Real Analysis Ph.D. Qualifying Exam

Temple University

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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Notation: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively, and dx indicates integration with respect to the Lebesgue measure on $\mathbb{R}$.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Let $E \subseteq \mathbb{R}$ be a set of Lebesgue measure zero. Show that there exists $a \in \mathbb{R}$ such that the set

$$E + a := \{x + a : x \in E\}$$

contains no rational numbers.

I.2. Define the function $f : [0,1] \longrightarrow \mathbb{R}$ by setting

$$f(x) := \begin{cases} x^3 \cos\left(\dfrac{\pi}{x^2}\right) & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the function f is of bounded variation on the interval $[0,1]$.

I.3. Problem 2. Let $I = [0,\infty)$ and for each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f_n(x) := \begin{cases} 1/n, & \text{on } [0,n]; \\ 0, & \text{otherwise.} \end{cases}$$

(a) For each $x \in \mathbb{R}$ find $f(x) := \lim_{n\to\infty} f_n(x)$.
(b) Is the convergence $\lim_{n\to\infty} f_n = f$ uniform on the interval I?
(c) Show that $\lim_{n\to\infty} \int_I f_n \, dx \neq \int_I f \, dx$.
(d) Show that the hypotheses of the Monotone Convergence Theorem are not satisfied.

I.4. For each $n \in \mathbb{N}$ consider the function $f_n : [0, +\infty) \longrightarrow \mathbb{R}$ given by

$$f_n(x) := \sin\left(\sqrt{4\pi^2 n^2 + x}\right), \qquad \forall\, x \geq 0.$$

Prove that

(1) f_n converges uniformly on each interval $[0, a]$ with $a > 0$;
(2) f_n does not converge uniformly on $[0, +\infty)$.

Part II. (Do 2 problems):

II.1. Show that if f is an essentially bounded function on the interval $[0, 1]$, then

$$\lim_{p\to\infty} \left(\int_{[0,1]} |f|^p \, dx \right)^{1/p} = \|f\|_\infty.$$

II.2. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of Lebesue measurable sets in $\mathbb{R}^n$ such that $|E_j \cap E_i| = 0$ for $j \neq i$ (here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$). Prove that

$$\left| \bigcup_{j=1}^{\infty} E_j \right| = \sum_{j=1}^{\infty} |E_j|.$$

II.3. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable functions on the interval $[0, 1]$ such that this converges in measure on $[0, 1]$ to the function f. Show that

$$\lim_{n\to\infty} \int_{[0,1]} \sin(f_n)\, dx = \int_{[0,1]} \sin(f)\, dx.$$