PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 2002

**Part I.** Do three (3) of these problems.

**I.1.** Suppose that \( f_n \to f \) a.e. on \( E \) and \( f_n \to g \) almost uniformly in \( E \).

(1) Give the definition of almost uniform convergence in \( E \).

(2) Prove that \( f = g \) a.e. on \( E \).

**I.2.** Let \( f : [0, +\infty) \to \mathbb{R}^+ \) be nondecreasing, and such that there exists a positive constant \( C \) satisfying

\[
\int_{2r}^{4r} f(t) \, dt \leq C \int_{r}^{2r} f(t) \, dt,
\]

for each \( r \geq 0 \). Prove that there exists a constant \( C' > 0 \) such that \( f(2r) \leq C' f(r) \) for all \( r \geq 0 \).

**I.3.** Let \( E \subset \mathbb{R}^n \) measurable such that \( |E| < \infty \). Prove that \( |E \cap B_R(0)^c| \to 0 \) as \( R \to \infty \); where \( B_R(0)^c \) denotes the complement of the Euclidean ball with center 0 and radius \( R \).

**I.4.** Consider the set \( C^{1/2} \) consisting of the functions \( f \)'s on \([0, 1]\) such that \( f(0) = 0 \) and

\[
\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/2}} : x \neq y \right\} < \infty.
\]

Prove that \( (C^{1/2}, \| \cdot \|) \) is complete.
Part II. Do two (2) of these problems.

II.1. Let $f_k$ be measurable and $f_k \to f$ a.e. in $\mathbb{R}^n$. Prove that there exists a sequence of measurable sets $\{E_j\}_{j=1}^\infty$ such that $|\mathbb{R}^n \setminus \bigcup_{j=1}^\infty E_j| = 0$ and $f_k \to f$ uniformly on each $E_j$.

II.2. Let $f : (a, b) \to \mathbb{R}$ be convex and $x \in (a, b)$.

1. (1) Prove that $\frac{f(x + h) - f(x)}{h}$, $h > 0$, decreases with $h$; and $\frac{f(x + h) - f(x)}{h}$, $h < 0$, increases with $h$.

2. (2) Prove that the one-sided derivatives

$$D^\pm f(x) = \lim_{h \to 0^\pm} \frac{f(x + h) - f(x)}{h}$$

exist and satisfy

$$\frac{f(x) - f(x - h)}{h} \leq D^- f(x) \leq D^+ f(x) \leq \frac{f(x + h) - f(x)}{h}, \quad h > 0.$$  

II.3. Let $f \in L^1(0, 1)$ and suppose that $\lim_{x \to 1^-} f(x) = A < \infty$. Prove that

$$\lim_{n \to \infty} n \int_0^1 x^n f(x) \, dx = A.$$