Part I. (Do 3 problems)

1. Solve the damped Burgers’ equation

\[ uu_x + u_y = -u, \quad \text{for} \quad x \in \mathbb{R}, y > 0, \]
\[ u(x, 0) = x. \]

2. Let \( u(x, t) \) solve the heat equation

\[ u_t = \Delta u, \quad \text{for} \quad x \in \mathbb{R}^n, t > 0, \]
\[ u = f \quad \text{for} \quad t = 0, \]

with the usual growth condition to guarantee uniqueness in place. Show that

\[ \|u(\cdot, t)\|_{L^p} \leq \|f\|_{L^p} \]

for any \( p \geq 1 \) and all \( t > 0 \).

3. Show that if \( f \in H^1(\Omega) \) for \( \Omega \subset \mathbb{R}^1 \), then \( f \) is Hölder continuous with exponent 1/2. Show that if \( \mathbb{R}^1 \) is replaced by \( \mathbb{R}^n, n > 1 \), then \( f \) need not even be continuous.

4. Let \( f \in L^1(\mathbb{R}^n) \) and its Fourier transform \( \hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i x \cdot y} \, dy \). If \( g(x) = |x| f(x) \) belongs to \( L^1(\mathbb{R}^n) \), then prove that \( \hat{f} \) satisfies the Lipschitz estimate

\[ |\hat{f}(x) - \hat{f}(y)| \leq 2 \pi \|g\|_1 |x - y| \quad \forall x, y \in \mathbb{R}^n. \]
Part II. (Do 2 problems)

1. Consider $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solution to the boundary value problem

$$\Delta u = c u - |\nabla u|^2, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Show that if $c(x) > 0$ for all $x \in \Omega$, then $u \equiv 0$ in $\Omega$.

2. Let $u = u(x, t) \in C^2([0, 1] \times [0, \infty))$ be a solution to

$$u_{tt} - u_{xx} = -\frac{u}{1 + u^2}, \text{ for } 0 < x < 1, t > 0,$$

$$u_t(1,t) u_x(1,t) - u_t(0,t) u_x(0,t) = 0, \text{ for } t > 0.$$

(a) Find a function $\phi$ so that the energy

$$E(t) = \int_0^1 (u_t^2 + u_x^2 + \phi(u)) \, dx$$

is constant in time.

(b) In addition, if $u(0, t) = 0$ for all $t > 0$, then conclude that there is a constant $c > 0$ so that $|u(x, t)| \leq c x^{1/2}$ for all $x \in [0, 1]$ and $t > 0$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $u$ solve the eigenvalue problem

$$-\Delta u = -u^3 + \lambda u \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega.$$ 

Here $u \neq 0$ and $\lambda \in \mathbb{R}$.

Prove the following

(a) $\lambda = \frac{\int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^4 \, dx}{\int_\Omega u^2 \, dx};$

(b) there cannot exist a sequence of eigen-pairs $(u_k, \lambda_k)$ such that $\lambda_k \to 0$. 

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