Ph.D. Comprehensive Examination
Real Analysis
Fall 2017

Part I. Do three of these problems.

I.1. Let $f(x) = \cos x$, for $0 \le x < 2\pi$, and $f(x) = \sin x$, for $2\pi \le x \le 4\pi$.

(a) Find the positive variation $P(x)$ and negative variation $N(x)$ of f on $[0, x]$ (without proofs). Sketch their graphs on the interval $0 \le x \le 4\pi$.

(b) Find
$$\int_0^{4\pi} N dP, \quad \int_0^{4\pi} P dN, \quad \text{and} \quad \int_0^{4\pi} N dP + \int_0^{4\pi} P dN.$$

I.2. Give an example of a closed subset F of $\mathbb{R}$ such that $0 < |F| < 1$ and F contains no open interval.

I.3. Let $f \in L^\infty(\mathbb{R})$. The essential range of f is defined by
$$R_f = \{y : \text{the measure of } \{x : |f(x) - y| < \epsilon\} \text{ is positive for all } \epsilon > 0\}.$$
Prove that

(a) R_f is compact.

(b) Find a relation between R_f and $||f||_\infty$.

I.4. Suppose $\mathcal{R}$ is a finite dimensional vector space (over $\mathbb{R}$) of continuous functions $\mathbb{R} \to \mathbb{R}$ that is also closed under multiplication:
$$f, g \in \mathcal{R} \implies fg \in \mathcal{R}.$$
Show that $\mathcal{R}$ consists only of constant functions.

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1. Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a finite measure space and $f : \mathcal{X} \to \mathbb{R}$ a measurable function. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(t) = \mu(\{x \in \mathcal{X} : f(x) < t\})$$

Show:

(a) F is continuous from below: for each $t_0 \in \mathbb{R}$, $\lim_{t \to t_0^-} F(t) = F(t_0)$.

(b) If $\mu(\{x : f(x) = t_0\}) = 0$, then F is continuous at t_0.

II.2. Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and let f be a complex valued function defined on $\mathbb{R} \times \mathcal{X}$. Let $\delta > 0$, and suppose that for $t \in [t_0 - \delta, t_0 + \delta]$,

(a) $u(t) = \displaystyle\int_{\mathcal{X}} f(t,x)\, d\mu(x)$ with $\displaystyle\int_{\mathcal{X}} |f(t,x)|\, d\mu(x) < \infty$,

(b) for each fixed $x \in \mathcal{X}$, $\dfrac{\partial f}{\partial t}(t,x)$ exists and is a continuous function of t,

(c) $\displaystyle\int_{\mathcal{X}} \sup_{-\delta \le \tau \le \delta} \left| \frac{\partial f}{\partial t}(t_0 + \tau, x) \right| d\mu(x) < \infty$.

Show that then $u'(t_0) = \displaystyle\int_{\mathcal{X}} \frac{\partial f}{\partial t}(t_0, x)\, d\mu(x)$.

II.3. Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a finite measure space and let $f : \mathcal{X} \to \mathbb{R}$ be a measurable function.

(a) Show that if f^n is integrable for each $n = 1, 2, 3, \ldots$ and $\lim \displaystyle\int f^n(x)\, d\mu(x)$ exists in $\mathbb{R}$, then $|f(x)| \le 1$ holds for almost all x.

(b) If f^n is integrable for each $n = 1, 2, 3, ...$, then show that $\displaystyle\int_X f^n\, d\mu(x) = c$ (a constant independent of n) for all $n = 1, 2, 3, ...$ if and only if $f = \chi_A$, the characteristic function of A, for some measurable subset A of X.