Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let $f \in C^1(\mathbb{R})$ with $|f'(x)| \le M$ for all x. Prove that
   (a) if $g \in BV[a,b]$, then the composition $f \circ g$ is of bounded variation in $[a,b]$.
   (b) if g is absolutely continuous on $[a,b]$, then $f \circ g$ is absolutely continuous on $[a,b]$.

2. Consider the sequence $f_n(x) = n^2\, x\, e^{-n\,x^2}$ on $[1,+\infty)$. Prove that
   (a) f_n converges uniformly on $[1,+\infty)$;
   (b) f_n converges in measure on $[1,+\infty)$;
   (c) $\displaystyle\int_1^\infty f_n(x)\,dx \to 0$ as $n \to \infty$.

3. Let $f \in L^\infty(\mathbb{R})$. The essential range of f is defined by
$$R_f = \{y \in \mathbb{R} : |\{x \in \mathbb{R} : |f(x) - y| < \epsilon\}| > 0,\ \text{for all } \epsilon > 0\}.$$
   Prove that
   (a) $R_f \subseteq [-\|f\|_\infty, +\|f\|_\infty]$;
   (b) R_f is compact.

4. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Is $f \in L^1([1,\infty)\times[1,\infty))$? Justify your answer.

Part II. (Do 2 problems)

1. Let f, f_k be measurable functions in $\mathbb{R}$ such that $f_k \to f$ a.e. Suppose there exist $g, g_k \in L^1(\mathbb{R})$ such that $|f_k| \le g_k$, $g_k \to g$, a.e., and $\lim_{k\to\infty}\int_{\mathbb{R}} g_k = \int_{\mathbb{R}} g$. Prove that $\lim_{k\to\infty}\int_{\mathbb{R}} |f_k - f| = 0$.
   Hint: $|f_k - f| \le g_k + |f|$, write $\int_{\mathbb{R}} \liminf_{k\to\infty}\left(g_k + |f| - |f_k - f|\right)\,dx$ and use Fatou's Lemma.

2. Let $f(t,x)$ be a function defined in $(a,b)\times\mathbb{R}$ such that:
   - for each fixed $t \in (a,b)$ the function $f(t,\cdot)$ is measurable;
   - for each fixed $x \in \mathbb{R}$ the function $f(\cdot,x)$ is continuous.

   Prove that the function $g(x) = \sup_{t\in(a,b)} f(t,x)$ is measurable.

3. Let $f \ge 0$ in $\mathbb{R}$. Prove that if $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ is in $L^1(\mathbb{R})$, then $f = 0$ a.e.