Ph.D. Comprehensive Examination (Sample I)
Partial Differential Equations

Part I. Do three of these problems.

I.1. Write a formula for a solution of the following initial-value problem
\[
\begin{cases}
ut + u_x = u^2 & \text{in } \{(x, t) : t > 0\} \\
u(x, 0) = g(x), & x \in \mathbb{R}
\end{cases}
\]

I.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfy \( \Delta u = u^3 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \). Prove that \( u \equiv 0 \) in \( \Omega \).

I.3. Let \( n > 1 \). Show that the function \( u(x) = \log \log \left( 1 + \frac{1}{|x|} \right) \) belongs to \( W^{1,n}(B_1(0)) \) where \( B_1(0) \) denotes the unit ball in \( \mathbb{R}^n \) centered at the origin.

I.4. Write a formula for a solution of
\[
\begin{cases}
ut - \Delta u + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) = g(x), & x \in \mathbb{R}^n.
\end{cases}
\]
Here \( c \in \mathbb{R} \) and \( a \in \mathbb{R}^n \) and \( g(x) \) is a bounded continuous function.

Part II. Do two of these problems.

II.1. Let \( u \) be a harmonic function on \( \mathbb{R}^n \). Prove that either \( u \) maps \( \mathbb{R}^n \) onto \( \mathbb{R} \) or it is constant.

II.2. Let \( u \in C^2(\mathbb{R} \times [0, \infty)) \) solve the wave equation:
\[
\begin{cases}
utt - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
u(x, 0) = g(x), u_t(x, 0) = h(x), & x \in \mathbb{R}.
\end{cases}
\]
Suppose \( g \) and \( h \) have compact support. The kinetic energy at time \( t \) is given by \( k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u^2_t(x, t) \, dx \) and the potential energy at time \( t \) is \( p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x, t) \, dx \).
Prove
\[
\begin{align*}
(1) & \quad k(t) + p(t) \text{ is constant in } t. \\
(2) & \quad \text{There exists } t_0 \text{ such that if } t \geq t_0, k(t) = p(t).
\end{align*}
\]

II.3. Let \( a \neq 0 \). For \( \xi \in \mathbb{R} \) compute the integral
\[
\int_{-\infty}^{\infty} e^{ix\xi} \frac{1}{x^2 + a^2} \, dx.
\]
Here \( i = \sqrt{-1} \).