Ph.D. Comprehensive Examination (Sample I)
Partial Differential Equations

Part I. Do three of these problems.

I.1. Write a formula for a solution of the following initial-value problem

$$\begin{cases} u_t + u_x = u^2 & \text{in } \{(x,t) : t > 0\} \\ u(x,0) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

I.2. Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u = u^3$ in Ω and $u = 0$ on $\partial\Omega$. Prove that $u \equiv 0$ in Ω.

I.3. Let $n > 1$. Show that the function $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(B_1(0))$ where $B_1(0)$ denotes the unit ball in $\mathbb{R}^n$ centered at the origin.

I.4. Write a formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Here $c \in \mathbb{R}$ and $a \in \mathbb{R}^n$ and $g(x)$ is a bounded continuous function.

Part II. Do two of these problems.

II.1. Let u be a harmonic function on $\mathbb{R}^n$. Prove that either u maps $\mathbb{R}^n$ onto $\mathbb{R}$ or it is constant.

II.2. Let $u \in C^2(\mathbb{R} \times [0,\infty)$ solve the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x),\ u_t(x,0) = h(x),\ x \in \mathbb{R}. \end{cases}$$

Suppose g and h have compact support. The kinetic energy at time t is given by $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the potential energy at time t is $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(1) $k(t) + p(t)$ is constant in t.
(2) There exists t_0 such that if $t \geq t_0$, $k(t) = p(t)$.

II.3. Let $a \neq 0$. For $\xi \in \mathbb{R}$ compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2 + a^2}\,dx.$$

Here $i = \sqrt{-1}$.