots, x_n \in [a, b] \). Show that there exists \( z \in [a, b] \) such that
\[
f(z) = f(x_1) + \cdots + f(x_n).
\]

I.3. Give an example of a function \( f \in L^p(\mathbb{R}), p \geq 1 \), such that
\[
\lim_{x \to \infty} f(x) \neq 0.
\]

I.4. (1) Let \( \{f_n\} \) be a subsequence of \( L^1(\mathbb{R}) \) such that \( \sum_{n=1}^{\infty} \|f_n\|_1 < \infty \). Show that \( \sum_{n=1}^{\infty} f_n \) converges absolutely a.e.

(2) Let \( (X, \mathcal{A}, \mu) \) be a measure space and let \( \{A_n\} \) be a subsequence of \( \mathcal{A} \). Show that if \( \sum_{n=1}^{\infty} \mu(A_m) < \infty \) then \( \mu(\limsup A_n) = 0 \), where \( \limsup A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \).

Part II. Do two (2) of these problems.

II.1. Let
\[
F(y) = \int_{0}^{\infty} e^{-2x} \cos(2xy)dx, \quad y \in \mathbb{R}.
\]
Show that \( F \) satisfies the differential equation
\[
F'(y) + 2yF(y) = 0.
\]
Justify the differentiation under the integral sign.

II.2. Let \( f \) be a real valued function defined on a closed bounded interval \([a, b]\). Establish the following:

(1) If \( f \) is continuous, \( f \) need not be of bounded variation. Consider
\[
f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}
\]
(2) If $f$ satisfies a Lipschitz condition, that is, $|f(x) - f(y)| \leq M|x - y|$ for some positive number $M$ and all $x, y \in [a, b]$, then $f$ is absolutely continuous.

(3) If $f'$ exists everywhere and is bounded on $[a, b]$, then $f$ is absolutely continuous.

II.3. Let $H$ be a Hilbert space and $y_0 \in H$. Show that there exists $\Lambda \in H^*$ (bounded linear functional on $H$) different from zero such that

$$\Lambda(y_0) = \|\Lambda\|_{H^*} \|y_0\|.$$ 

(Hint: Either apply the Hahn-Banach theorem with the sublinear functional $p(x) = \|x\|$, or construct a bounded linear functional in terms of $y_0$).