Part I. (Do 3 problems)

1. Let \( f \in C^2(a, b) \). Prove that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x)
\]
for each \( x \in (a, b) \).

2. Let \( f_n(x) = \frac{n x}{x^2 + n^2}, x \in \mathbb{R} \). Show that
   (a) \( f_n \) does not converge uniformly in \( \mathbb{R} \);
   (b) \( f_n \) does not converge in measure in \( \mathbb{R} \).

3. Let \( f(x, y) = \frac{x - y}{(x + y)^3} \). Show that \( f \notin L^1([1, \infty) \times [1, \infty)) \). Justify your answer.

4. Find
\[
\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} \, dx.
\]
Justify your answer.

Part II. (Do 2 problems)

1. Let \( f \) be absolutely continuous on \([a, b] \). Prove that
   (a) if \( E \subset [a, b] \) with \( |E| = 0 \), then \( |f(E)| = 0 \);
   (b) if \( E \) is measurable, then \( f(E) \) is measurable.

2. Let \( f_1, \ldots, f_k \) be continuous real valued functions on the interval \([a, b] \). Show that the set \( \{f_1, \ldots, f_k\} \) is linearly dependent on \([a, b]\) if and only if the \( k \times k \) matrix with entries
\[
\langle f_i, f_j \rangle = \int_a^b f_i(x) f_j(x) \, dx
\]
has determinant zero.

3. Let \( f_n : E \to \mathbb{R} \) be a sequence of measurable functions and \( a \in \mathbb{R} \). Suppose that
\[
\sum_{n=1}^{\infty} |\{x \in E : f_n(x) > a\}| < \infty.
\]
Prove that \( \lim_{n \to \infty} f_n(x) \leq a \) for a.e. \( x \in E \).

HINT: let \( E_n = \{x \in E : f_n(x) > a\} \) and \( S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \). Show \( |S| = 0 \), and if \( \lim_{n \to \infty} f_n(x) > a \), then \( x \in S \).