Real Analysis Ph.D. Qualifying Exam, Temple University, August 25, 2010

Part I. (Select 3 questions.)

1. Prove that the series $f(x) = \sum_{n=0}^{\infty} \frac{\cos(\sqrt{n}\,x)}{1+n^2}$ converges uniformly in $\mathbf{R}$, and show that

$$\lim_{R\to+\infty} \frac{1}{2R}\int_{-R}^{R} f(x)\,dx = 1.$$

2. Given two sets $A, B \subset \mathbf{R}^n$ define $A + B = \{x + y, x \in A, y \in B\}$. Prove that

   (a) if A is open or B is open, then $A + B$ is open,

   (b) if A is compact and B is closed, then $A + B$ is closed.

   (c) in $\mathbf{R}^2$ take $A = \{(x, 0) : x \in \mathbf{R}\}$ and $B = \{(y, 1/y) : y > 0\}$, show A and B are both closed and $A + B$ is not.

3. Let f be a continuous function in the closed interval $[0, 1]$. Prove that

$$\lim_{a\to 0^+} a\int_0^1 x^{a-1} f(x)\,dx = f(0).$$

4. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous. Suppose that $f\left(\sum_{i=1}^N \lambda_i x_i\right) \leq \sum_{i=1}^N \lambda_i f(x_i)$ for all $x_i \in \mathbf{R}$ and for all $\lambda_i \geq 0$ such that $\sum_{i=1}^N \lambda_i = 1$, that is, f is convex. Prove that

$$f(x) \leq \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt,$$

for all $x \in \mathbf{R}$ and $h > 0$. Hint: consider the partition of $[x-h, x+h]$ given by $t_i = x - h + \frac{2h}{N}i$ for $i = 0, 1, \cdots, N$. Estimate the corresponding Riemann sum using the convexity.

Part II. (Select 2 questions.)

1. If $f_n : [0, 1] \to \mathbf{R}$ are measurable with $f_n \to 0$ a.e, and $\left|\int_0^1 f_n(x)\,dx\right| \leq C$ for all n, then can you conclude that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$?

2. Let $f \in L^2[0, 1]$. Show that $F(x) = \int_0^x f(t)\,dt$ is continuous in $[0, 1]$ and satisfies

$$\lim_{x\to 0^+} x^{-1/2} F(x) = 0.$$

3. Let $A, B \subset [0, 1]$ measurable sets with $|A| > 1/2$ and $|B| > 1/2$.

   (a) Prove that $|A \cap (1 - B)| > 0$ (recall $1 - B = \{1 - x : x \in B\}$).

   (b) Show that the function $f(x) = \int_{\mathbf{R}} \chi_A(y)\chi_B(x - y)\,dy$ is well defined for all $x \in \mathbf{R}$.

   (c) Conclude that there exist $y \in A$ and $z \in B$ such that $y + z = 1$.