PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 1996

**Part I.** Do three (3) of these problems.

**I.1.** Determine all the values of $p$ for which the limit

$$
\lim_{x \to 0} \frac{\sin(|\sin x|^p)}{x}
$$

exists and calculate its value. Justify your answer.

**I.2.** Show that the series

$$
\sum_{n=1}^{\infty} \left( \cos \frac{1}{n} \right)^{n^2}
$$

diverges.

*Hint:* write $\cos \frac{1}{n} = \sqrt{1 - \left( \sin \frac{1}{n} \right)^2}$ and use that $0 \leq \sin \frac{1}{n} \leq \frac{1}{n}$.

**I.3.** Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of equicontinuous functions, that is: for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$, for all $n$.

Show that the set

$$
\{ x \in \mathbb{R} : \{ f_n(x) \} \text{ is a Cauchy sequence} \}
$$

is closed.

**I.4.**

(a) Give an example of a function $f \in L^2(\mathbb{R})$ such that $f \notin L^1(\mathbb{R})$.

(b) Show that if $f \in L^2(0, 1)$ then $f \in L^1(0, 1)$.
II.1. Let \( f(x, y) = \frac{xy}{x^2 + y^2} \).

(a) Show that
\[
\int_{-1}^{1} \left( \int_{-1}^{1} f(x, y) \, dx \right) \, dy = \int_{-1}^{1} \left( \int_{-1}^{1} f(x, y) \, dy \right) \, dx = 0.
\]

(b) Prove that \( f \) is Lebesgue integrable on \([-1, 1] \times [-1, 1]\) and calculate the value of the integral
\[
\int_{[-1,1] \times [-1,1]} f(x, y) \, dxdy.
\]

II.2. Let \( f_n(x) = \frac{1}{\ln(n+1)} \left( \frac{n x}{1 + n^2 x^4} \right), 0 \leq x \leq 1 \). Prove that

(a) \( f_n(x) \to 0 \) pointwise in \([0, 1]\).

(b) \( f_n(x) \) do not converge uniformly to 0 in \([0, 1]\).

(c) \( f_n(x) \to 0 \) in measure in \([0, 1]\).

II.3. Show that for any real \( \theta \) not a multiple of \( 2\pi \), the sequences of partial sums of the series
\[
\sum_{n=1}^{\infty} \cos n\theta, \quad \sum_{n=1}^{\infty} \sin n\theta
\]
are bounded. (Hint: consider \( \sum_{n=1}^{N} e^{in\theta} \).)

Show that the series
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \cos n\theta
\]
converges for all \( s > 0 \) when \( \theta \) is not a multiple of \( 2\pi \). (Hint: use the formula of summation by parts: \( \sum_{n=1}^{N} a_n b_n = \left( \sum_{k=1}^{N} a_k \right) b_{N+1} - \sum_{k=1}^{N} \left( \sum_{r=1}^{k} a_r \right) (b_{k+1} - b_k) \).)