Real Analysis Ph.D. Qualifying Exam
Temple University
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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do 3 problems)

I.1. Let A_k be a sequence of measurable subsets of $[0,1]$ such that, for every finite set of indices $i_1 < i_2 < \cdots < i_k$,

$$m(A_{i_1} \cap A_{i_2} \cap \cdots A_{i_k}) = m(A_{i_1})m(A_{i_2})\dots m(A_{i_k})$$

where m stands for Lebesgue measure.

(a) Show that the sequence $B_k = [0,1] \setminus A_k$ has the same property.

(b) Suppose in addition that the series $\sum m(A_k)$ diverges. Show that

$$m\Big(\cup_{k=1}^{\infty} A_k\Big) = 1.$$

I.2. Let $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2}$, $t > 0, x \in R$. It is known that $\int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$. Let $f \in L^\infty(R)$ and $u(t,x) = f * p_t(x)$.

Show that $\frac{\partial}{\partial t} u(t,x) = \int_R f(y) \frac{\partial}{\partial t} p_t(x-y) dy$, $t > 0, x \in R$.

I.3. Let r_n be the sequence of all rational numbers and

$$f(x) = \sum_{n: r_n < x} \frac{1}{2^n}.$$

Prove that

(a) f is continuous at irrational numbers x.

(b) f is discontinuous at rational numbers r_n.

(c) Calculate $\int_0^1 f$.

I.4. Consider the expression

$$\int_0^\infty \frac{\sin x}{x^\alpha} dx.$$

Does there exist an $\alpha > 0$ such that this exists an improper Riemann integral but does not exist as a Lebesgue integral? Prove your answer.

Part II (Do 2 problems)

II.1. Assume that $f : [0,1] \mapsto \mathrm{R}$ is an absolutely continuous function with $\int_0^1 f(x)dx = 0$. Prove that for any $y \in [0,1]$ it holds

$$\left| \int_0^1 (y-x) f'(x) dx \right| \leq \sup_{0 \leq x \leq 1} |f(x)|.$$

II.2. Let $T_\theta : R^2 \to R^2$ be a mapping given by

$$T_\theta(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta).$$

Show that $\|f \circ T_\theta - f\|_p \to 0$, as $\theta \to 0$, for all $f \in L^p(R^2)$, $0 < p < \infty$.

II.3. If $\{f_1, f_2, \ldots\}$ is a complete orthonormal set in $L^2[0,1]$ and A is an arbitrary subset of positive Lebesgue measure in $[0,1]$ show that

$$1 \leq \int_A \sum_{i=1}^{\infty} |f_i(x)|^2 dx.$$