Mathematics Real Analysis Ph.D. Qualifying Exam
Temple University
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All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable and all integrals are against Lebesgue measure. You may not use or refer to the Riemann integral in any of your answers; everything must be justified within the context of the Lebesgue theorems (MCT, DCT, LDT, ...).

Part I. (Select 3 questions.)

1. We say $f : \mathbf{R} \to \mathbf{R}$ is *superlinear* if

$$\lim_{x \to \pm\infty} \frac{f(x)}{|x|} = +\infty.$$

   Show that f superlinear and differentiable implies $f'(\mathbf{R}) = \mathbf{R}$.

2. Given $a_0 > b_0 > 0$, let

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n \geq 0.$$

   Show that (a_n) is decreasing, (b_n) is increasing, and both sequences converge to the same limit.

3. Use the geometric series to show that

$$\sum_{n=1}^{\infty} \frac{n^k}{2^n}$$

   is an integer for $k = 1, 2, 3, \ldots$.

4. Given a set $E \subset \mathbf{R}^n$ let $O_k = \{x \in \mathbf{R}^n : \operatorname{dist}(x, E) < 1/k\}$. Prove that O_k is open and if E is compact, then $|E| = \lim_{k\to\infty} |O_k|$.

Part II. (Select 2 questions.)

1. Define the Lebesgue measure $|A|$ of a set $A \subset \mathbf{R}^d$. Show that, if $|A| > 0$ and $\epsilon > 0$, there is a product of intervals $Q = I_1 \times I_2 \times \cdots \times I_d$ satisfying

$$|Q \cap A| > (1 - \epsilon)|Q|.$$

2. Construct a sequence of functions $f_n : [0, 1] \to \mathbf{R}$ such that $\int_0^1 |f_n(x)|\, dx \to 0$ and $f_n(x)$ does not converge for any $x \in [0, 1]$.

3. Let f_k be a sequence of measurable functions on E. Show that $\sum_{k=1}^{\infty} f_k$ converges absolutely a.e. in E if $\sum_{k=1}^{\infty} \int_E |f_k| < \infty$. Use this to prove that if $\{r_k\}$ denotes the rational numbers in $[0, 1]$ and $\{a_k\}$ satisfies $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} \dfrac{a_k}{|x - r_k|^{1/2}}$ converges absolutely a.e. in $[0, 1]$.