PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 1996

Part I. Do three (3) of these problems.

I.1. Determine all the values of p for which the limit

$$\lim_{x\to 0}\frac{\sin(|\sin x|^p)}{x}$$

exists and calculate its value. Justify your answer.

I.2. Show that the series

$$\sum_{n=1}^{\infty}\left(\cos\frac{1}{n}\right)^{n^2}$$

diverges.

Hint: write $\cos\frac{1}{n}=\sqrt{1-\left(\sin\frac{1}{n}\right)^2}$ and use that $0\le\sin\frac{1}{n}\le\frac{1}{n}$.

I.3. Let $f_n:\mathbb{R}\to\mathbb{R}$ be a sequence of equicontinuous functions, that is: for every $\epsilon>0$ there exists $\delta>0$ such that if $|x-y|<\delta$ then $|f_n(x)-f_n(y)|<\epsilon$, for all n.

Show that the set

$$\{x\in\mathbb{R}:\{f_n(x)\}\quad\text{is a Cauchy sequence}\}$$

is closed.

I.4.

(a) Give an example of a function $f\in L^2(\mathbb{R})$ such that $f\notin L^1(\mathbb{R})$.
(b) Show that if $f\in L^2(0,1)$ then $f\in L^1(0,1)$

Part II. Do two (2) of these problems.

II.1. Let $f(x,y) = \dfrac{xy}{x^2+y^2}$.

(a) Show that

$$\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)\,dx\right)dy = \int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)\,dy\right)dx = 0.$$

(b) Prove that f is Lebesgue integrable on $[-1,1]\times[-1,1]$ and calculate the value of the integral

$$\int_{[-1,1]\times[-1,1]} f(x,y)\,dxdy.$$

II.2. Let $f_n(x) = \dfrac{1}{\ln(n+1)}\left(\dfrac{n\,x}{1+n^2\,x^4}\right)$, $0 \le x \le 1$. Prove that

(a) $f_n(x) \to 0$ pointwise in $[0,1]$.
(b) $f_n(x)$ do not converge uniformly to 0 in $[0,1]$.
(c) $f_n(x) \to 0$ in measure in $[0,1]$.

II.3. Show that for any real θ not a multiple of 2π, the sequences of partial sums of the series

$$\sum_{n=1}^{\infty} \cos n\theta, \qquad \sum_{n=1}^{\infty} \sin n\theta$$

are bounded. (Hint: consider $\sum_{n=1}^{N} e^{in\theta}$.)

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cos n\theta$$

converges for all $s > 0$ when θ is not a multiple of 2π. (Hint: use the formula of summation by parts: $\sum_{n=1}^{N} a_n b_n = \left(\sum_{k=1}^{N} a_k\right) b_{N+1} - \sum_{k=1}^{N}\left(\sum_{r=1}^{k} a_r\right)(b_{k+1} - b_k)$.)