Ph.D. Comprehensive Examination
Real Analysis

Fall 2016

Part I. Do three of these problems.

I.1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with asymptotes $y = a_+x + b_+$ as $x \to +\infty$ and $y = a_-x + b_-$ as $x \to -\infty$ ($a_\pm$, $b_\pm$ some real numbers). Show that f is uniformly continuous.

I.2. Show that

$$\lim_{r\to\infty} \int_0^r \frac{\sin t}{t}\, dt$$

converges, but

$$\lim_{r\to\infty} \int_0^r \frac{|\sin t|}{t}\, dt$$

does not.

I.3. Let $\{f_k\}_{k=1}^\infty$ be a sequence of continuous functions $(-1,1) \to \mathbb{R}$ such that $|f_k(x)| \le M$, for all x and k. Let $g_k : (-1,1) \to \mathbb{R}$ be defined by

$$g_k(x) = \int_0^x f_k(t)\, dt.$$

Show that $\{g_k\}_{k=1}^\infty$ has a uniformly convergent subsequence.

I.4. Give an example of a sequence of functions $f_n(x)$ in $[0,1]$ such that $\int_0^1 |f_n(x)|dx \to 0$, as $n \to \infty$, but $f_n(x)$ does not converge to zero for any x in $[0,1]$. Can you make the functions f_n continuous?

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\{a_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive numbers with $c_0 a_k \leq a_{k+1} \leq c_1 a_k$ for some constants c_0, c_1, both > 1. Suppose $f : X \to \mathbb{R}$ is measurable. Show:

$$\sum_{k=1}^{\infty} a_k \, \mu\{x : a_k \leq |f(x)|\} < \infty \iff f \in L^1(X, \mathcal{M}, \mu)$$

II.2. Let $\mathbb{R}[x]$ be the set of polynomials on $\mathbb{R}$, let $F = \{p(x)e^{-x^2/2} : p \in \mathbb{R}[x]\}$. Show that F is dense in $L^2(\mathbb{R})$.

II.3. Let $f(x, y), 0 \leq x, y \leq 1$, be a function such that for each x, $f(x, y)$ is an integrable function of y, and $(\partial f(x, y)/\partial x)$ is a bounded function of (x, y). Show that $(\partial f(x, y)/\partial x)$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial}{\partial x} f(x, y) dy.$$