Part I. (Select 3 questions.)

1. We say \( f : \mathbb{R} \to \mathbb{R} \) is superlinear if
\[
\lim_{x \to \pm \infty} \frac{f(x)}{|x|} = +\infty.
\]
Show that \( f \) superlinear and differentiable implies \( f'(\mathbb{R}) = \mathbb{R} \).

2. Given \( a_0 > b_0 > 0 \), let
\[
a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n \geq 0.
\]
Show that \( (a_n) \) is decreasing, \( (b_n) \) is increasing, and both sequences converge to the same limit.

3. Use the geometric series to show that
\[
\sum_{n=1}^{\infty} \frac{n^k}{2^n}
\]
is an integer for \( k = 1, 2, 3, \ldots \).

4. Given a set \( E \subset \mathbb{R}^n \) let \( O_k = \{ x \in \mathbb{R}^n : \text{dist}(x, E) < 1/k \} \). Prove that \( O_k \) is open and if \( E \) is compact, then \( |E| = \lim_{k \to \infty} |O_k| \).

Part II. (Select 2 questions.)

1. Define the Lebesgue measure \( |A| \) of a set \( A \subset \mathbb{R}^d \). Show that, if \( |A| > 0 \) and \( \epsilon > 0 \), there is a product of intervals \( Q = I_1 \times I_2 \times \cdots \times I_d \) satisfying
\[
|Q \cap A| > (1 - \epsilon)|Q|.
\]

2. Construct a sequence of functions \( f_n : [0, 1] \to \mathbb{R} \) such that \( \int_{0}^{1} |f_n(x)| \, dx \to 0 \) and \( f_n(x) \) does not converge for any \( x \in [0, 1] \).

3. Let \( f_k \) be a sequence of measurable functions on \( E \). Show that \( \sum_{k=1}^{\infty} f_k \) converges absolutely a.e. in \( E \) if \( \sum_{k=1}^{\infty} \int_{E} |f_k| < \infty \). Use this to prove that if \( \{r_k\} \) denotes the rational numbers in \( [0, 1] \) and \( \{a_k\} \) satisfies \( \sum_{k=1}^{\infty} |a_k| < \infty \), then \( \sum_{k=1}^{\infty} \frac{a_k}{|x - r_k|^{1/2}} \) converges absolutely a.e. in \( [0, 1] \).