Justify your answers thoroughly.
You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let \( A_k \) be a sequence of measurable subsets of \([0, 1]\) such that, for every finite set of indices \( i_1 < i_2 < \cdots < i_k \),
\[
m(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = m(A_{i_1})m(A_{i_2}) \cdots m(A_{i_k})
\]
where \( m \) stands for Lebesgue measure.

(a) Show that the sequence \( B_k = [0, 1] \setminus A_k \) has the same property.

(b) Suppose in addition that the series \( \sum m(A_k) \) diverges. Show that
\[
m(\bigcup_{k=1}^{\infty} A_k) = 1
\]

I.2. Let \( p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2}, t > 0, x \in \mathbb{R} \). It is known that \( \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2} dx = 1 \). Let \( f \in L^\infty(\mathbb{R}) \) and \( u(t, x) = f \ast p_t(x) \).

Show that \( \frac{\partial}{\partial t} u(t, x) = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial t} p_t(x-y) dy, t > 0, x \in \mathbb{R} \).

I.3. Let \( r_n \) be the sequence of all rational numbers and
\[
f(x) = \sum_{n, r_n < x} \frac{1}{2^n}.
\]
Prove that

(a) \( f \) is continuous at irrational numbers \( x \).

(b) \( f \) is discontinuous at rational numbers \( r_n \).

(c) Calculate \( \int_0^1 f \).

I.4. Consider the expression
\[
\int_0^{\infty} \frac{\sin x}{x^\alpha} dx.
\]

Does there exist an \( \alpha > 0 \) such that this exists an improper Riemann integral but does not exist as a Lebesgue integral? Prove your answer.
Part II (Do 2 problems)

II.1. Assume that $f : [0, 1] \mapsto \mathbb{R}$ is an absolutely continuous function with $\int_0^1 f(x)dx = 0$. Prove that for any $y \in [0, 1]$ it holds

$$\left| \int_0^1 (y - x)f'(x)dx \right| \leq \sup_{0 \leq x \leq 1} |f(x)|.$$

II.2. Let $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a mapping given by

$$T_\theta(x, y) = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta).$$

Show that $\| f \circ T_\theta - f \|_p \to 0$, as $\theta \to 0$, for all $f \in L^p(\mathbb{R}^2)$, $0 < p < \infty$.

II.3. If $\{f_1, f_2, \ldots\}$ is a complete orthonormal set in $L^2[0, 1]$ and $A$ is an arbitrary subset of positive Lebesgue measure in $[0, 1]$ show that

$$1 \leq \int_A \sum_{i=1}^\infty |f_i(x)|^2 dx.$$